<u>**Forty-five minutes**</u>

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

14/11/16

Time: 12.00

Marking Scheme Included

Do not publish

This is a CLOSED book examination

The use of electronic calculators is <u>not</u> permitted.

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1. a) Consider the following function:

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{R}^+ \\ z &\longmapsto z \cdot \overline{z} \end{aligned}$$

Is this function injective? Is it surjective? Justify your answers. (5 marks)

Model answer and marking scheme for Q1a
In order to see what this function does it is easiest to calculate its action on a complex number given by its real and imaginary part. For this purpose we calculate

$$a+bi \longmapsto (a+bi)(a-bi) = a^2+b^2.$$

This function is not injective but it is surjective.
It is easy to check that the numbers 1 and -1 are both mapped to 1, which provides an example to show that this function isn't injective. Numerous alternative counterexamples exist.
To see that it is surjective, assume we have $r \in \mathbb{R}^+$. Then we may calculate the (positive) square root of r, that is $\sqrt{r}$, and we can see that $\sqrt{r}+0\cdot i$ is mapped to $\sqrt{r}^2+0^2=r$.
One mark each for the two correct answers, one for a counterexample to injectivity and two for the proof that it is surjective.
I expect student answers to be less organized. If the argument they give is largely correct they should get both marks for the proof.

b) Consider the binary operation on the set

$$\{a,b,c\}$$

given by the following table.

$\circledast$	a	b	c
a	a	a	a
b	a	b	c
c	c	c	c

Is this operation associative? Is it commutative? Justify your answers. (5 marks)

Model answer and marking scheme for Q1b

The operation is associative. In order to show this one can either write out all twenty-seven cases, or one can look more closely at the table to realize that these cases can be summarized in a sensible way.

The fastest way of checking all twenty-seven possibilities is to write them out similarly to a truth table. This is also a good way to spot any patterns, in which case one wouldn't complete the whole table.

s	s'	s''	$s \circledast s'$	$(s \circledast s') \circledast s''$	$s' \circledast s''$	$s \circledast (s' \circledast s'')$
a	a	a	a	a	a	a
a	a	b	a	a	a	a
a	a	c	a	a	a	a
a	b	a	a	a	a	a
a	b	b	a	a	b	a
a	b	c	a	a	c	a
a	c	a	a	a	c	a
a	c	b	a	a	c	a
a	c	c	a	a	c	a
b	a	a	a	a	a	a
b	a	b	a	a	a	a
b	a	c	a	a	a	a
b	b	a	b	a	a	a
b	b	b	b	b	b	b
b	b	c	b	c	c	c
b	c	a	c	c	c	c
b	c	b	c	c	c	c
b	c	c	c	c	c	c
c	a	a	c	c	a	c
c	a	b	c	c	a	c
c	a	c	c	c	a	c
c	b	a	c	c	a	c
c	b	b	c	c	b	c
c	b	c	c	c	c	c
c	c	a	c	c	c	c
c	c	b	c	c	c	c
c	c	c	c	c	c	c

Model answer and marking scheme for Q1b
Here is the formal argument one might make once the pattern has been spotted (with or wihtout constructing some of the table). For x, y and z from the given set we have that

$$(x \circledast y) \circledast z = \begin{cases} a \circledast z = a & x = a \\ y \circledast z & x = b \\ c \circledast y = c & \text{else.} \end{cases}$$

while

$$x \circledast (y \circledast z) = \begin{cases} a & x = a \\ y \circledast z & x = b \\ c & \text{else.} \end{cases}$$

Since we get the same answer in all three cases this establishes that the operation is associative.
The operation is not commutative. We note that

$$a \circledast c = a \neq c = c \cdot a,$$

which is sufficient to establish the claim.
One mark each for the correct answer, one mark for the counterexample of commutativity, two for the proof of associativity. *Just writing that all cases have been checked is not sufficient to get the two marks for the proof of associativity. A student recognizing the pattern but describing it less formally should get most or all of the marks.*

2. a) i) Construct a truth table for the formula: (3 marks)

$$(P \wedge \neg Q) \leftrightarrow \neg(\neg P \vee Q).$$

ii) Determine if the formula is a tautology. Explain your answer. (1 mark)

Model answer and marking scheme for Q2a

i)

P	Q	$\neg Q$	$P \wedge \neg Q$	$\neg P$	$\neg P \vee Q$	$\neg(\neg P \vee Q)$	Fml
1	**1**	**0**	**0**	**0**	**1**	**0**	**1**
1	**0**	**1**	**1**	**0**	**0**	**1**	**1**
0	**1**	**0**	**0**	**1**	**1**	**0**	**1**
0	**0**	**1**	**0**	**1**	**1**	**0**	**1**

Three marks for correct truth table. One or two marks for small mistakes.

ii) The formula is a tautology, since all entries for the formula are **1**. **One mark for correct answer and explanation.**

b) Answer **one** of the following: (2 marks)

i. Briefly explain **one** key difference between the Boolean semantics and the power set semantics of propositional formulas.

ii. Give **two** reasons why transformation to conjunctive normal form is useful.

Model answer and marking scheme for Q2b
Answers will obviously vary.

i. Possible answers: In the Boolean semantics, propositional formulas are interpreted as true or false, while in the power set semantics, they are interpreted as subsets of some set.
Or: In the Boolean semantics the interpretation of the connectives is defined by their standard truth tables, while in the power set semantics the connectives are interpreted as set operations. **Two marks for mentioning two corresponding underlined concepts. One mark for one underlined concept or a minor mistake.**

ii. Possible reasons from lecture notes and slides:

- Simplify formulas; make them more understandable.
- Eliminate superfluous connectives.
- Can be used to test equivalence of two formulas.
- Exploited in some deduction systems; exploited in implementations of automated reasoning tools

Two marks for two correct reasons, one mark for one correct reason or minor mistake.

c) Use our CNF algorithm to transform this formula into conjunctive normal form. Simplify your answer as much as possible. (4 marks)

$$(P \wedge \neg R) \to \neg(Q \to \neg(P \to Q)).$$

Model answer and marking scheme for Q2c

$$\begin{array}{lll} (P \wedge \neg R) \to \neg(Q \to \neg(P \to Q)) & & \\ \quad \equiv & \neg(P \wedge \neg R) \vee \neg(\neg Q \vee \neg(\neg P \vee Q)) & \text{Step 1/Elim} \to \text{repeatedly} \\ \quad \equiv & (\neg P \vee \neg\neg R) \vee (\neg\neg Q \wedge \neg\neg(\neg P \vee Q)) & \text{Step 2/De Morgan repeatedly} \\ \quad \equiv & (\neg P \vee R) \vee (Q \wedge (\neg P \vee Q)) & \text{Step 3/Elim } \neg\neg \\ \quad \equiv & \neg P \vee R \vee (Q \wedge (\neg P \vee Q)) & \text{flattening/assoc. } \vee \\ \quad \equiv & (\neg P \vee R \vee Q) \wedge (\neg P \vee R \vee \neg P \vee Q) & \text{Step 4/distr. and flattening/assoc. } \vee \\ \quad \equiv & (\neg P \vee R \vee Q) \wedge (\neg P \vee R \vee Q) & \text{comm. } \vee\text{, idempotence} \\ \quad \equiv & \neg P \vee R \vee Q & \text{idempotence} \end{array}$$

A shorter derivation that does simplification steps early:

$$\begin{array}{lll} (P \wedge \neg R) \to \neg(Q \to \neg(P \to Q)) & & \\ \quad \equiv & \neg(P \wedge \neg R) \vee \neg(\neg Q \vee \neg(\neg P \vee Q)) & \text{Step 1/Elim} \to \text{repeatedly} \\ \quad \equiv & (\neg P \vee \neg\neg R) \vee (\neg\neg Q \wedge \neg\neg(\neg P \vee Q)) & \text{Step 2/De Morgan repeatedly} \\ \quad \equiv & (\neg P \vee R) \vee (Q \wedge (\neg P \vee Q)) & \text{Step 3/Elim } \neg\neg \\ \quad \equiv & \neg P \vee R \vee (Q \wedge (\neg P \vee Q)) & \text{flattening/assoc. } \vee \\ \quad \equiv & \neg P \vee R \vee (Q \wedge (Q \vee \neg P)) & \text{commutativity } \vee \\ \quad \equiv & \neg P \vee R \vee Q & \text{absorption} \end{array}$$

Four marks for correct answer and correct justifications of each step, subtract one mark for each mistake. Only perfect answer should receive full marks. Maximally two marks, if truth table method was used: one mark for correct answer, one mark for correct truth table.

END OF EXAMINATION