Forty-five minutes

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

12/11/18

Time: 11.00

Marking Scheme Included

Do not publish

Please answer all TWO Questions

This is a CLOSED book examination

The use of electronic calculators is not permitted.

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1. a) Consider the following function:

$$f\colon [1,\infty) \longrightarrow [0,1]$$
$$x \longmapsto x^{-1}.$$

Is this function injective? Is it surjective? Justify your answers. (5 marks)

Model answer and marking scheme for Q1a
This function is injective but not surjective.
Assume that we have two inputs x and x' in $[1,\infty)$ with

$$x^{-1} = fx = fx' = x'^{-1}.$$

We know by Fact 6 from the notes that x^{-1} and x'^{-1} are the respective unique multiplicative inverses of x and x', so their being equal requires x and x' to be equal as well.
For surjectivity we note that no element of the source set is mapped to 0 in the target set since 0 does not have a multiplicative inverse.
One mark each for the two correct answers, two for the proof of to injectivity and one for a counterexample to surjectivity.
I expect the student answers not to make explicit the facts about real numbers they use. For the counterexample to surjectivity they should get the mark if they identify 0 as not being in the image of f; to get both marks for the proof of injectivity I expect the basic property to be correctly identified and there to be a sensible attempt to justify it that does not contain any false claims.

b) Consider the binary operation on non-empty binary strings defined as follows: To obtain $s \circledast s'$, take s and replace its last symbol with the last symbol of s'. For example we have

$$01001 \circledast 110 = 01000.$$

Is this operation associative? Is it commutative? (5 marks)

Model answer and marking scheme for Q1b

The operation is associative. We can see that in order to form

$$(s \circledast s') \circledast s''$$

we take s, and first replace its last symbol with the last symbol of s', and then we replace that same symbol with the last symbol of s'', so overall we have s where the last symbol has been replaced by the last symbol of s''.

If we look at

$$s \circledast (s' \circledast s'')$$

we can see that it arises from taking s and replacing its last symbol with the last symbol of $s' \circledast s''$, which is the last symbol of s'' and so we arrive at the same binary string.

The operation is not commutative. To find a counterexample take any two strings with different last symbols. For example we have

$$0 \circledast 1 = 1 \neq 0 = 1 \circledast 0.$$

One mark each for the correct answer, two marks for the argument that it is associative, one for the counterexample to commutativity. *I expect the students to struggle a bit to write up their argument. If you can understand what they're trying to say then please give them both marks even if the write-up is not that good, but if the write-up contains wrong statements then please withhold one of the marks.*

2. a) Consider the formula

$$\neg(Q \rightarrow P) \wedge \neg(\neg R \rightarrow P).$$

   i) Construct the truth table for this formula. (3 marks)

   ii) Read off the disjunctive normal form of the formula from the truth table. (1 mark)

Model answer and marking scheme for Q2a

i)

P	Q	R	$\neg R$	$Q \rightarrow P$	$\neg(Q \rightarrow P)$	$\neg R \rightarrow P$	Fml
1	**1**	**1**	**0**	**1**	**0**	**1**	**0**
1	**1**	**0**	**1**	**1**	**0**	**1**	**0**
1	**0**	**1**	**0**	**1**	**0**	**1**	**0**
1	**0**	**0**	**1**	**1**	**0**	**1**	**0**
0	**1**	**1**	**0**	**0**	**1**	**1**	**0**
0	**1**	**0**	**1**	**0**	**1**	**0**	**1**
0	**0**	**1**	**0**	**1**	**0**	**1**	**0**
0	**0**	**0**	**1**	**1**	**0**	**0**	**0**

Three marks for correct answer and truth table. One or two marks for small mistakes.

ii) The extracted DNF is $\neg P \wedge Q \wedge \neg R$.
One mark for the correct answer.

b) For **one** of the following notions explain one main use in propositional logic. (2 marks)

   i) binding precedence

   ii) Substitution Theorem

Model answer and marking scheme for Q2b
Instructions. *Answers will obviously vary. Give 1 mark for each underlined aspects if they appear in their answers and what is written makes sense and is not complete non-sense.*

i) The connectives are assumed to have binding precedence which allow brackets to be omitted from formulas.

ii) The significance of the Substitution Theorem is that it gives us the generalised forms of the fundametal laws/equivalences.

c) Consider this propositional formula.

$$(\neg A \to B) \vee \neg\big(\neg B \to (A \wedge C)\big)$$

i. Use our CNF algorithm to transform the formula into conjunctive normal form.
ii. Simplify your answer as much as possible.

Justify all the steps in your derivations. (4 marks)

Model answer and marking scheme for Q2c

i)

$$\begin{aligned}
& (\neg A \to B) \vee \neg\big(\neg B \to (A \wedge C)\big) & \\
\equiv\ & (\neg\neg A \vee B) \vee \neg\big(\neg\neg B \vee (A \wedge C)\big) & \text{Step 1/Elim. } \to \\
\equiv\ & A \vee B \vee \neg\big(B \vee (A \wedge C)\big) & \text{Step 2/Elim } \neg\neg\text{, flattening/assoc. } \vee \\
\equiv\ & A \vee B \vee \big(\neg B \wedge \neg(A \wedge C)\big) & \text{De Morgan} \\
\equiv\ & A \vee B \vee (\neg B \wedge (\neg A \vee \neg C)) & \text{De Morgan} \\
\equiv\ & (A \vee B \vee \neg B) \wedge (A \vee B \vee \neg A \vee \neg C) & \text{Step 4}
\end{aligned}$$

ii) We continue the derivation above.

$$\begin{aligned}
\equiv\ & (A \vee \top) \wedge (\top \vee B \vee \neg C) & \text{law of excl. middle (twice), commutativity } \vee \\
\equiv\ & \top \wedge \top & A \vee \top \equiv \top \equiv \top \vee A \\
\equiv\ & \top & \top \wedge \top \equiv \top
\end{aligned}$$

Four marks for correct answer and correct justifications of each step, .5 for each type of law used, 2.5 + 1.5 marks for (i) and (ii). Only perfect answer with justifications should receive full marks.

END OF EXAMINATION