The Type Theoretic Concept of Set

Set Theory, Classical and Constructive

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Plan of the talk

- Introduction
- The classical conception of set
- The type theoretic conception of set
- A quick look at the type theory
- Some references

Some set theories

$$\begin{array}{lclcl} CZF & \Rightarrow & IZF & \Rightarrow & ZF \\ \Uparrow & & \Uparrow & & \Updownarrow \\ CZF_{R,E} & \Rightarrow & IZF_R & & (CZF_{R,E} + EM) \end{array}$$

$$ZF \leq_{\neg\neg} IZF$$

$$CZF \;\sim\; ID_1 \;\sim\; KP \;<<\; IZF \;\sim\; ZF$$

The type-theoretic interpretations

$$CZF \leq_{tt} (ML + \mathbb{U} + \mathbb{V})$$

$$ZF \leq_{tt} (ML + \mathbb{U} + \mathbb{V} + EM)$$

$$ZF \leq_{\neg\neg} IZF \leq_{tt} (ML + \mathbf{\Omega} + \mathbb{U} + \mathbb{V})$$

The cumulative hierarchy in ZF

- $V = \bigcup_{\alpha \in On} V_\alpha$,
  where $V_\alpha = \bigcup_{\beta < \alpha} Pow(V_\beta)$ for $\alpha \in On$.
- $V_0 \subseteq V_1 \subseteq \cdots V_\alpha \subseteq V_{\alpha+1} \subseteq \cdots$
- $V_0 = \emptyset, V_1 = \{\emptyset\}, V_{\alpha+1} = Pow(V_\alpha)$
- $V_\alpha =$ the set of sets formed by stage α
- $\{x \in V_\beta \mid \cdots x \cdots\} \in V_\alpha$ if $\beta < \alpha$

The classical iterative conception

Zermelo, Scott, Schoenfield, Boolos, Parsons,...

- Sets are extensional.
- Sets are formed in stages, α, out of elements formed at earlier stages.
- A set is formed by collecting together (into a whole) its elements.
- There are lots of stages:
  1. There is a stage.
  2. For each stage there is a later stage.
  3. There is a stage, ω, reflecting 1,2.
  4. If $\{\alpha_i\}_{i \in I}$ is a family of stages indexed by a set I then there is a stage later than all the α_i.

The formation of powersets

Suppose X is an infinite set formed at some stage α.

- Then each element of X will have been formed at some stage before α.
- So each subset of X will have been formed at or before stage α. But can the elements of each subset of X, however the subset is formed, really be collected into a whole at or before stage α?
- So the powerset, $Pow(X)$, will be formed at any stage after α. But can the subsets of X be collected into a whole so easily?

Types, sets and classes

- Any discussion concerning the concept of set must distinguish between the three distinct notions of type, set and class.
- For example the intended universe of ZF set theory is a type, the objects in that universe are sets and $\{x \mid x \notin x\}$ is a class.
- The notion of type is perhaps best taken as a pre-mathematical philosophical notion.
- The notion of set of ZF is an iterative combinatorial notion.
- The notion of class is a logical notion - the extension of a predicate.

Types and Classes

- A mathematical object is always given as an object of some type.
- We write $a : A$ for the judgement that a is an object of type A.
- A class on a type is the extension of a propositional function on the type.
- If B is a propositional function on the type A then its extension is the class $C = \{x : A \mid B(x)\}$.
- For $a : A$ the proposition that a is in the class C is $B(a)$.
- If also $C' = \{x : A \mid B'(x)\}$ then $(C = C')$ is the proposition

$$(\forall x : A)[B(x) \leftrightarrow B'(x)].$$

What is a set of elements of a type?

It is a collection into a whole of objects chosen from the type.

- e.g. given the type $\mathbb{N}$ of natural numbers we have sets of natural numbers such as $\{0\}, \{0,1\}, \{0,3,18\}, \{\}$
- and sets $\{0,2,4,6,\ldots,92\}$, $\{2i \mid i < n\}$ for $n : \mathbb{N}$,
- and infinite sets such as $\{0,2,4,6,\ldots\} = \{2i \mid i : \mathbb{N}\}$.
- In general we can form sets of natural numbers $\{a_i \mid i \in I\}$ with $a_i : \mathbb{N}$ for $i : I$, where I is an index-type.

Index-types

- I use the word index-type for something like
  - Bishop's constructive notion of set,
  which I think is also something like
  - Martin-Löf's type-theoretic notion of set or data-type
  and something like
  - the category theorists' notion of set
  when they talk about a category of sets.
- I need a distinct word in order to avoid confusion with the combinatorial notion of set, which is what axiomatic set theory is about.
- A set is formed out of its elements. But an index-type is an object that is conceptually prior to its elements.
- The index-types form a type $\mathbb{U}$.

Sets, of elements of a type, 1

- Given a type A, a set of elements of A is given by:
  1. an index-type I, the index-type of the set,
  2. a function $f : I \rightarrow A$, also thought of as a family of elements of A, $\{a_i\}_{i:I}$, where $a_i = f(i) : A$ for $i : I$.
- We may write the set as $(\mathsf{set}\ i : I) f(i)$ or $[a_i \mid i : I]$.
- The chosen elements of the set are the a_i for $i : I$.
- The sets of elements of A form a type $Sub(A)$.
- A itself may be an index-type, so that we may form the set $[x \mid x : A] : Sub(A)$ of all the objects of A.
- But, in general, the type A need not be an index-type.

Sets, of elements of a type, 2

- An equality relation, $=_A$, on a type A is an assignment of a proposition $(b =_A c)$ to $b, c : A$ so that the laws for an equivalence relation hold; i.e.
  - $(\forall x : A)[x =_A x]$, $\quad (\forall x, y : A)[x =_A y \to y =_A x]$,
  - $(\forall x, y, z : A)[x =_A y \to (y =_A z \to x =_A z)]$.
- Given an equality relation $=_A$ on a type A we may define the membership relation $\in_A$ and extensional equality relation $=_{Sub(A)}$ as follows:
  - If $\alpha : Sub(A)$ is $[a_i \mid i : I]$ then, for $a : A$, $(a \in_A \alpha)$ is the proposition $(\exists i : I)[a =_A a_i]$.
  - If also $\beta : Sub(A)$ is $[b_j \mid j : J]$ then $(\alpha =_{Sub(A)} \beta)$ is the proposition

$$(\forall i : I)(\exists j : J)[a_i =_A b_j] \wedge (\forall j : J)(\exists i : I)[a_i =_A b_j].$$

The type of iterative sets

- The type $\mathbb{V}$ of iterative sets is the inductive type obtained by iterating the set-of operation.
- The iterative sets are generated using the following rule.

Any set-of objects in $\mathbb{V}$ is an object in $\mathbb{V}$

- In Constructive Type Theory $\mathbb{V}$ is the inductive type having the introduction rule

$$\frac{I : \mathbb{U} \quad f : I \to \mathbb{V}}{(\text{set } i : I) f(i) : \mathbb{V}}$$

- So we have $Sub(\mathbb{V}) = \mathbb{V}$.

Equality and membership on $\mathbb{V}$

- We can recursively define $(\alpha =_{\mathbb{V}} \beta)$ for $\alpha, \beta : V$ using the rule

$$\frac{(\forall i : I)(\exists j : J)[a_i =_{\mathbb{V}} b_j] \wedge (\forall j : J)(\exists i : I)[a_i = \mathbb{V} b_j]}{\alpha =_{\mathbb{V}} \beta}$$

  where $\alpha = [a_i \mid i : I]$ and $\beta = [b_j \mid j : J]$.
- Also

$$\alpha \in_V \beta \ := \ (\exists j : J)(\alpha =_{\mathbb{V}} b_j).$$

ZF justified in type theory

- The logic of Type theory is obtained using the Propositions-as-types paradigm.
- The logic using this paradigm is intuitionistic logic.
- Also the type-theoretic axiom of choice holds:

$$(\forall x : A)(\exists y : B(x))R(x, y) \\ \rightarrow (\exists f : (\Pi x : A)B(x))(\forall x : A)R(x, f(x))$$

- The type $\mathbb{V}$ of iterative sets provides an interpretation of CZF (Constructive ZF).
- If the logic of type theory is made classical then $\mathbb{V}$ provides an interpretation of ZF.
- But this last step is not constructive.

The basic type theory ML, 1

The primitive forms of type

$$\mathbb{0}, \mathbb{1}, \mathbb{B}, \mathbb{N} \text{ type} \qquad \frac{A \text{ type} \qquad B(x) \text{ type } (x{:}A)}{(\Pi x{:}A)B(x), (\Sigma x{:}A)B(x) \text{ type}}$$

The introduction rules

$$\star : \mathbb{1} \qquad \mathbb{t}, \mathbb{f} : \mathbb{B} \qquad \frac{b(x){:}B(x) \ (x{:}A)}{(\lambda x{:}A)b(x){:}(\Pi x{:}A)B(x)}$$

$$0 : \mathbb{N} \qquad \frac{n{:}\mathbb{N}}{\mathsf{succ}(n){:}\mathbb{N}} \qquad \frac{a{:}A \qquad b{:}B(a)}{\mathsf{pair}(a,b){:}(\Sigma x{:}A)B(x)}$$

Defined types

$$(A_1 \to A_2) := (\Pi \_ : A_1)A_2 \qquad (A_1 \times A_2) := (\Sigma \_ : A_1)A_2$$

No type dependency so far!

The basic type theory ML, 2

There are standard elimination rules for each form of type e.g. the standard elimination rule for $\mathbb{B}$ is

$$\frac{A(x)\ \mathsf{type}\ (x{:}\mathbb{B}) \qquad c{:}\mathbb{B} \qquad a_1{:}A(\mathbb{t}) \qquad a_2{:}A(\mathbb{f})}{\begin{cases} \mathsf{cases}(c, a_1, a_2) : A(c) \\ \mathsf{cases}(\mathbb{t}, a_1, a_2) = a_1 : A(\mathbb{t}) \\ \mathsf{cases}(\mathbb{f}, a_1, a_2) = a_2 : A(\mathbb{f}) \end{cases}}$$

In order to have dependent types we also use

$$\frac{c{:}\mathbb{B} \qquad A_1{,}A_2\ \mathsf{type}}{\begin{cases} \mathsf{Cases}(c, A_1, A_2)\ \mathsf{type} \\ \mathsf{Cases}(\mathbb{t}, A_1, A_2) = A_1 \\ \mathsf{Cases}(\mathbb{f}, A_1, A_2) = A_2 \end{cases}}$$

The basic type theory ML, 3

More defined types

$$(A_1 + A_2) \quad := (\Sigma x : \mathbb{B})\mathsf{Cases}(x, A_1, A_2) \qquad (A_1, A_2 \text{ type})$$

$$\mathbb{T}rue(c) \quad := \mathsf{Cases}(c, \mathbb{1}, \mathbb{0}) \qquad (c : \mathbb{B})$$

So we have the derived introduction rules

$$\frac{a{:}A_1}{\mathsf{pair}(\mathbb{t},a){:}A_1{+}A_2} \qquad \frac{a{:}A_2}{\mathsf{pair}(\mathbb{f},a){:}A_1{+}A_2}$$

and the equalities

$$\mathbb{T}rue(\mathbb{t}) = \mathbb{1} \qquad \mathbb{T}rue(\mathbb{f}) = \mathbb{0}$$

The basic type theory ML, 4

Propositions-as-Types (à la Curry-Howard)

Proposition = Type

$Prop$	$\bot$	$\top$	$A_1 \supset A_2$	$A_1 \wedge A_2$	$A_1 \vee A_2$
$Type$	$\mathbb{0}$	$\mathbb{1}$	$A_1 \to A_2$	$A_1 \times A_2$	$A_1 + A_2$

$Prop$	$(\forall x : A)B(x)$	$(\exists x : A)B(x)$
$Type$	$(\Pi x : A)B(x)$	$(\Sigma x : A)B(x)$

Adding Excluded Middle

$$\frac{A \text{ type}}{\sigma(A) : (A + (A \rightarrow \mathbb{O}))}$$

or equivalently

$$\frac{A \text{ type}}{\begin{cases} \tau(A) & : \mathbb{B} \\ t(A) & : (\mathbb{T}rue(\tau(A)) \leftrightarrow A) \end{cases}}$$

Note: $(B \leftrightarrow A) := (B \rightarrow A) \times (A \rightarrow B)$

Adding a type Ω

$$\Omega \text{ type} \qquad \frac{a : \Omega}{\mathbb{T}(a) \text{ type}} \qquad \frac{A \text{ type}}{\begin{cases} \tau(A) & : \Omega \\ t(A) & : (\mathbb{T}(\tau(A)) \leftrightarrow A) \end{cases}}$$

- Think of Ω as a type of predicative ‘propositions’, or perhaps truth values. $\mathbb{T}(a)$ is the proposition that expresses that a is a true predicative ‘proposition’. These rules are analogous to Bertrand Russell’s Axiom of Reducibility and express that each proposition A is logically equivalent to the proposition $\mathbb{T}(\tau(A))$.

- Excluded Middle is the extreme version of Ω where $\Omega = \mathbb{B}$ and $\mathbb{T}(a) = \mathbb{T}rue(a)$.

Adding a type universe $\mathbb{U}$

- The rules of $\mathbb{U}$ express that $\mathbb{U}$ is a type of types that reflects the previous type forming rules.
- Reflecting the rules of ML we get

$$\mathbb{U}\ \text{type} \qquad \frac{A{:}\mathbb{U}}{A\ \text{type}} \qquad \mathbb{0}, \mathbb{1}, \mathbb{B}, \mathbb{N} : \mathbb{U} \qquad \frac{A{:}\mathbb{U} \qquad B(x){:}\mathbb{U}\ (x{:}A)}{(\Pi x{:}A)B(x), (\Sigma x{:}A)B(x){:}\mathbb{U}}$$

- Reflecting the $\mathbb{\Omega}$ rules we get

$$\mathbb{\Omega} : \mathbb{U} \qquad \frac{a{:}\mathbb{\Omega}}{\mathbb{T}(a){:}\mathbb{U}}$$

The type-theoretic interpretations

$$CZF \leq_{tt} (ML + \mathbb{U} + \mathbb{V})$$

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$$ZF \leq_{\neg\neg} IZF \leq_{tt} (ML + \mathbb{\Omega} + \mathbb{U} + \mathbb{V})$$

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