A constructive version of the Lusin Separation Theorem

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Contents

1 Introduction

The aim of this note is to state and prove a constructive version of the following classical result.

The Lusin Separation Theorem: *Disjoint analytic sets of Baire space are Borel separable.*

We recall the classical definitions. *Baire space* is the topological space $\mathcal{N} = {}^{\mathbb{N}}\mathbb{N}$ of infinite sequences of natural numbers, which has the product topology, $\mathbb{N}$ having the discrete topology. A subset of Baire space is *analytic* if it is a projection of a closed subset of the product space $\mathcal{N}\times\mathcal{N}$ or equivalently a continuous image of a closed subset of $\mathcal{N}$. The *Borel sets* of Baire space form the smallest σ-algebra on Baire space that includes the open sets. Here a *σ-algebra* on a set is a class of subsets of the set that contains the empty set and is closed under complements and countable unions. A pair of subsets of Baire space is *Borel separable* if there is a Borel set that includes one subset and is disjoint from the other.

The Lusin Separation Theorem has, as an immediate consequence, Suslin's fundamental classical theorem that if a subset of Baire space is both analytic and coanalytic; i.e. the complement of an analytic set, then it is Borel. In fact the converse is also classically true, so that the Borel sets are exactly the sets that are both analytic and coanalytic. According to [9] Suslin's Theorem was anounced in [10] without any proof; the first published proof being in [6], another being in [7]. It seems that the separation theorem was only established in [4]. Moschovakis, in [9], gives two proofs of the separation theorem, the first a highly non-constructive argument by contradiction, followed by a second constructive proof using Bar Induction and Bar Recursion. Such proofs were first given in [5]. The proof of the main lemma in this paper has been based on this second proof. It is interesting to note that Brouwer, in stating his bar theorem in [2] was probably inspired by the separation theorem[1].

Our aim is to prove a version of Lusin's result in the constructive set theory **CZF**. One of the standard classical proofs, see [9], using Bar Induction and Bar Recursion would appear to be essentially constructive. But care is needed in formulating and using the asumption that the analytic subsets A_1, A_2 of Baire space, $\mathcal{N}$, are disjoint. The straightforward formulation that $A_1 \cap A_2$ is empty, may be stated as follows.

$$\neg(\exists\alpha_1 \in A_1)(\exists\alpha_2 \in A_2)[\alpha_1 = \alpha_2].$$

[1]See the discussion in the introduction to the translation of [2] in [3].

The equality $\alpha_1 = \alpha_2$ may be written $\neg(\exists n \in \mathbb{N})[\alpha_1 n \neq \alpha_2 n]$ so that, using constructively correct logical equivalences we may restate the disjointness of A_1, A_2 as

$$(\forall \alpha_1 \in A_1)(\forall \alpha_2 \in A_2)\ \neg\neg(\exists n \in \mathbb{N})[\alpha_1 n \neq \alpha_2 n].$$

We get a stronger notion by removing the double negation from the above. We then write that A_1, A_2 are *positively disjoint*. Wim Veldman,[12], considers that this is the natural constructive notion of disjointness. Note that the two notions become equivalent when Markov's Principle **(MP)** is assumed. This is the following principle.

Markov's Principle (MP): For each decidable subset R of $\mathbb{N}$; i.e. $(\forall n \in \mathbb{N})[(n \in R) \vee \neg(n \in R)]$,

$$\neg\neg(\exists n \in \mathbb{N})(n \in R) \ \Rightarrow\ (\exists n \in \mathbb{N})(n \in R).$$

But we do not consider **MP** to be constructively acceptable, although it is accepted by the Russian school of recursive constructive mathematics.

Even the assumption that the analytic sets are positively disjoint does not seem to be strong enough to deduce constructively, using the standard constructive forms of Bar Induction[2] and Bar Recursion, their Borel separation. Wim Veldman has shown how to overcome this problem when the analytic sets are strictly analytic, a restricted notion of analytic set introduced by Veldman, [11, 12]. Here we overcome the problem in another way by strengthening the disjointness assumption even further. We will formulate a notion of *barred disjointness* for pairs of analytic sets. We will recapture a version of Veldman's result as a consequence of our main result.

While Bar Induction is an acceptable principle of Brouwer's Intuitionistic mathematics it is not an accepted principle of Bishop's constructive mathematics and is not provable in **CZF**. In order to avoid using Bar Induction we will strengthen the premiss of Lusin's Separation Theorem even further by using a strong point-free formulation of the disjointness property for analytic sets. This point-free notion of disjointness is equivalent to barred disjointness when Bar Induction is assumed. To compensate for the strengthening of the assumption we will also strengthen the conclusion. So we will define when analytic sets A_1, A_2 are *strongly disjoint* and when they are *strongly Borel separable* and prove the following result.

Theorem: 1 (Constructive Lusin Separation Theorem) *Strongly disjoint analytic sets of Baire space are strongly Borel separable.*

[2] In the literature this standard form of Bar Induction is often refered to as BI_D.

In our point-free approach to the separation theorem we will use trees to represent analytic sets and Borel codes to represent Borel sets. We will define a relation $\leq$ between trees T and Borel codes b such that if $T \leq b$ then the analytic set represented by the tree T will be a subset of the Borel set represented by the code b. We will define a binary operation on trees that assigns to trees T_1, T_2 a tree $T_1 \wedge T_2$ to represent the analytic set $A_1 \cap A_2$, where A_1, A_2 are the analytic sets represented by T_1, T_2 respectively. Also we will define a unary operation on Borel codes that assigns to each Borel code b a Borel code $-b$ that represents a Borel set disjoint from the Borel set represented by b. Using these notions we can define the key concepts used in the statement of our constructive separation theorem.

Definition: 2

1. *Analytic sets A_1, A_2 are* strongly disjoint *if there are trees T_1, T_2, representing A_1, A_2 respectively, such that*

$$T_1 \wedge T_2 \leq \Box,$$

   where $\Box$ is a Borel code for the empty Borel set.

2. *Analytic sets A_1, A_2 are* strongly Borel separable *if there are trees T_1, T_2, representing A_1, A_2 respectively, such that there is a Borel code b such that*

$$T_1 \leq b \text{ and } T_2 \leq -b.$$

With this definition Theorem 1 is an immediate consequence of the main lemma.

Main Lemma: *If T_1, T_2 are trees such that $T_1 \wedge T_2 \leq \Box$ then there is a Borel code b such that $T_1 \leq b$ and $T_2 \leq -b$.*

Our point-free approach to Borel sets was inspired by Per Martin-Löf's constructive recursive treatment in the book [8] and the constructive approach to point-free topology developed there and in the literature on formal topology. The book formulates a point-free version of the subset relation between Borel sets on Cantor space. Here we have chosen to focus on Baire space. The book might easily have contained an extra chapter on analytic sets and the separation theorem. In fact Martin-Löf had envisioned[3] such a chapter at the time of writing his book.

[3] Private Communication

A key feature of point-free topology has been the aim to prove constructive versions of classical results while avoiding any use of Bar Induction. Here we also seek to avoid Bar Induction in proving a version of a classic result of descriptive set theory.

The above theorem will be proved informally in the constructive set theory **CZF**, which we consider to be entirely compatible with Brouwer's Intuitionism. The reader is referred to [1] for an introduction to **CZF**. An important feature of our proof is the extensive use of inductive definitions of classes. One advantage of working in the set theory **CZF** is that the theory allows a flexible application of such inductive definitions.

By assuming Bar Induction we get an intuitionist separation result that assumes only that the analytic sets are barred-disjoint, a notion we introduce in section 5. By assuming both Bar Induction and Markov's Principle we get a proof of the classical formulation of the separation theorem, and as **ZF**+**DC** has all the theorems of **CZF** + **BI** + **MP** we recapture the classical result. The formal system **CZF** does not have any form of choice principle, not even countable choice, which is usually accepted in constructive mathematics. We avoid needing countable choice by working with codes of Borel sets. We prefer to avoid any form of choice whenever possible, thereby making our results more compatible with the mathematics generally true in a topos. It should be noted that when countable choice is not assumed then the standard proof that all Borel sets are analytic no longers works.

In section 2 we present our constructive approach to the definition of the Borel sets, which uses an inductive definition of a class of codes for Borel sets, and is not essentially very different from the approaches that may be found in [8, 11, 12]. Section 3 contains a review of the **CZF** approach to inductive definitions and their application to the definition and properties of Borel sets discussed in section 2. The notion of a tree plays an important role in the theory of analytic sets and these are discussed in section 4. The well-founded trees are defined inductively and well-founded tree induction and well-founded tree recursion are shown to hold in **CZF**. These are variants of Bar Induction and Bar Recursion for trees that can be proved in **CZF** because the usual assumption that the tree is barred is replaced by the assumption that the tree is well-founded. The analytic sets are defined in section 5, and four notions of disjointness for pairs of analytic sets are considered, all being equivalent if one assumes both Bar Induction and Markov's Principle. The strong inductive point-free form of Borel separation is introduced in section 6 and the main lemma concerning trees and Borel codes is stated and proved in section 7. Theorem 1 is an easy consequence of this main lemma. Section 8 characterises when a pair of trees represent positively disjoint analytic sets. The result is then used in section 9 to obtain Veldman's result concerning positively disjoint strictly analytic sets as another consequence of the main lemma.

In this paper attention has been limited to the constructive analysis of Lusin's separation Theorem for Baire space only. But modern classical Descriptive Set Theory is a theory concerning Polish spaces; i.e. separable completely metrizable spaces. It is very plausible that our constructive treatment of Lusin's theorem for Baire space should carry over in a fairly routine way to Polish spaces. But we have not had time to examine this matter. Also, Lusin's Theorem is only one of a number of classical results in Descriptive Set Theory that should be amenable to a constructive treatment. In fact it should be worthwhile to develop a constructive descriptive set theory developing further the approach taken in this paper and relating it to Veldman's Intuitionistic approach. One potential application, pointed out to me by Yiannis Moschovakis is the possibility to apply results proved constructively to both the classical and recursive versions of descriptive set theory by using suitable realisability models. This remains to be studied.

I am grateful to Yiannis Moschovakis for recently drawing my attention to the Lusin Separation Theorem and its constructive character. I am grateful to Wim Veldman for drawing my attention to his work on Intuitionistic Descriptive Set Theory, [11, 12], and in particular to his separation result for positively disjoint strictly analytic sets. The anonymous referee made some useful suggestions which has helped to improve the presentation of this paper.

2 Constructive Borel sets

From now on we restrict our attention to Baire space; i.e. the space $\mathcal{N}$ of all infinite sequences of natural numbers, given the product topology where $\mathbb{N}$ is given the discrete topology. So a natural basis of clopen sets for the topology can be indexed by the set $\mathbb{N}^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ of finite sequences of natural numbers. With each index $a \in \mathbb{N}^*$ of length n is associated the clopen set $G_a = \{\alpha \in \mathcal{N} \mid \overline{\alpha}n = a\}$ where if $\alpha \in \mathcal{N}$ then $\overline{\alpha}n = (\alpha 0, \ldots, \alpha(n-1))$. We may call these the *elementary clopen sets*. Note that the complement of an elementary clopen set will not be elementary. Each open set of Baire space is determined by a subset X of $\mathbb{N}^*$, and then the open set has the form

$$G_X = \{\alpha \in \mathcal{N} \mid (\exists n \in \mathbb{N})\ \overline{\alpha}n \in X\}.$$

The elementary clopen sets are not closed under taking complements. So, for our purposes a nicer basis of clopen sets uses, as indices, pairs (n, X) where $n \in \mathbb{N}$ and X is a decidable subset of $\mathbb{N}^n$. Let S be the set of such pairs. With each index $s = (n, X) \in S$ we can associate the clopen set $G_s = \{\alpha \in \mathcal{N} \mid \overline{\alpha}n \in X\}$ and its complement G_{-s}, where $-s = (n, \mathbb{N}^n - X) \in S$. We will call such sets the *simple clopen sets*. These form a base for the topology that is closed under taking complements, finite intersections and finite unions.

Classically the Borel sets can be defined to be the closure of the simple clopen sets under countable unions and intersections. Moreover every open set can be represented as a countable union of simple clopen sets and so is a Borel set. But in **CZF** we cannot expect every open set to be a countable union of simple clopen sets. This is because, although the simple clopen sets form a countable set and every open set is a union of a subset of that countable set, the argument that every subset of a countable set is countable is not constructively acceptable. Let us call an open set *countably open* if it is a countable union $\bigcup_{n\in\mathbb{N}} A_n$ of an $\mathbb{N}$-indexed family $\{A_n\}_{n\in\mathbb{N}}$ of simple clopen sets A_n. Similarily we may define the *countably closed* sets to be the countable intersections $\bigcap_{n\in\mathbb{N}} A_n$ of $\mathbb{N}$-indexed families $\{A_n\}_{n\in\mathbb{N}}$ of simple clopen sets A_n.

We can get a notion of Borel set by taking the Borel sets to be obtained from the simple clopen sets by repeatedly taking countable intersections and countable unions. Because we do not want to assume countable choice we will not use that definition, but instead work with a more constructive notion by first inductively generating indices for the Borel sets.

Definition: 3 *The class $\mathcal{B}$ of* Borel codes *is defined to be the smallest class such that*

1. $(0, s) \in \mathcal{B}$ *for each index* $s \in S$ *for a simple clopen set,*
2. *If* $f : \mathbb{N} \to \mathcal{B}$ *then* $(i, f) \in \mathcal{B}$ *for* $i = 1, 2$.

With each Borel code $b \in \mathcal{B}$ we associate a set $\mathbb{B}_b \subseteq \mathcal{N}$ by recursion following the inductive definition so that

1. $\mathbb{B}_b = G_s$ *if* $b = (0, s)$ *where* $s \in S$.
2. $\mathbb{B}_b = \bigcup_{n\in\mathbb{N}} \mathbb{B}_{fn}$ *if* $b = (1, f)$ *where* $f : \mathbb{N} \to \mathcal{B}$,
3. $\mathbb{B}_b = \bigcap_{n\in\mathbb{N}} \mathbb{B}_{fn}$ *if* $b = (2, f)$ *where* $f : \mathbb{N} \to \mathcal{B}$.

We define the constructive Borel sets *to be the sets* $\mathbb{B}_b$ *for* $b \in \mathcal{B}$.

Note that every countably closed set is Borel.

Definition: 4 *The duality operation* $- : \mathcal{B} \to \mathcal{B}$ *on Borel codes is the unique class function such that*

1. $-(0, s) = (0, (n, \mathbb{N}^n - X))$ *for each index* $s = (n, X) \in S$.
2. $-(i, f) = (3 - i, (\lambda n \in \mathbb{N}) - (fn))$ *for* $i = 1, 2$ *and* $f : \mathbb{N} \to \mathcal{B}$.

Let $\square$ be the Borel code $(0, (0, \emptyset))$. Then $\mathbb{B}_\square = \emptyset$ and $\mathbb{B}_{-\square} = \mathcal{N}$.

Proposition: 5 *For all* $b \in \mathcal{B}$

1. $--b = b$,
2. $\mathbb{B}_b \cap \mathbb{B}_{-b} = \emptyset$.

Definition: 6 *The* complementary Borel pairs *are the pairs of Borel sets of the form*

$$\mathbb{B}_b \ , \ \mathbb{B}_{-b}$$

for $b \in \mathcal{B}$.

3 Inductive definitions in CZF

Our definition of the class $\mathcal{B}$ of Borel codes was an inductive definition. We now state the result, which can be proved in **CZF**, which justifies that inductive definition and the associated recursive definitions that assign to each Borel code b the Borel set $\mathbb{B}_b$ and the dual Borel code $-b$.

In general we take an inductive definition in **CZF** to be given as a class Φ of pairs (X, a). We call such pairs *steps*, X being the set of *premisses* of the step and a being the *conclusion* of the step. Given an inductive definition Φ we define a class Y to be Φ-*closed* if

$$X \subseteq Y \Rightarrow a \in Y$$

for every step (X, a). A proof of the following result may be found in [1].

Theorem: 7 (CZF) *For each inductive definition* Φ *there is a (uniquely determined) smallest* Φ*-closed class* $I(\Phi)$*, the* class inductively defined by Φ. *More generally, for each class* A *there is a unique smallest* Φ*-closed class that includes the class* A*. We will write* $I(\Phi, A)$ *for this class.*

Using this result we can take the class $\mathcal{B}$ of Borel codes to be the class $I(\Phi, \{0\} \times S)$ where Φ is the inductive definition whose steps are the pairs $(\{fn \mid n \in \mathbb{N}\}, (i, f))$ for $(i, f) \in Q$ where Q is the class of pairs (i, f) where f a function with domain $\mathbb{N}$ and $i = 1, 2$. Also we can take the class function assigning the Borel set $\mathbb{B}_b$ to each Borel code $b \in \mathcal{B}$ to be the class $I(\Phi', A')$ where

$$A' = \{((0, s), G_s) \mid s \in S\},$$

and

$$\begin{aligned}\Phi' = \quad & \{(\{(fn, gn) \mid n \in \mathbb{N}\}, ((i, f), B)) \mid \\ & (i, f) \in Q \;\;\&\;\; g : \mathbb{N} \to Pow(\mathcal{N}) \;\;\& \\ & [(i = 1 \;\&\; B = \textstyle\bigcup_{n \in \mathbb{N}} gn) \;\vee\; (i = 2 \;\&\; B = \textstyle\bigcap_{n \in \mathbb{N}} gn)]\}.\end{aligned}$$

Of course it is necessary to prove that $I(\Phi', A')$ is a class function $\mathcal{B} \to Pow(\mathcal{N})$ and this can be done by induction on Φ.

Next, the (graph of the) duality operator $-$ is the class $I(\Phi'', A'')$ where

$$A'' = \{((0, s), (0, -s) \mid s \in S\}$$

and

$$\Phi'' = \quad \{(\{(fn, gn) \mid n \in \mathbb{N}\}, ((i, f), (3 - i, g))) \mid (i, f), (3 - i, g) \in Q\} \;.$$

Again it is necessary to prove that $I(\Phi'', A'')$ is a function $\mathcal{B} \to \mathcal{B}$ and this can be done by induction on Φ. Also Proposition 5 can be proved by induction on Φ.

4 Trees on $\mathbb{N}$

Definition: 8

- *If* $a = (x_0, \ldots, x_{n-1}) \in \mathbb{N}^n$ *and* $b = (y_0, \ldots, y_{m-1}) \in \mathbb{N}^m$ *then their* concatenation *is the sequence* $a^\frown b = (x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}) \in \mathbb{N}^{n+m}$

- *If* $a \in \mathbb{N}^*$ *then a* prefix *of* a *is a sequence* $b \in \mathbb{N}^*$ *such that* $a = b^\frown c$ *for some* $c \in \mathbb{N}^*$.

- *By a* tree *we shall always mean a decidable prefix-closed subset of* $\mathbb{N}^*$*; i.e.* $T \subseteq \mathbb{N}^*$ *is a tree if, for all* $a \in \mathbb{N}^*$,

  - *either* $a \in T$ *or* $a \notin T$, *and*
  - *if* $a \in T$ *then every prefix of* a *is also in* T.

- *For each tree* T *let*

$$[T] = \{\alpha \in \mathcal{N} \mid (\forall n \in \mathbb{N})\ [\overline{\alpha}n \in T]\}.$$

Note that for each tree T the set $[T]$ is always a countably closed set.

Definition: 9 *A tree* T *is* barred *if*

$$(\forall \alpha \in \mathcal{N})(\exists n \in \mathbb{N})\ [\overline{\alpha}n \notin T]$$

and is weakly barred *if*

$$(\forall \alpha \in \mathcal{N})\neg\neg(\exists n \in \mathbb{N})\ [\overline{\alpha}n \notin T].$$

Note that a tree T is weakly barred iff $[T] = \emptyset$.

We use an inductive definition to formulate a stronger point-free version of the notion of a barred tree. Let Θ be the inductive definition whose steps are the pairs $(\{a^\frown(n) \mid n \in \mathbb{N}\}, a)$ for $a \in \mathbb{N}^*$.

Definition: 10 *A tree* T *is* well-founded (wf) *if the empty sequence* $()$ *is in* $I(\Theta, (\mathbb{N}^* - T))$.

Proposition: 11 *If* T *is a wf tree then* $\mathbb{N}^* \subseteq I(\Theta, (\mathbb{N}^* - T))$.

Proof: Let T be a wf tree and let $I = I(\Theta, (\mathbb{N}^* - T))$. Let Y be the class $\{a \in \mathbb{N}^* \mid (\forall c \in \mathbb{N}^*)[a^\frown c \in I\}$. Then $Y \subseteq I$ and it suffices to show that $() \in Y$. In fact, as $() \in I$, it is enough to show that $I \subseteq Y$, which we do by showing that

1. $(\mathbb{N}^* - T) \subseteq Y$,
2. Y is Θ-closed.

For 1 observe that if $a \in (\mathbb{N}^* - T)$ then, as T is prefix-closed,

$$a^\frown c \in (\mathbb{N}^* - T) \subseteq I,$$

for any $c \in \mathbb{N}^*$, so that $a \in Y$.

For 2 let $(\forall n \in \mathbb{N})[a^\frown(n) \in Y]$. Then, as $Y \subseteq I$ and I is Θ-closed, $a^\frown() = a \in I$. Also, for all $n \in \mathbb{N}$ and all $c \in \mathbb{N}^*$

$$a^\frown(n)^\frown c \in I.$$

So $a^\frown e \in I$ for every $e \in \mathbb{N}^*$, as every such e is either $()$ or else has the form $(n)^\frown c$. Thus $a \in Y$, as desired.
■

We may restate this result as the following principle.

Well-Founded Tree Induction *Let T be a wf tree. Let $Y \subseteq \mathbb{N}^*$ be a class such that*

1. $(\mathbb{N}^* - T) \subseteq Y$,

2. Y *is* Θ*-closed.*

Then $\mathbb{N}^* \subseteq Y$.

The following result provides an alternative characterisation of the wf trees. If T_n is a tree for each $n \in \mathbb{N}$ then let $\Sigma_{n \in \mathbb{N}} T_n$ be the tree

$$\{()\} \cup \bigcup_{n \in \mathbb{N}} \{(n)^\frown a \mid a \in T_n\}.$$

Proposition: 12 *The class of wf trees is the smallest class $\mathbb{W}$ of trees such that*

1. $\emptyset \in \mathbb{W}$,

2. *If $T_n \in \mathbb{W}$ for $n \in \mathbb{N}$ then $\Sigma_{n \in \mathbb{N}} T_n \in \mathbb{W}$.*

Proof: Let $\mathbb{W}'$ be the class of wf trees. We must show that $\mathbb{W}' = \mathbb{W}$. To show that $\mathbb{W} \subseteq \mathbb{W}'$ it suffices to observe that

1. $\emptyset$ is a wf tree,

2. If T_n is a wf tree for each $n \in \mathbb{N}$ then $\Sigma_{n \in \mathbb{N}} T_n$ is a wf tree.

For 1, observe that

$$() \in (\mathbb{N}^* - \emptyset) \subseteq I(\Theta, (\mathbb{N}^* - \emptyset)).$$

For 2 assume that, for each $n \in \mathbb{N}$, T_n is a wf tree. Let $T = \Sigma_{n \in \mathbb{N}} T_n$ and, for each $n \in \mathbb{N}$ let

$$Y_n = \{a \in \mathbb{N}^* \mid (n)^\frown a \in I(\Theta, (\mathbb{N}^* - T))\}.$$

Observe that each Y_n is Θ-closed. Also each Y_n includes $(\mathbb{N}^* - T_n)$, as

$$a \in (\mathbb{N}^* - T_n) \Rightarrow (n)^\frown a \in (\mathbb{N}^* - T) \Rightarrow (n)^\frown a \in I(\Theta, (\mathbb{N}^* - T)) \Rightarrow a \in Y_n.$$

So, for each $n \in \mathbb{N}$, by Well-Founded Tree Induction on the wf tree T_n, $\mathbb{N}^* \subseteq Y_n$. In particular, for each $n \in \mathbb{N}$, $() \in Y_n$; i.e. $(n) \in I(\Theta, (\mathbb{N}^* - T))$. So $() \in I(\Theta, (\mathbb{N}^* - T))$; i.e. the tree T is wf as desired.

To show that $\mathbb{W}' \subseteq \mathbb{W}$, let $T \in \mathbb{W}'$. So, by Proposition 11,

$$\mathbb{N}^* \subseteq I(\Theta, (\mathbb{N}^* - T)).$$

For $a \in \mathbb{N}^*$ let $T_a = \{c \in \mathbb{N}^* \mid a^\frown c \in T\}$. Clearly each T_a is a tree. As $T_{()} = T$ it suffices to show that each T_a is in $\mathbb{W}$; i.e. that $\mathbb{N}^* \subseteq Y$, where $Y = \{a \in \mathbb{N}^* \mid T_a \in \mathbb{W}\}$. Observe that (i) if $a \in (\mathbb{N}^* - T)$ then $T_a = \emptyset \in \mathbb{W}$ and (ii) if $T_{a^\frown(n)} \in \mathbb{W}$ for all $n \in \mathbb{N}$ then $T_a = \Sigma_{n\in\mathbb{N}} T_{a^\frown(n)} \in \mathbb{W}$. It follows that $\mathbb{N}^* \subseteq I(\Theta, (\mathbb{N}^* - T)) \subseteq Y$.
■

Proposition: 13 *If T is a tree then*

1. *T is wf $\Rightarrow$ T is barred.*
2. *T is barred $\Rightarrow$ T is weakly barred.*

Proof: For 1, by proposition 12, it suffices to observe that $\emptyset$ is a barred tree and if T_n is a barred tree for all $n \in \mathbb{N}$ then the tree $\Sigma_{n\in\mathbb{N}} T_n$ is barred. Part 2 is trivial.
■

We are ready to formulate the principle of (decidable) Bar Induction, which is the converse of part 1 of the previous proposition.

Bar Induction: *Every barred tree is wf.*

The converse to part 2 of the previous proposition seems to be less constructive. Nevertheless it is an immediate consequence of **(MP)**.

Proposition: 14 (Assuming MP) *Every weakly barred tree is barred.*

In fact it is not hard to see that the statement that every weakly barred tree is barred is equivalent to (**MP**).

The following result is a variant of Bar Recursion obtained by replacing the assumption that a tree T is barred by the assumption that T is wf. Of course Bar Recursion is an immediate consequence of the theorem using Bar Induction.

Theorem: 15 (Well-Founded Tree Recursion) *Let T be a wf tree. Let Y be a class. If $g : (\mathbb{N}^* - T) \to Y$ and $Q^a : {}^{\mathbb{N}}Y \to Y$ for each $a \in T$ then there is a unique $F : \mathbb{N}^* \to Y$ such that, for $a \in \mathbb{N}^*$,*

$$Fa = \begin{cases} ga & \text{if } a \notin T \\ Q^a((\lambda n \in \mathbb{N})F(a^\frown(n))) & \text{if } a \in T. \end{cases}$$

In fact $F = I(\Psi, g)$ where Ψ is the inductive definition with steps $(f, (a, Q^a f))$ for $a \in T$ and $f : \mathbb{N} \to Y$. It is necessary to prove that this is indeed a single valued function defined on $\mathbb{N}^*$.

5 Analytic sets

We will use a fixed bijection $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with associated projections $\pi_1, \pi_2 : \mathbb{N} \to \mathbb{N}$ such that $\pi_i(\pi(x_1, x_2)) = x_i$ for $x_1, x_2 \in \mathbb{N}$ and $i = 1, 2$. A standard example is the bijection π with definition

$$\pi(x_1, x_2) = (x_1 + x_2)(x_1 + x_2 + 1)/2 + x_2$$

for $x_1, x_2 \in \mathbb{N}$.

We lift these functions to bijections $\mathbb{N}^n \times \mathbb{N}^n \to \mathbb{N}^n$ and associated projections as follows. If $a_1 = (x_0, \ldots, x_{n-1}) \in \mathbb{N}^n$ and $a_2 = (y_0, \ldots, y_{n-1}) \in \mathbb{N}^n$ then let

$$\pi(a_1, a_2) = (\pi(x_0, y_0), \ldots, \pi(x_{n-1}, y_{n-1})) \in \mathbb{N}^n.$$

Also, for $i = 1, 2$ if $a = (x_0, \ldots, x_{n-1}) \in \mathbb{N}^n$ then let

$$\pi_i a = (\pi_i x_0, \ldots, \pi_i x_{n-1}) \in \mathbb{N}^n.$$

We also lift these functions to bijections $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$ and associated projections as follows. If $\alpha_1, \alpha_2 \in \mathcal{N}$ then let $\pi(\alpha_1, \alpha_2) \in \mathcal{N}$ be given by

$$\pi(\alpha_1, \alpha_2)n = \pi(\alpha_1 n, \alpha_2 n)$$

for $n \in \mathbb{N}$. Also, for $i = 1, 2$, if $\alpha \in \mathcal{N}$ let $\pi_i \alpha \in \mathcal{N}$ be given by

$$(\pi_i \alpha)n = \pi_i(\alpha n)$$

for $n \in \mathbb{N}$.

Definition: 16 *A set $A \subseteq \mathcal{N}$ is defined to be* analytic *if there is a tree T such that*

$$\begin{aligned} A = \pi_1[T] &= \{\pi_1 \gamma \mid \gamma \in [T]\} \\ &= \{\alpha \in \mathcal{N} \mid (\exists \beta \in \mathcal{N})(\forall n \in \mathbb{N})\, \pi(\overline{\alpha} n, \overline{\beta} n) \in T\}. \end{aligned}$$

We then call T a tree representation *of the analytic set A.*

We will also need a bijection $\tau : \mathbb{N} \hat{\times} \mathbb{N} \to \mathbb{N}$ and associated projections $\tau_1, \tau_2 : \mathbb{N} \to \mathbb{N}$, where

$$\mathbb{N} \hat{\times} \mathbb{N} = \{(n_1, n_2) \in \mathbb{N} \times \mathbb{N} \mid \pi_1 n_1 = \pi_2 n_2\}.$$

We first define $\tau' : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and let τ be the restriction of τ' to the set $\mathbb{N} \hat{\times} \mathbb{N}$.

- $\tau'(n_1, n_2) = \pi(\pi_1 n_1, \pi(\pi_2 n_1, \pi_2 n_2))$, for $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$,
- $\tau_i n = \pi(\pi_1 n, \pi_i(\pi_2 n))$ for $i = 1, 2$ and $n \in \mathbb{N}$.

As with π, π_1, π_2 these functions can be lifted to each $\mathbb{N}^n$ and to $\mathcal{N}$.

Proposition: 17 *$\tau : \mathbb{N}\hat{\times}\mathbb{N} \to \mathbb{N}$ is a bijection with projections $\tau_1, \tau_2 : \mathbb{N} \to \mathbb{N}$.*

Proof: Routine computations using the definitions show that,
for $n, n_1, n_2 \in \mathbb{N}$,

$$[(n_1, n_2) \in \mathbb{N}\hat{\times}\mathbb{N} \;\;\&\;\; n = \tau(n_1, n_2)] \quad \Leftrightarrow \quad [n_1 = \tau_1 n \;\;\&\;\; n_2 = \tau_2 n].$$

■

Proposition: 18 *Let T_1, T_2 be trees representing the analytic sets A_1, A_2. Then*

$$T_1 \wedge T_2 = \{a \in \mathbb{N}^* \mid \tau_1 a \in T_1 \;\&\; \tau_2 a \in T_2\}$$

is a tree that represents the analytic set $A_1 \cap A_2 = \pi_1[T_1 \wedge T_2]$. Moreover A_1, A_2 are disjoint iff $T_1 \wedge T_2$ is weakly barred.

Proof: That $T_1 \wedge T_2$ is decidable and prefix-closed follows easily from the corresponding properties of T_1 and T_2.

Next we must show that

$$\gamma \in \pi_1[T_1 \wedge T_2] \quad \Leftrightarrow \quad \gamma \in \pi_1[T_1] \cap \pi_1[T_2].$$

Assuming the left hand side, $\gamma = \pi_1 \alpha$ for some $\alpha \in [T_1 \wedge T_2]$ so that, if $\alpha_i = \tau_i \alpha$ then $\alpha_i \in [T_i]$ and $\gamma = \pi_1 \alpha = \pi_1 \alpha_i \in \pi_1[T_i]$ for $i = 1, 2$, giving us the right hand side.

Assuming the right hand side, there are $\alpha_1, \alpha_2 \in \mathcal{N}$ such that $\gamma = \pi_1 \alpha_1 = \pi_1 \alpha_2$ and for all $n \in \mathbb{N}$,

$$\overline{\alpha}_1 n \in T_1 \text{ and } \overline{\alpha}_2 n \in T_2.$$

We may define $\alpha = \tau(\alpha_1, \alpha_2)$ and observe that $\gamma = \pi_1 \alpha$ and $\tau_i \alpha = \alpha_i \in [T_i]$ for $i = 1, 2$; i.e. the left hand side.

Finally,

$$\begin{aligned} A_1, A_2 \text{ are disjoint} \quad &\Leftrightarrow \;\; \pi_1[T_1 \wedge T_2] = \emptyset \\ &\Leftrightarrow \;\; [T_1 \wedge T_2] = \emptyset \\ &\Leftrightarrow \;\; (\forall \alpha \in \mathcal{N})\neg\neg(\exists n \in \mathbb{N})[\overline{\alpha} n \notin T_1 \wedge T_2] \\ &\Leftrightarrow \;\; T_1 \wedge T_2 \text{ is weakly barred.} \end{aligned}$$

■

Definition: 19 *Analytic sets A_1, A_2 are defined to be* strongly disjoint (barred-disjoint) *if trees T_1, T_2 representing them can be chosen such that $T_1 \wedge T_2$ is wf (barred).*

Proposition: 20 *For analytic sets A_1, A_2 we have the implications*

$$D1 \Rightarrow D2 \Rightarrow D3 \Rightarrow D4$$

where

D1: *A_1, A_2 are strongly disjoint,*

D2: *A_1, A_2 are barred disjoint,*

D3: *A_1, A_2 are positively disjoint,*

D4: *A_1, A_2 are disjoint,*

Moreover, assuming ($\mathbf{BI}$), ($\mathbf{D1}$) and ($\mathbf{D2}$) are equivalent and, assuming ($\mathbf{MP}$), ($\mathbf{D2}$), ($\mathbf{D3}$) and ($\mathbf{D4}$) are equivalent.

Proof:

D1$\Rightarrow$ D2 By part 1 of proposition 13.

D2$\Rightarrow$ D3 Let A_1, A_2 be barred-disjoint analytic sets. So there are trees T_1, T_2 such that $A_i = \pi_1[T_i]$ for $i = 1, 2$ such that $T_1 \wedge T_2$ is barred; i.e. for all $\alpha \in \mathcal{N}$ there is $n \in \mathbb{N}$ such that $\overline{\alpha}n \notin T_1 \wedge T_2$. Now let $\alpha_i \in [T_i]$ for $i = 1, 2$ and let $\alpha = \tau'(\alpha_1, \alpha_2)$. So there is $n \in \mathbb{N}$ such that $\overline{\alpha}n \notin T_1 \wedge T_2$; i.e.

$$[\tau_1(\overline{\alpha}n) \notin T_1] \text{ or } [\tau_2(\overline{\alpha}n) \notin T_2].$$

Note that, if $\pi_1(\overline{\alpha}_1 n) = \pi_1(\overline{\alpha}_2 n)$ then

$$\overline{\alpha}_i n = \tau_i(\overline{\alpha}n) \notin T_i$$

for $i = 1, 2$. So

$$[\pi_1(\overline{\alpha}_1 n) = \pi_1(\overline{\alpha}_2 n)] \;\Rightarrow\; [\overline{\alpha}_1 n \notin T_1 \text{ or } \overline{\alpha}_2 n \notin T_2]$$

and hence

$$[\overline{\alpha}_1 n \in T_1 \text{ and } \overline{\alpha}_2 n \in T_2] \;\Rightarrow\; [\pi_1(\overline{\alpha}_1 n) \neq \pi_1(\overline{\alpha}_2 n)].$$

As $\alpha_i \in [T_i]$ for $i = 1, 2$,

$$[\overline{\alpha}_1 n \in T_1 \text{ and } \overline{\alpha}_2 n \in T_2]$$

so that $[\pi_1(\overline{\alpha}_1 n) \neq \pi_1(\overline{\alpha}_2 n)]$.

We have shown that

$$(\forall \alpha_1 \in [T_1])(\forall \alpha_2 \in [T_2])(\exists n \in \mathbb{N})[(\pi_1(\overline{\alpha}_1 n) \neq \pi_1(\overline{\alpha}_2 n)].$$

It follows that

$$(\forall \beta_1 \in A_1)(\forall \beta_2 \in A_2)(\exists n \in \mathbb{N})[\overline{\beta}_1 n \neq \overline{\beta}_2 n];$$

i.e. A_1, A_2 are positively disjoint.

D3$\Rightarrow$ **D4** Trivial.

The final assertion about the equivalences first assuming (**BI**) and then assuming (**MP**) follows because (**BI**) expresses that every barred tree is wf and (**MP**) implies that every weakly barred tree is barred.
■

6 Strong Borel Separation

Recall that, classically, analytic sets A_1, A_2 are Borel separable if there is a Borel set B such that $A_1 \subseteq B$ and $A_2 \subseteq \mathcal{N} - B$, where the complement of B, $\mathcal{N} - B$, is a Borel set. In our constructive context we do not know that $\mathcal{N} - B$ is itself a Borel set. Instead we will use complementary pairs, $\mathbb{B}_b, \mathbb{B}_{-b}$ of Borel sets determined by a Borel code $b \in \mathcal{B}$. In fact we will work with a strengthened notion of Borel Separation obtained by using an inductively defined point-free relation, '$T \leq b$', between tree representations T of analytic sets and Borel codes $b \in \mathcal{B}$, instead of the relation '$A \subseteq B$' between analytic sets A and Borel sets B.

Given a tree T let

$$\mathcal{G}_T = \{(a, b) \in \mathbb{N}^* \times \mathcal{B} \mid A_a \subseteq \mathbb{B}_b\}$$

where $A_a = \pi_1([T] \cap G_a)$. Note that $A_{()} = \pi_1[T]$ is the analytic set with tree representation T and, in general, for $a \in \mathbb{N}^*$, $A_a = \pi_1[T_a]$ is the analytic set with tree representation

$$T_a = \{a' \in T \mid a' \leq a \ \vee \ a \leq a'\}.$$

So $(a,b) \in \mathcal{G}_T$ iff the analytic set A_a is a subset of the Borel set $\mathbb{B}_b$. Our aim now is to give an inductive definition

$$\mathcal{F}_T = I(\Phi, Y_T),$$

of a subclass $\mathcal{F}_T$ of the set $\mathcal{G}_T$ so that we can then define the desired relation between tree-codes T and Borel codes b as follows.

$$T \leq b \;\Leftrightarrow\; ((), b) \in \mathcal{F}_T.$$

We first define the base set Y_T of the inductive definition and show that $Y_T \subseteq \mathcal{G}_T$.

Definition: 21 *For each tree T let*

$$Y_T = \{(a, (0, (n, X))) \mid a \in \mathbb{N}^*, (n, X) \in S \text{ and } [a \notin T \;\vee\; a \in \hat{X}]\},$$

where, for each $(n, X) \in S$,

$$\hat{X} = \{a \in \mathbb{N}^* \mid \exists a' \leq a \; \pi_1 a' \in X\}.$$

Proposition: 22 $Y_T \subseteq \mathcal{G}_T$.

Proof: Let $(a, (0, (n, X))) \in Y_T$. So $a \in \mathbb{N}^*, (n, X) \in S$ and either (i) $a \notin T$ or (ii) $a \in \hat{X}$.

If (i) then $[T] \cap G_a = \emptyset$ so that $A_a = \pi_1 \emptyset = \emptyset \subseteq \mathbb{B}_b$.

If (ii) then $a \geq a'$ for some $a' \in \mathbb{N}^*$ such that $\pi_1 a' \in X$. So, as $\mathbb{B}_b = \bigcup_{a \in X} G_a$,

$$\begin{aligned} \beta \in [T] \cap G_a \; &\Rightarrow \beta \in G_{a'} \\ &\Rightarrow \pi_1 \beta \in G_{\pi_1 a'} \subseteq \mathbb{B}_b. \end{aligned}$$

It follows that

$$\begin{aligned} \alpha \in A_a \; &\Rightarrow \alpha = \pi_1 \beta \text{ for some } \beta \in [T] \cap G_a \\ &\Rightarrow \alpha \in \mathbb{B}_b. \end{aligned}$$

In either case $(a, b) \in \mathcal{G}_T$.
■

We now turn to the definition of the class Φ of steps of the inductive definition of $\mathcal{F}_T$ and the proof that $\mathcal{G}_T$ is Φ-closed. Given $a \in \mathbb{N}^*$ and $b \in \mathcal{B}$ let

$$X^0_b(a) = \{(a^\frown(m), b) \mid m \in \mathbb{N}\}.$$

Also, if $f : \mathbb{N} \to \mathcal{B}$ let

$$X^1_{f,n}(a) = \{(a, f(n))\} \text{ for each } n \in \mathbb{N}$$

and

$$X^2_f(a) = \{(a, f(m)) \mid m \in \mathbb{N}\}.$$

Definition: 23 *Let Φ be the class of all pairs $(X,(a,b))$ such that $(a,b) \in \mathbb{N}^* \times \mathcal{B}$ and either $X = X^0_b(a)$ or $b = (i,f)$, with $i \in \{1,2\}$ and $f : \mathbb{N} \to \mathcal{B}$, and if $i = 1$ then $X = X^1_{f,n}(a)$ for some $n \in \mathbb{N}$ and if $i = 2$ then $X = X^2_f(a)$.*

Proposition: 24 *$\mathcal{G}_T$ is Φ-closed.*

Proof: Let $(X,(a,b)) \in \Phi$ such that $X \subseteq \mathcal{G}_T$. Then $(a,b) \in \mathbb{N}^* \times \mathcal{B}$ and we must show that $(a,b) \in \mathcal{G}_T$. There are three cases.

$X = X^0_b(a)$: By assumption, $(a^\frown(m), b) \in \mathcal{G}_T$ for all $m \in \mathbb{N}$; i.e.

$$A_{a^\frown(m)} \subseteq \mathbb{B}_b \text{ for all } m \in \mathbb{N}.$$

Observe that

$$A_a = \bigcup_{m\in\mathbb{N}} A_{a^\frown(m)}.$$

It follows that $A_a \subseteq \mathbb{B}_b$.

$X = X^1_{f,n}(a)$, with $n \in \mathbb{N}$, $b = (1,f)$ and $f : \mathbb{N} \to \mathcal{B}$:
In this case, as $(a, f(n)) \in \mathcal{G}_T$,

$$A_a \subseteq \mathbb{B}_{f(n)} \subseteq \bigcup_{m\in\mathbb{N}} \mathbb{B}_{f(m)} = \mathbb{B}_b.$$

$X = X^2_f(a)$, with $b = (2,f)$ and $f : \mathbb{N} \to \mathcal{B}$:
As $(a, f(m)) \in \mathcal{G}_T$ for all $m \in \mathbb{N}$,

$$A_a \subseteq \bigcap_{m\in\mathbb{N}} \mathbb{B}_{f(m)} = \mathbb{B}_b.$$

In all three cases we have shown that $(a,b) \in \mathcal{G}_T$.
■

By Propositions 22 and 24 we get the following result.

Proposition: 25 *$\mathcal{F}_T \subseteq \mathcal{G}_T$.*

Recall the definition $T \le b \iff ((),b) \in \mathcal{F}_T$. We get the following corollary, using Definition 10 for part 2.

Corollary: 26 *Let T be a tree.*

1. *If $b \in \mathcal{B}$ such that $T \le b$ then $\pi_1[T] \subseteq \mathbb{B}_b$.*

2. *The tree T is wf iff $T \leq \Box$.*

Definition: 27 *Analytic sets A_1, A_2 are* strongly Borel separable *if trees T_1, T_2 representing them and $b \in \mathcal{B}$ can be chosen such that $T_1 \leq b$ and $T_2 \leq -b$.*

For each $b \in \mathcal{B}$ let

$$\mathcal{F}_T(b) = \{a \in \mathbb{N}^* \mid (a, b) \in \mathcal{F}_T\}.$$

The following proposition is just a reformulation of the fact that, for any tree T, the class $\mathcal{F}_T$ is Φ-closed.

Proposition: 28 *For all $a \in \mathbb{N}^*$, $b \in \mathcal{B}$ and $f : \mathbb{N} \to \mathcal{B}$,*

F0 $(\forall m \in \mathbb{N})[a^\frown(m) \in \mathcal{F}_T(b)] \;\Rightarrow\; a \in \mathcal{F}_T(b)$,

F1 $(\exists n \in \mathbb{N})[a \in \mathcal{F}_T(f(n))] \;\Rightarrow\; a \in \mathcal{F}_T((1, f))$,

F2 $(\forall n \in \mathbb{N})[a \in \mathcal{F}_T(f(n))] \;\Rightarrow\; a \in \mathcal{F}_T((2, f))$.

The next result expresses the crucial idea behind the constructive proof of the Lusin theorem.

Proposition: 29 *Given trees T_1, T_2, if $a_1, a_2 \in \mathbb{N}^*$ and $h : \mathbb{N} \times \mathbb{N} \to \mathcal{B}$ such that*

$$(*) \quad a_1{}^\frown(n_1) \in \mathcal{F}_{T_1}(h(n_1, n_2)) \;\&\; a_2{}^\frown(n_2) \in \mathcal{F}_{T_2}(-h(n_1, n_2))$$

for all $n_1, n_2 \in \mathbb{N}$ then

$$a_1 \in \mathcal{F}_{T_1}(b) \;\&\; a_2 \in \mathcal{F}_{T_2}(-b)$$

where

$$b = (1, (\lambda n_1 \in \mathbb{N})(2, (\lambda n_2 \in \mathbb{N})h(n_1, n_2))).$$

Proof: Let $f = (\lambda k \in \mathbb{N})(2, h_k)$ where $h_k = (\lambda m \in \mathbb{N})h(k, m)$. Then $b = (1, f)$.

By (*), F2) and F1), for all $n_1 \in \mathbb{N}$,

$$a_1{}^\frown(n_1) \in \bigcap_{m \in \mathbb{N}} \mathcal{F}_{T_1}(h_{n_1}(m)) \subseteq \mathcal{F}_{T_1}((2, h_{n_1})) = \mathcal{F}_{T_1}(f(n_1)) \subseteq \mathcal{F}_{T_1}(b),$$

So by F0), $a_1 \in \mathcal{F}_{T_1}(b)$.

Let $f^- = (\lambda k \in \mathbb{N})(1, h_k^-)$ where $h_k^- = (\lambda m \in \mathbb{N}) - h(k, m)$. Then $b = (2, f^-)$.

By (*) and F1), for all $n_1, n_2 \in \mathbb{N}$,

$$a_2{}^\frown(n_2) \in \mathcal{F}_{T_2}(h^-_{n_1}(n_2)) \subseteq \mathcal{F}_{T_2}((1, h^-_{n_1})) = \mathcal{F}_{T_2}(f^-(n_1)).$$

So, by F0),

$$a_2 \in \mathcal{F}_{T_2}(f^-(n_1)) \text{ for all } n_1 \in \mathbb{N},$$

so that, by F2),

$$a_2 \in \mathcal{F}_{T_2}((2, f^-)) = \mathcal{F}_{T_2}(-b).$$

7 The Main Lemma

Theorem 1 is an easy consequence of the following point-free result.

Lemma: 30 (Main Lemma) *If T_1, T_2 are trees such that $T_1 \wedge T_2 \leq \Box$ then $T_1 \leq b$ and $T_2 \leq -b$ for some $b \in \mathcal{B}$.*

Note: The converse result, that if $T_1 \leq b$ and $T_2 \leq -b$ then $T_1 \wedge T_2 \leq \Box$, is plausible, but has not been proved yet.

We now start the proof of the Main Lemma. We will need functions $h^a : \mathbb{N} \times \mathbb{N} \to \mathcal{B}$ for $a \in \mathbb{N}^*$, given by

$$h^a(n_1, n_2) = \begin{cases} q^a n_1 & \text{if } \pi_1 n_1 \neq \pi_1 n_2 \\ f(\tau(n_1, n_2)) & \text{if } \pi_1 n_1 = \pi_1 n_2 \end{cases}$$

for $n_1, n_2 \in \mathbb{N}$ where, if $a \in \mathbb{N}^m$ then $q^a : \mathbb{N} \to \mathcal{B}$ is given by

$$q^a n = (0, (m + 1, X_n))$$

for $n \in \mathbb{N}$, where $X_n = \{c^\frown(\pi_1 n) \mid c \in \mathbb{N}^m\}$.

Lemma: 31 *For each tree T, if $a \in \mathbb{N}^*$ and $n, n' \in \mathbb{N}$ such that $\pi_1 n \neq \pi_1 n'$ then*

$$a^\frown(n) \ \in \ \mathcal{F}_T(q^a n) \ \subseteq \ \mathcal{F}_T(-q^a n').$$

Proof: Let $a \in \mathbb{N}^m$. As $\pi_1 a \in \mathbb{N}^m$,

$$\pi_1(a^\frown(n)) = \pi_1 a{}^\frown(\pi_1 n) \in X_n$$

so that $a^\frown(n) \in \hat{X}_n \subseteq \mathcal{F}_T(q^a n)$.

Now let $n, n' \in \mathbb{N}$ such that $\pi_1 n \neq \pi_1 n'$. It remains to show that $\mathcal{F}_T(q^a n) \subseteq \mathcal{F}_T(-q^a n')$. Observe that

$$\begin{aligned} \mathcal{F}_T(q^a n) &= (\mathbb{N}^* - T) \cup \hat{X}_n, \\ \mathcal{F}_T(-q^a n') &= (\mathbb{N}^* - T) \cup \hat{Y}_{n'}, \end{aligned}$$

where $Y_{n'} = (\mathbb{N}^{m+1} - X_{n'})$. So it suffices to show that

$$(*) \qquad \hat{X_n} \subseteq \hat{Y_{n'}}.$$

Observe that

$$\begin{array}{lll} a \in \hat{X_n} & \Leftrightarrow & (\exists c \in \mathbb{N}^m)(\exists n_0 \in \mathbb{N})(\exists d \in \mathbb{N}^*)[a = c^\frown(n_0)^\frown d \text{ and } \pi_1 n_0 = \pi_1 n], \\ a \in \hat{Y_{n'}} & \Leftrightarrow & (\exists c \in \mathbb{N}^m)(\exists n_0' \in \mathbb{N})(\exists d \in \mathbb{N}^*)[a = c^\frown(n_0')^\frown d \text{ and } \pi_1 n_0' \neq \pi_1 n']. \end{array}$$

As $\pi_1 n \neq \pi_1 n'$ we get $(*)$ using $n_0' = n_0$.
■

Let T_1, T_2 be trees such that $T = T_1 \wedge T_2 \leq$ so that T is wf. We define $F : \mathbb{N}^* \to \mathcal{B}$ by Well-Founded Tree Recursion, Theorem 15, on T so that for $a \in \mathbb{N}^*$

$$Fa = \begin{cases} ga & \text{if } a \notin T \\ Q^a((\lambda n \in \mathbb{N})F(a^\frown(n))) & \text{if } a \in T \end{cases}$$

where, if $a \notin T$ then

$$ga = \begin{cases} \Box & \text{if } \tau_1 a \notin T_1 \\ -\Box & \text{if } \tau_1 a \in T_1 \end{cases}$$

and, if $a \in T$ and $f : \mathbb{N} \to \mathcal{B}$ then

$$Q^a f = (1, (\lambda n \in \mathbb{N})(2, (\lambda m \in \mathbb{N})h^a(n, m)))$$

To complete the proof of the main lemma it is enough to apply the following lemma with $a = ()$ and put $b = F()$.

Lemma: 32 *For all $a \in \mathbb{N}^*$*

$$(*) \qquad \tau_1 a \in \mathcal{F}_{T_1}(Fa) \quad \textit{and} \quad \tau_2 a \in \mathcal{F}_{T_2}(-Fa).$$

Proof: Let Y be the class of $a \in \mathbb{N}^*$ such that $(*)$. By Well-Founded Tree Induction on the wf tree T it suffices to show that

1. $(\mathbb{N}^* - T) \subseteq Y$,
2. Y is Θ-closed

For 1: Let $a \in (\mathbb{N}^* - T)$ so that $a \notin T$ and hence $Fa = ga$ and either $\tau_1 a \notin T_1$ or $\tau_2 a \notin T_2$.

Case 1 $\tau_1 a \notin T_1$: As $\tau_1 a \in \mathbb{N}^* - T_1$ we have $Fa = ga = \Box$ so that $\tau_1 a \in (\mathbb{N}^* - T_1) \subseteq \mathcal{F}_{T_1}(Fa)$. Observe that, as $\{\hat{()}\} = \mathbb{N}^*$,

$$\mathcal{F}_{T_2}(-Fa) = \mathcal{F}_{T_2}(-\Box) = (\mathbb{N}^* - T_2) \cup \{\hat{()}\} = \mathbb{N}^*.$$

It follows that $\tau_2 a \in \mathcal{F}_{T_2}(-Fa)$.

Case 2 $\tau_1 a \in T_1$ & $\tau_2 a \notin T_2$**:** In this case $-ga = \Box$ and we can argue as in case 1 interchanging the roles of the subscripts 1,2.

For 2: Let $a \in \mathbb{N}^*$ such that $a\frown(n) \in Y$ for all $n \in \mathbb{N}$. We want to show that $a \in Y$. By 1 we may assume that $a \in T$ so that $Fa = Q^a f$ where $f = (\lambda n \in \mathbb{N})\ F(a\frown(n))$. Let $a_1 = \tau_1 a$, $a_2 = \tau_2 a$. By our initial assumption that $a\frown(n) \in Y$ for all $n \in \mathbb{N}$ we get that if $n_1, n_2 \in \mathbb{N}$ such that $\pi_1 n_1 = \pi_1 n_2$ then

$$a_1\frown(n_1) \in \mathcal{F}_{T_1}(f(\tau(n_1, n_2))) \quad \& \quad a_2\frown(n_2) \in \mathcal{F}_{T_2}(-f(\tau(n_1, n_2))).$$

Also observe that, if $n_1, n_2 \in \mathbb{N}$ such that $\pi_1 n_1 \neq \pi_1 n_2$ then, by Lemma 31 below,

$$a_1\frown(n_1) \in \mathcal{F}_{T_1}(q^a n_1) \quad \& \quad a_2\frown(n_2) \in \mathcal{F}_{T_2}(-q^a n_1).$$

It follows that for all $n_1, n_2 \in \mathbb{N}$

$$a_1\frown(n_1) \in \mathcal{F}_{T_1}(h^a(n_1, n_2)) \quad \& \quad a_2\frown(n_2) \in \mathcal{F}_{T_2}(-h^a(n_1, n_2))$$

so that, by Proposition 29, $a \in Y$.
■

8 Positive Disjointness

If T_1, T_2 are trees representing the analytic sets A_1, A_2 then we may characterise that A_1, A_2 are positively disjoint, as defined in Section 1, in terms of a relative notion of barred tree as follows. When **MP** is assumed then the relative notion of barred subtree is equivalent to the unrelativised notion of barred tree.

Definition: 33 *If T is a tree then a subtree T' is a* barred subtree of T *if*

$$(\forall \alpha \in [T])(\exists n \in \mathbb{N})\ \overline{\alpha}n \notin T'.$$

If T_1, T_2 are trees then let $T_1 \times T_2$ be the tree

$$\{a \in \mathbb{N}^* \mid \pi_1 a \in T_1 \ \&\ \pi_2 a \in T_2\}$$

and let $T_1 \hat{\times} T_2$ be the subtree

$$\{a \in T_1 \times T_2 \mid \pi_1(\pi_1 a) = \pi_1(\pi_2 a)\}.$$

Proposition: 34 *If T_1, T_2 are trees representing the analytic sets A_1, A_2 then A_1, A_2 are positively disjoint analytic sets iff $T_1 \hat{\times} T_2$ is a barred subtree of the tree $T_1 \times T_2$.*

Proof: Let $A_i = \pi_1[T_i]$ where T_i is a tree, for $i = 1, 2$. Note that

$$\begin{array}{rcl} \gamma \in [T_1 \times T_2] & \Leftrightarrow & (\forall n \in \mathbb{N})[\pi_1(\overline{\gamma}n) \in T_1 \;\&\; \pi_2(\overline{\gamma}n) \in T_2] \\ & \Leftrightarrow & \pi_1\gamma \in [T_1] \;\&\; \pi_2\gamma \in [T_2]. \end{array}$$

So, A_1, A_2 are positively disjoint

$$\begin{array}{rl} \Leftrightarrow & (\forall \alpha_1 \in A_1)(\forall \alpha_2 \in A_2)(\exists n \in \mathbb{N})[\alpha_1 n \neq \alpha_2 n] \\ \Leftrightarrow & (\forall \gamma_1 \in [T_1])(\forall \gamma_2 \in [T_2])(\exists n \in \mathbb{N})[\pi_1(\gamma_1 n) \neq \pi_1(\gamma_2 n)] \\ \Leftrightarrow & (\forall \gamma \in [T_1 \times T_2])(\exists n \in \mathbb{N})[\pi_1(\pi_1(\gamma n)) \neq \pi_1(\pi_2(\gamma n))] \\ \Leftrightarrow & (\forall \gamma \in [T_1 \times T_2])(\exists n \in \mathbb{N})[\pi_1(\pi_1(\overline{\gamma} n)) \neq \pi_1(\pi_2(\overline{\gamma} n))] \\ \Leftrightarrow & (\forall \gamma \in [T_1 \times T_2])(\exists n \in \mathbb{N})[\overline{\gamma} n \notin T_1 \hat{\times} T_2] \\ \Leftrightarrow & T_1 \hat{\times} T_2 \text{ is a barred subtree of } T_1 \times T_2. \end{array}$$

■

Proposition: 35 (Assuming MP) *If T' is a subtree of a tree T then T' is a barred subtree of T iff T' is a barred tree.*

Proof: The implication from right to left is trivial. For the other direction let T' be a barred subtree of T; i.e. $T' \subseteq T$ such that, for all $\alpha \in \mathcal{N}$,

$$(*) \quad (\forall n \in \mathbb{N})[\overline{\alpha}n \in T] \Rightarrow (\exists n \in \mathbb{N})[\overline{\alpha}n \notin T'].$$

Given $\alpha \in \mathcal{N}$ we must show that $(\exists n \in \mathbb{N})[\overline{\alpha} \notin T']$. We have

$$\begin{array}{rll} \neg(\exists n \in \mathbb{N})[\overline{\alpha}n \notin T'] & \Rightarrow (\forall n \in \mathbb{N})[\overline{\alpha}n \in T'], & \text{as } T' \text{ is a decidable subset of } \mathbb{N}^*, \\ & \Rightarrow (\forall n \in \mathbb{N})[\overline{\alpha}n \in T], & \text{as } T' \subseteq T, \\ & \Rightarrow (\exists n \in \mathbb{N})[\overline{\alpha}n \notin T'], & \text{by } (*). \end{array}$$

Hence $\neg\neg(\exists n \in \mathbb{N})[\overline{\alpha}n \notin T']$. So, by **MP**, $(\exists n \in \mathbb{N})[\overline{\alpha}n \notin T']$.
■

Corollary: 36 (Assuming MP) *If T_1, T_2 are tree representations of the analytic sets A_1, A_2 respectively then A_1, A_2 are positively disjoint iff the tree $T_1 \hat{\times} T_2$ is (weakly) barred.*

9 Strictly analytic sets

Using Bar Induction we get the following point-set Corollary of Theorem 1.

Theorem: 37 (Assuming BI) *Barred-disjoint analytic sets are strongly Borel separable and hence Borel separable.*

If we also assume Markov's Principle then we regain (a slight strengthening of) the classical result.

Theorem: 38 (Assuming both BI and MP) *Disjoint analytic sets are strongly Borel separable.*

Note that every theorem of $\mathbf{CZF} + \mathbf{BI} + \mathbf{MP}$ is a theorem of $\mathbf{ZF} + \mathbf{DC}$.

We apply Theorem 37 to get a version, in our setting, of a result of Wim Veldman, see Theorem 9.2 of [12].

Definition: 39 *A tree T is a* spread tree *if* $() \in T$ *and*

$$(\forall a \in T)(\exists n \in \mathbb{N})\ a^\frown(n) \in T.$$

An analytic set is strictly analytic *if it can be represented by a spread tree.*

Note that an analytic set need not be strictly analytic, as the empty set is analytic but any strictly analytic set is a continuous image of the whole of Baire space and so is always an inhabited set. A construction for the continuous function is given in the proof of part (1) of Lemma 42 below.

Theorem: 40 (Assuming BI) *Positively disjoint strictly analytic sets are strongly Borel separable.*

It is a consequence of Theorem 37 and the following result.

Theorem: 41 *Positively disjoint strictly analytic sets are barred disjoint.*

Proof: By Proposition 34 this is a consequence of the following lemma.

Lemma: 42

1. *If T is a spread tree then*

$$(\forall \alpha \in \mathcal{N})(\exists \beta \in \mathcal{N})(\forall n \in \mathbb{N})\ [\overline{\beta}n \in T\ \&\ (\overline{\alpha}n \in T \Rightarrow \overline{\alpha}n = \overline{\beta}n)].$$

2. *If T_1, T_2 are spread trees such that $T_1 \hat{\times} T_2$ is a barred subtree of $T_1 \times T_2$ then $T_1 \hat{\times} T_2$ is a barred tree.*

3. *For all trees T_1, T_2, the tree $T_1 \wedge T_2$ is barred iff $T_1 \hat{\times} T_2$ is barred.*

Proof:

1. Given $\alpha \in \mathcal{N}$, define $\beta \in [T]$ by primitive recursion as follows. For each $n \in \mathbb{N}$ let $\beta n = \alpha n$ if $\overline{\alpha}(n+1) \in T$. If $\overline{\alpha}(n+1) \notin T$ then let βn be the least $j \in \mathbb{N}$ such that $\overline{\beta}n^\frown(j) \in T$. Such a j will always exist as $\overline{\beta}n \in T$ and T is a spread tree.

2. Let T_1, T_2 be spread trees such that $T_1 \hat{\times} T_2$ is a barred subtree of $T_1 \times T_2$.

   Given $\alpha_1, \alpha_2 \in \mathcal{N}$ choose $\beta_1, \beta_2 \in \mathcal{N}$ by part (1), such that for all $n \in \mathbb{N}$, $\overline{\beta_1}n \in T_1, \overline{\beta_2}n \in T_2$, and

   $$(\overline{\alpha_1}n \in T_1 \Rightarrow \overline{\alpha_1}n = \overline{\beta_1}n) \text{ and } (\overline{\alpha_2}n \in T_1 \Rightarrow \overline{\alpha_2}n = \overline{\beta_2}n)$$

   As $\beta_1 \in [T_1]$ and $\beta_2 \in T_2$, $\pi(\beta_1, \beta_2) \in [T_1 \times T_2]$ so that there is $n \in \mathbb{N}$ such that $\pi(\overline{\beta_1}n, \overline{\beta_2}n) \notin T_1 \hat{\times} T_2$ and hence $\pi_1(\overline{\beta_1}n) \neq \pi_1(\overline{\beta_2}n)$. It follows that if $\overline{\alpha_1}n \in T_1$ and $\overline{\alpha_2}n \in T_2$ then $\overline{\alpha_1}n = \overline{\beta_1}n$ and $\overline{\alpha_2}n = \overline{\beta}n$ so that $\pi_1(\overline{\alpha_1}n) \neq \pi_1(\overline{\alpha_2}n)$. Thus $\overline{\pi(\alpha_1, \alpha_2)}n \notin T_1 \hat{\times} T_2$.

   We have shown that $T_1 \hat{\times} T_2$ is a barred tree.

3. Observe that if $a_1, a_2 \in \mathbb{N}^*$ have the same length then

   $$\pi(a_1, a_2) \in T_1 \hat{\times} T_2 \iff \tau(a_1, a_2) \in T_1 \wedge T_2 \ \& \ (\pi_1 a_1 = \pi_1 a_2).$$

   The result easily follows.

■

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