Introduction to Constructive Set Theory

30 June, 2011, Maloa, Leeds

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Plan of lectures

1: Background to CST

2: The axiom system CZF

3: The number systems in CZF

4: Inductive definitions in CST

1: Background to CST

Some brands of constructive mathematics

B1: Intuitionism (Brouwer, Heyting, ..., Veldman)

B2: 'Russian' constructivism (Markov,...)

B3: 'American' constructivism (Bishop, Bridges,...)

B4: 'European' constructivism (Martin-Löf, Sambin,...)

- B1,B2 contradict classical mathematics; e.g.

  $B1:$ All functions $\mathbb{R} \to \mathbb{R}$ are continuous,

  $B2:$ All functions $\mathbb{N} \to \mathbb{N}$ are recursive (i.e. CT).

- B3 is compatible with each of classical maths, B1,B2 and forms their common core.
- B4 is a more philosophical foundational approach to B3.
- All B1-B4 accept RDC and so DC and CC.

Some liberal brands of mathematics using intuitionistic logic

B5: Topos mathematics (Lawvere, Johnstone,...)

B6: Liberal Intuitionism (Mayberry,...)

- B5 does not use any choice principles.
- B6 accepts Restricted EM.

B7: Core explicit Mathematics (CeM)

i.e. a minimalist, non-ideological approach. The aim is to do as much mainstream constructive mathematics as possible in a weak framework that is common to all brands, and explore the variety of possible extensions.

Some settings for constructive mathematics

type theoretical

category theoretical

set theoretical

Some contrasts

classical logic versus intuitionistic logic

impredicative versus predicative

some choice versus no choice

intensional versus extensional

consistent with EM versus inconsistent with EM

Mathematical Taboos

A mathematical taboo is a statement that we may not want to assume false, but we definately do not want to be able to prove.
For example Brouwer's weak counterexamples provide taboos for most brands of constructive mathematics; e.g. if

$$DPow(A) = \{b \in Pow(A) \mid (\forall x \in A)[(x \in b) \vee (x \notin b)]\}$$

then

$$(\forall b \in DPow(\mathbb{N}))[\ (\exists n)[n \in b] \vee \neg(\exists n)[n \in b]\ ]$$

is the Limited Excluded Middle (LEM) taboo.

Warning!

There are two meanings of the word theory in mathematics that can be confused.

mathematical topic: e.g. (classical) set theory

formal system: e.g. ZF set theory

I will use constructive set theory (CST) as the name of a mathematical topic and constructive ZF (CZF) as a specific first order axiom system for CST.

Introducing CST

- It was initiated (using a formal system called CST) by John Myhill in his 1975 JSL paper.
- In 1976 I introduced CZF and gave an interpretation of CZF+RDC in Martin-Löf's dependent type theory. In my view the interpretation makes explicit a constructively acceptable foundational understanding of a constructive iterative notion of set.
- By not assuming any choice principles, CZF allows reinterpretations in sheaf models so that mathematics developed in CZF will apply to such models.
- CST allows the development of constructive mathematics in a purely extensional way exploiting the standard set theoretical representation of mathematical objects.

2: The axiom system CZF

The axiom systems ZF and IZF

- These axiom systems are formulated in predicate logic with equality and the binary predicate symbol $\in$.
- ZF uses classical logic and IZF uses Intuitionistic logic for the logical operations $\wedge, \vee, \rightarrow, \bot, \forall, \exists$.
- $ZF = IZF + EM$
- ZF has a $\neg\neg$-translation into IZF (H. Friedman).

The non-logical axioms and schemes of ZF and IZF

Extensionality

Pairing

Union

Separation

Infinity

Powerset

Collection (classically equivalent to Replacement)

Set Induction (classically equivalent to Foundation)

Collection $(\forall x \in a)\exists y \phi(x,y) \;\rightarrow\; \exists b(\forall x \in a)(\exists y \in b)\phi(x,y)$

Set Induction $\forall a[(\forall x \in a)\theta(x) \;\rightarrow\; \theta(a)] \;\rightarrow\; \forall a \theta(a)$

The axiom system CZF

This is the axiom system that is like IZF except that

- the Separation scheme is restricted,
- the Collection scheme is strengthened,
- and the Powerset axiom is weakened to the Subset Collection scheme.

- $CZF \subseteq IZF$ and $CZF + EM = ZF$.
- CZF has the same proof theoretic strength as Kripke-Platek set theory (KP) or the system ID_1 (i.e. Peano Arithmetic with axioms for an inductive definition of Kleene's second number class $\mathcal{O}$).

The Restricted Separation Scheme

Restricted Quantifiers We write

$$(\forall x \in a)\theta(x) \equiv \forall x[x \in a \to \theta(x)]$$
$$(\exists x \in a)\theta(x) \equiv \exists x[x \in a \land \theta(x)]$$

A formula is restricted (bounded,Δ_0) if every quantifier in it is restricted.

The Scheme: $\exists b \forall x[x \in b \leftrightarrow (x \in a \land \theta(x, \ldots))]$
for each restricted formula $\theta(x, \ldots)$.

- We write $\{x \in a \mid \theta(x, \ldots)\}$ for the set b.

Collection Principles of $CZF, 1$

- We write $(\forall\exists\ \frac{x\in a}{y\in b})\,\theta$ for

$$(\forall x\in a)(\exists y\in b)\theta\ \wedge\ (\forall y\in b)(\exists x\in a)\theta.$$

- Strong Collection

$$(\forall x\in a)\exists y\phi(x,y)\ \rightarrow\ \exists b(\forall\exists\ \frac{x\in a}{y\in b})\,\phi(x,y).$$

- Subset Collection

$$\exists c\forall z[(\forall x\in a)(\exists y\in b)\phi(x,y,z)$$

$$\rightarrow\ (\exists b'\in c)(\forall\exists\ \frac{x\in a}{y\in b'})\,\phi(x,y,z)].$$

Collection Principles of $CZF, 2$

- Strong Collection can be proved in IZF using Collection and Separation.
- For if b is the set given by Collection then we get the set

$$\{y \in b \mid \exists x \in a\ \phi(x,y)\}$$

  by Separation, which gives Strong Collection if used instead of b.
- Replacement can be proved in CZF using Strong Collection.
- For if $\forall x \in a\ \exists! y\ \phi(x,y)$ and b is a set such that $(\forall\exists\ \frac{x \in a}{y \in b})\,\phi(x,y)$ then

$$b = \{y \mid \exists x \in a\ \phi(x,y)\}.$$

Classes

- Class terms: $\{x \mid \phi(x, \ldots)\}$
- $$a \in \{x \mid \phi(x, \ldots)\} \;\leftrightarrow\; \phi(x, \ldots)$$
- Identify each set a with the class $\{x \mid x \in a\}$.
- $$[A = B] \equiv \forall x[x \in A \;\leftrightarrow\; x \in B]$$

Some Examples

$$\begin{aligned}
V &= \{x \mid x = x\} \\
\bigcup A &= \{x \mid \exists y \in A\; x \in y\} \\
\bigcap A &= \{x \mid \forall y \in A\; x \in y\} \\
Pow(A) &= \{x \mid x \subseteq A\} \\
A \times B &= \{x \mid (\exists a \in A)(\exists y \in B) x = (a, b)\}
\end{aligned}$$

where $(a, b) = \{\{a\}, \{a, b\}\}$.

Classes -more examples

$$\begin{aligned} \{x \in A \mid \phi(x, \ldots)\} &= \{x \mid x \in A \wedge \phi(x, \ldots)\} \\ \{\ldots x \ldots \mid x \in A\} &= \{y \mid \exists x \in A\ y = \ldots x \ldots\} \end{aligned}$$

Class functions For classes F, A, B let $F : A \to B$ if $F \subseteq A \times B$ such that

$$(\forall x \in A)(\exists! y \in B)[(x, y) \in F].$$

Also, if $a \in A$ then let $F(a)$ be the unique $b \in B$ such that $(a, b) \in F$. By Replacement, if A is a set then so is

$$\{F(x) \mid x \in A\}.$$

The Fullness axiom

- For classes A, B, C let $C : A \succ\!\!- B$ if $C \subseteq A \times B$ such that

  $(\forall x \in A)(\exists y)[(x, y) \in C]$.

- For sets a, b let

$$mv(a, b) = \{r \in Pow(a \times b) \mid r : a \succ\!\!- b\}.$$

The Axiom

$$(\exists c \in Pow(mv(a, b)))(\forall r \in mv(a, b))(\exists s \in c)[s \subseteq r]$$

Theorem: Given the other axioms and schemes of CZF, the Subset Collection scheme is equivalent to the

Fullness axiom.

Myhill's Exponentiation Axiom

- If a is a set and B is a class let ${}^aB \equiv \{f \mid f : a \to B\}$.
- If $F : a \to B$ then $\{F(x) \mid x \in a\}$ is a set, and so is F, as

$$F = \{(x, F(x)) \mid x \in a\}.$$

So $F \in {}^aB$.

The axiom: ${}^ab$ is a set for all sets a, b.

- This is an immediate consequence of the Fullness axiom and so a theorem of CZF.
- For if $c \subseteq mv(a, b)$ is given by Fullness then ${}^ab = \{f \in c \mid f : a \to b\}$ is a set by Restricted Separation.

‘Truth Values’

- Let $0 = \emptyset$, $1 = \{0\}$ and $\Omega = Pow(1)$.
- For each formula θ we may associate the class $< \theta >= \{x \in 1 \mid \theta\}$, where x is not free in θ. Then

  $$\theta \ \leftrightarrow < \theta >= 1$$

  and if θ is a restricted formula then $< \theta >$ is a set in Ω.
- It is natural to call $< \theta >$ the truth value of θ.
- the Powerset axiom is equivalent to “The class Ω is a set”,
- the full Separation scheme is equivalent to “Each subclass of 1 is a set and so in Ω”.
- With classical logic each subclass of 1 is either 0 or 1, so that the powerset axiom and the full separation scheme hold; i.e. we have ZF.

Set Terms, 1

We can conservatively extend CZF to a theory CZF$_{st}$ by adding set terms, t, given by the syntax equation:

$$t ::= x \mid \emptyset \mid \{t, t\} \mid \cup t \mid t \cap t \mid \{t \mid x \in t\},$$

where free occurrences of x in t_1 are bound in $\{t_1 \mid x \in t_2\}$, and adding the following axioms.

$$\begin{array}{lcl} y \in \emptyset & \leftrightarrow & \bot \\ y \in \{t_1, t_2\} & \leftrightarrow & [y = t_1 \vee y = t_2] \\ y \in \cup t & \leftrightarrow & (\exists x \in t)\ y \in x \\ y \in t_1 \cap t_2 & \leftrightarrow & [y \in t_1 \wedge y \in t_2] \\ y \in \{t_1 \mid x \in t_2\} & \leftrightarrow & (\exists x \in t_2)\ y = t_1 \end{array}$$

Set Terms, 2

Theorem: For each restricted formula $\theta(x)$ and set term a there is a set term t such that

$$\text{CZF}_{st} \vdash t = \{x \in a \mid \theta(x)\}.$$

Corollary: Given the other axioms and schemes of CZF, the Restricted Separation Scheme is equivalent to the conjunction of the axioms

Emptyset: the empty class $\emptyset$ is a set,

Binary Intersection: the intersection class $a \cap b$ of sets a, b is a set.

The Infinity Axiom

Call a class A inductive if $\emptyset \in A$ and $(\forall x \in A)[x^+ \in A]$, where $x^+ = x \cup \{x\}$.

- Infinity Axiom: There is an inductive set.
- Strong Infinity Axiom: There is a smallest inductive set, $\omega = \cap\{x \mid x$ is an inductive set$\}$.
- Full Infinity Scheme: There is a smallest inductive set that is a subset of each inductive class.

In CZF, by making essential use of the Set Induction Scheme, each instance of the full infinity scheme can be derived.

3: The number systems in CZF

$$\mathbb{N} \mapsto \mathbb{Z} \mapsto \mathbb{Q} \mapsto \mathbb{R}$$

Peano structures,1

- Call $(\mathbb{N}, 0, S)$ a Peano structure if the Dedekind-Peano axioms hold; i.e. $\mathbb{N}$ is a set, $0 \in \mathbb{N}$, $S : \mathbb{N} \to \mathbb{N}$ is injective such that $(\forall n \in \mathbb{N})[S(n) \neq 0]$ and, for all sets $A \subseteq \mathbb{N}$

$$[0 \in A] \wedge (\forall n \in A)[S(n) \in A] \ \rightarrow \ (\forall n \in \mathbb{N})[n \in A].$$

  It is a full Peano structure if this holds for all classes A.

- In CZF, $(\omega, \emptyset, s)$ is a Peano structure, where $s : \omega \to \omega$ is given by $s(n) = n^+$ for $n \in \omega$.

Peano structures,2

- Theorem: In CZF, any Peano structure $(\mathbb{N}, 0, S)$ is full and functions can be defined on $\mathbb{N}$ by iteration and, more generally by primitive recursion.
- Iteration Scheme: For classes A and $F : A \to A$, if $a_0 \in A$ then there is a unique $H : \mathbb{N} \to A$ such that $H(0) = a_0$ and $(\forall n \in \mathbb{N})[H(S(n)) = F(H(n))]$.
- Corollary 1: In CZF, given a Peano structure $(\mathbb{N}, 0, S)$ all the primitive recursive functions on $\mathbb{N}$ exist. So Heyting Arithmetic can be interpreted in CZF.
- Corollary 2: In CZF, any two Peano structures are isomorphic. So the axioms for a Peano structure form a categorical axiom system.

Number sets $\mathbb{Z}$ and $\mathbb{Q}$ in CST, 1

- Starting with the Peano structure $(\mathbb{N}, 0, S)$, the successive construction of first the ordered ring $(\mathbb{Z}, \ldots)$ of integers and then the ordered field $(\mathbb{Q}, \ldots)$ of rationals can be carried out in weak systems of CST much as in classical set theory.
- Both the constructions $\mathbb{N} \mapsto \mathbb{Z}$ and $\mathbb{Z} \mapsto \mathbb{Q}$ can be obtained using a quotient $(A \times B)/R$, where A, B are suitably chosen sets and R is a set equivalence relation on the set $A \times B$.

Number sets $\mathbb{Z}$ and $\mathbb{Q}$ in CST, 2

- $A \times B$ is the set $X = \cup\{\cup\{(a,b) \mid a \in A\} \mid b \in B\}$ and the quotient X/R is the set $\{[x] \mid x \in X\}$ where $[x] = \{x' \in X \mid (x,x') \in R\}$.
- Only the Union and Pairing axioms and the Replacement and Restricted Separation schemes are needed to get these sets.

Archimedean pseudo-ordered rings

A relation $<$ on a set R is a pseudo-ordering if, for all $x, y, z \in R$,

1. $\neg[x < y \ \wedge \ y < x]$,
2. $[x < y] \ \rightarrow \ [x < z \ \vee \ z < y]$,
3. $\neg[x < y \ \vee \ y < x] \ \rightarrow \ [x = y]$.

A pseudo-ordered ring is a ring R with a pseudo-ordering compatible with the ring structure; i.e. for all $x, y, z \in R$,

1. $[x < y] \ \rightarrow \ [x + z < y + z]$,
2. $[x < y \ \wedge \ 0 < z] \ \rightarrow \ [xz < yz]$.

It is Archimedean if, for all $a \in R$ there is $n \in \mathbb{N}$ such that

$$a < \overbrace{1 + \cdots + 1}^{n}.$$

More on pseudo-orderings

- Let $<$ be a pseudo-ordering of a set R. Define $\leq$ on R:

$$x \leq y \quad \leftrightarrow \quad \neg[y < x].$$

- Then $\leq$ is a partial ordering of R; i.e. it is reflexive, transitive and antisymmetric.

Cauchy Completeness

- Theorem (CZF+CC): $(\mathbb{R}_c, \ldots)$ is the unique, up to isomorphism Archimedean pseudo-ordered field that is Cauchy complete.
- A pseudo-ordered ring, R, is Cauchy complete if every Cauchy sequence of elements of R converges to an element of R.
- $f : \mathbb{N} \to R$ is a Cauchy sequence if

$$(\forall \epsilon > 0)(\exists n)(\forall m \geq n)\ [f(n) - \epsilon < f(m) < f(n) + \epsilon],$$

and converges to $a \in R$ if

$$(\forall \epsilon > 0)(\exists n)(\forall m \geq n)\ [a - \epsilon < f(m) < a + \epsilon].$$

Dedekind Completeness

- An Archimedean pseudo-ordered field, R, is Dedekind complete if every upper-located subset has a supremum.
- A subset X of R is upper-located if

$$(\forall \epsilon > 0)(\exists x \in X))(\forall y \in X)[y < x + \epsilon].$$

and $a \in R$ is a supremum of X if

$$(\forall x \in X)[x \leq a] \ \wedge \ (\forall \epsilon > 0)(\exists x \in X)[a < x + \epsilon].$$

- Note: If a is a supremum of X then it is the lub of X; i.e.

$$(\forall x \in X)[x \leq a] \ \wedge \ (\forall b \in R)[(\forall x \in X)[x \leq b] \ \rightarrow \ [a \leq b].$$

The continuum without choice

- Proposition (CZF): Let R be an Archimedean pseudo-ordered field. Then
  1. If R is Dedekind complete then it is Cauchy complete.
  2. Assuming CC, if R is Cauchy complete then R is Dedekind complete.
- Theorem (CZF): There is a unique, up to isomorphism Dedekind complete, Archimedean, pseudo-ordered field.
- An upper-located $X \subseteq \mathbb{Q}$ is a Dedekind cut if $X = X^{<}$, where $X^{<} = \{y \in \mathbb{Q} \mid (\exists x \in X)[y < x]\}$.
- Theorem (CZF): The class $\mathbb{R}_d$ of all Dedekind cuts forms a set that can be made into a Dedekind complete, Archimedean, pseudo-ordered field.

4: Inductive Definitions

Examples of inductive definitions

- ω is the smallest class I such that $\emptyset \in I$ and
$$(\forall x \in I)\ x^+ \in I, \text{ where } x^+ = x \cup \{x\}.$$
- HF is the smallest class I such that, for all $n \in \omega$,
$$(\forall f \in {}^n I)\ ran(f) \in I.$$
- HC is the smallest class I such that, for all $a \in \omega^+$
$$(\forall f \in {}^a I)\ ran(f) \in I.$$
- For each class A, $H(A)$ is the smallest class I such that, for all $a \in A$, $(\forall f \in {}^a I)\ ran(f) \in I$.
- $\omega = H(2), \quad HF = H(\omega), \quad HC = H(\omega^+)$

Recall $0 = \emptyset$, $1 = 0^+$ and $2 = 1^+$. Note that ω and HF, but not HC, can be proved to be sets in CZF.

What is an inductive definition?

An *inductive definition* is a class of pairs. A pair (X, a) in an inductive definition will usually be written X/a and called an *(inference) step* of the inductive definition, with *conclusion* a and set X of *premisses*.

- If Φ is an inductive definition, a class I is *Φ-closed* if $X \subseteq I$ implies $a \in I$ for each step X/a of Φ.

Theorem: *There is a smallest Φ-closed class;*
i.e. a class I such that (i) I is Φ-closed and, for each class B, (ii) if B is Φ-closed then $I \subseteq B$. class.
The smallest Φ-closed class is unique and is called the

class inductively defined by Φ and is written $I(\Phi)$.

More Examples

- The Set Induction Scheme expresses that V is the smallest class I such that $a \subseteq I \Rightarrow a \in I$.
- If R is a subclass of $A \times A$ such that $R_a = \{x \mid (x, a) \in R\}$ is a set for each $a \in A$ then $Wf(A, R)$ is the smallest subclass I of A such that $\forall a \in A\, [R_a \subseteq I \Rightarrow a \in I]$.

  Note that $Wf(A, R) = I(\Phi)$, where Φ is the class of steps R_a/a for $a \in A$.
- If B_a is a set for each $a \in A$ then $W_{x \in A} B_x$ is the smallest class I such that $a \in A \,\&\, f : B_a \to I \;\Rightarrow\; (a, f) \in I$.

  Note that $W_{x \in A} B_x = I(\Phi)$, where Φ is the class of steps $ran(f)/(a, f)$ for $a \in A$ and $f : B_a \to V$.

Proof of the theorem

- Given a class Φ of steps X/a, for each class Y let ΓY be the class of a such that there is a step X/a of Φ with $X \subseteq Y$. So Y is Φ-closed iff $\Gamma Y \subseteq Y$.
- Γ is monotone; i.e. $Y_1 \subseteq Y_2 \ \Rightarrow \ \Gamma Y_1 \subseteq \Gamma Y_2$ and what is wanted is a *least pre-fixed point* of Γ.
- The idea for the proof is to iterate the operator Γ into the transfinite so that it ultimately closes up.
- Call a class J of pairs an *iteration class* for Φ if, for all sets a, $J^a = \Gamma J^{\in a}$ where $J^a = \{x \mid (a,x) \in J\}$ and $J^{\in a} = \bigcup_{x \in a} J^x$.

Lemma: *Every inductive definition has an iteration class.*

Proof of the lemma

A set G of ordered pairs is defined to be *good* if

$$(*) \quad G^a \subseteq \Gamma G^{\in a} \text{ for all sets } a.$$

Let J be the union of all good sets.

- We must show that $J^a = \Gamma J^{\in a}$.
- If $y \in J^a$ then, for some good set G,

$$y \in G^a \subseteq \Gamma G^{\in a} \subseteq \Gamma J^{\in a}.$$

Thus $J^a \subseteq \Gamma J^{\in a}$. For the converse let $y \in \Gamma J^{\in a}$ so that X/a is a step of Φ for some $X \subseteq J^{\in a}$. So

$$\forall y' \in X \; \exists G \; [\; G \text{ is good and } y' \in G^{\in a} \;].$$

By Strong Collection there is a set Z of good sets such that

$$\forall y' \in X \; \exists G \in Z \; y' \in G^{\in a}.$$

Let $G = \{(a, y)\} \cup \bigcup Z$. Then G is good so that $y \in G^a \subseteq J^a$. Thus $\Gamma J^{\in a} \subseteq J^a$.

Definition of $I(\Phi)$

We show that $J^\infty = \bigcup_{a \in V} J^a$ is the smallest Φ-closed class.

- To show that J^∞ is Φ-closed let X/y be a step of Φ for some set $X \subseteq J^\infty$. We must show that $y \in J^\infty$.
  $--$ As $\forall y' \in X\ \exists x\ y' \in J^x$, by Collection, there is a set a such that $\forall y' \in X\ \exists x \in a\ y' \in J^x$; i.e. $X \subseteq J^{\in a}$. Hence $y \in \Gamma J^{\in a} = J^a \subseteq J^\infty$. Thus J^∞ is Φ-closed.
- Let I be Φ-closed, to show that $J^\infty \subseteq I$ we show that $J^a \subseteq I$ by set-induction on a. So we may assume the induction hypothesis that $J^x \subseteq I$ for all $x \in a$. It follows that $J^{\in a} \subseteq I$ so that $J^a = \Gamma J^{\in a} \subseteq \Gamma I \subseteq I$, the inclusions holding because Γ is monotone and I is Φ-closed. Thus $J^\infty \subseteq I$

So we define $I(\Phi) = J^\infty$.

Local Inductive Definitions

An inductive definition Φ is defined to be *local* if ΓY is a set for each set Y.

Proposition: If Φ is local then J^a and $J^{\in a}$ are sets for all a.

This has an easy proof by Set Induction.

When is the class $I(\Phi)$ a set?

- A class B is a *bound* for Φ if, whenever X/y is a step of Φ then $X = ran(f)$ for some $f \in \bigcup_{b \in B} {}^{b}X$.
- Φ is *bounded* if Φ has a set bound and, for each set X, the class of conclusions y of steps X/y in Φ is a set.
- Note that if Φ is a set then it is bounded.
- $CZF^{+} = CZF + REA$, where REA is the

Regular Extension Axiom

Theorem (CZF^{+}): If Φ is bounded then it is local and $I(\Phi)$ is a set. Examples: For each set A,

- $H(A)$ is a set,
- $Wf(A, R)$ is a set, if R is a set,
- $W_{x \in A} B_x$ is a set, if B_x is a set for each $x \in A$.

Regular Extension Axiom (REA)

- A set A is regular if $(A, \in \cap (A \times A))$ is a transitive model of the Strong Collection Scheme; i.e. it is an inhabited set such that $A \subseteq Pow(A)$ and if $a \in A$ and $R : a \succ\!\!- A$ then there is $b \in A$ such that

$$\forall x \in a\ \exists y \in b\ (x, y) \in R \text{ and } \forall y \in b\ \exists x \in a\ (x, y) \in R$$

- The axiom REA: Every set is a subset of a regular set.
- Classically, if α is a regular ordinal then V_α is a regular set.

Set Compactness

Theorem (CZF^+): For each set S and each set $P \subseteq Pow(S)$ there is a set B of subsets of $P \times S$ such that, for each class $\Phi \subseteq P \times S$,

$$a \in I(\Phi) \iff a \in I(\Phi_0) \text{ for some } \Phi_0 \in B \text{ such that } \Phi_0 \subseteq \Phi.$$

Definition: For each class X let

$$I(\Phi, X) = I(\Phi \cup (\{\emptyset\} \times X)).$$

Theorem (CZF^+): If Φ is a subset of $Pow(S) \times S$, where S is a set, then there is a set B of subsets of S such that, for each class X,

$$a \in I(\Phi, X) \iff a \in I(\Phi, X_0) \text{ for some } X_0 \in B \text{ such that } X_0 \subseteq X.$$