Forty-five minutes

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

13/11/17

Time: 12.00

Marking Scheme Included

Do not publish

Please answer all TWO Questions

This is a CLOSED book examination

The use of electronic calculators is not permitted.

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1. a) Consider the following function:

$$f\colon \mathbb{N} \longrightarrow \{k \in \mathbb{Z} \mid k \text{ is even}\}$$
$$n \longmapsto \begin{cases} n & n \text{ even} \\ -2n & \text{else.} \end{cases}$$

Is this function injective? Is it surjective? Justify your answers. (5 marks)

Model answer and marking scheme for Q1a
This function is injective but not surjective.
For injectivity we note first of all that since even numbers are mapped to non-negative integers, and odd numbers to negative ones, two numbers can be mapped to the same output by this function only if they are either both even or both odd.
For the former case assume that n and n' are two even numbers. Then if we have

$$n = fn = fn' = n'$$

we clearly have $n = n'$. For the latter case assume that n and n' are two odd numbers. If we have

$$-2n = f(n) = f(n') = -2n',$$

we my divide both sides by 2 and deduce that $n = n'$. In both cases we have derived $n = n'$ which establishes injectivity.
For surjectivity consider $-4 = 2\cdot(-2)$ in the target set. Since this is a negative number it can only be the image of an odd number. But it would have to be equal to the negative of twice an odd number to occur in the image of f, whereas it is the negative of twice 2, which is an even number. Hence no element is mapped to -4.
One mark each for the two correct answers, two for the proof of to injectivity and one for a counterexmaple to surjectivity.
I expect student answers to be less organized. I expect them to have difficulties with the case distinction in the proof, but if it is largely correct then please give them both marks. If they have a counterexample to surjectivity but struggle to show it's a counterexample they should still get the second mark.

b) Consider the binary operation on the set

$$\{0,1,2\}$$

given by the assignment

$$m \circledast n = (mn+1) \text{ mod } 3.$$

Is this operation associative? Is it commutative? Justify your answers.

(5 marks)

Model answer and marking scheme for Q1b
The operation is not associative. It shouldn't be too hard to find a counterexample. (In the worst case one would start a table until one stumbles across one.)

$$(0 \circledast 1) \circledast 1 = 1 \circledast 1 = 2,$$

while

$$0 \circledast (1 \circledast 1) = 0 \circledast 2 = 1.$$

The operation is commutative. This isn't hard to prove. Let m and n be elements of the given set. Then

$$\begin{aligned} m \circledast n &= (mn+1) \text{ mod } 3 && \text{def } \circledast \\ &= (nm+1) \text{ mod } 3 && \text{Fact 1} \\ &= n \circledast m && \text{def } \circledast . \end{aligned}$$

One mark each for the correct answer, two marks for the counterexample to associativity, one for the proof of commutativity. *I think finding the counterexample is more time-consuming, hence the allocation. If the students are missing justifications in their proof then please point that out but give them the one mark.*

2. a) Let A be the following propositional formula.

$$P_1 \leftrightarrow (P_2 \to P_1)$$

i) Construct a truth table for the formula. (2 marks)

ii) Describe in a sentence for which valuations the formula is true. (1 mark)

Model answer and marking scheme for Q2a

P_1	P_2	$P_2 \to P_1$	$P_1 \leftrightarrow (P_2 \to P_1)$
1	**1**	**1**	**1**
1	**0**	**1**	**1**
0	**1**	**0**	**1**
0	**0**	**1**	**0**

The interpretation of A is **1** when $\mathrm{v}\,P_1 = \mathbf{1}$ or $\mathrm{v}\,P_2 = \mathbf{1}$. Alternative: The interpretation of A is **1** when the valuation of P_1 and P_2 are not both **0**. Or something equivalent.
(i) Two marks for correct truth table. One mark for small mistake.
(ii) One mark for correct description.

b) Give a brief explanation of **one** of the following. (2 marks)

i) tautology

ii) atomic formula (in this case, also give an example)

iii) Boolean function

Model answer and marking scheme for Q2b
Instructions. *Answers will obviously vary. Give 1 mark for each underlined aspects if they appear in their answers and what is written makes sense and is not complete non-sense.*

i) A <u>propositional formula</u> is a tautology if <u>its interpretation is **1**</u> for all possible truth values/valuation of the propositional variables occurring in the formula (in the Boolean semantics).

ii) An atomic formula is a <u>propositional variable</u>. <u>Example: P</u>

iii) A Boolean function f is a <u>function</u> defined over the <u>truth value set $\mathbb{B}$</u> mapping tuples in $\mathbb{B}^n$ to single values in $\mathbb{B}$.

c) Consider this propositional formula.

$$\neg(\neg R \vee P) \vee (P \wedge R).$$

i) Use our CNF algorithm to transform the formula into conjunctive normal form. (2 marks)

ii) Simplify your answer as much as possible. (3 marks)

Model answer and marking scheme for Q2c

i) $\neg(\neg R \vee P) \vee (P \wedge R)$

$\equiv (\neg\neg R \wedge \neg P) \vee (P \wedge R)$ Step 2/De Morgan

$\equiv (R \wedge \neg P) \vee (P \wedge R)$ Step 3/Elim $\neg\neg$

$\equiv (R \vee P) \wedge (R \vee R) \wedge (\neg P \vee P) \wedge (\neg P \vee R)$ Step 4/distr.

Two marks for correct answer and correct justifications of each step, subtract one mark for every 1-2 mistakes. Only perfect answer should receive full marks.

ii) We continue the derivation above.

$\neg(\neg R \vee P) \vee (P \wedge R)$

$\equiv (R \vee P) \wedge R \wedge \top \wedge (\neg P \vee R)$ idempotency, excl. middle

$\equiv R \wedge (R \vee P) \wedge (\neg P \vee R)$ commutativity, $A \wedge \top \equiv A$

$\equiv R$ absorption twice, commutativity

Three marks for correct answer and correct justifications of each step, subtract one mark for every 1-2 mistakes. Only perfect answer should receive full marks.

END OF EXAMINATION