On the T_1 Axiom and Other Separation Properties in Constructive Point-free and Point-set Topology

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Abstract

In this note a T_1 formal space (T_1 set-generated locale) is a formal space whose points are closed as subspaces. Any regular formal space is T_1. We introduce the more general notion of T_1^* formal space, and prove that the class of points of a weakly set-presentable T_1^* formal space is a set in the constructive set theory **CZF**. The same also holds in constructive type theory. We then formulate separation properties T_i^* for constructive topological spaces (ct-spaces), strengthening separation properties discussed elsewhere. Finally we relate the T_i^* properties for ct-spaces with corresponding properties of formal spaces.

Introduction

There is no unanimously adopted localic analogue of the T_1 axiom for topological spaces. Unordered (T_U) locales [10, 11], and subfit/conjunctive locales [17, 5] have been considered as candidates. However neither of these two notions is regarded as entirely satisfactory, primarily because both fail to coincide with the T_1 property in the spatial case. For example there are

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Hausdorff spaces whose frame of open subsets is not unordered [11], and subfit sober spaces that fail to be T_1 [12]. One should then add that these notions have been discussed mostly in a classical setting, so that a further flourishing of distinct concepts has to be expected in constructive contexts.

In this note we use the notion of T_1 locale as one whose points are closed considered as sublocales. This concept of T_1 locale is studied classically in [15] in the form that a locale is T_1 iff its primes are dual atoms. The justification for this choice is that sublocales of the frame of a sober T_1 space are precisely locales that enjoy this property. Observe, however, that according to this notion all locales without points are T_1.

The definitions and results in this note are carried out in the setting of constructive set theory. More specifically the definitions and theorems can be represented in the formal system **CZF**, [4], and, with minor adjustments, can also be represented in constructive type theory, [13]. The system **CZF** can be formulated in the same formal language as **ZF**, but uses intuitionistic logic rather than classical logic and uses some modifications of the set theoretic axioms of **ZF**. We use class notation and terminology following the standard approach used in **ZF**. So each class can be presented as $\{x \mid \phi(x, ...)\}$ where $\phi(x, ...)$ is a formula of the first order language of axiomatic set theory which may have free occurrences of the variable x and possibly other variables treated as parameters intended to represent fixed sets.

Class notation and terminology can often be useful in set theory and is particularly useful in constructive set theory. This is because the Powerset axiom is not available and nor is the full Separation scheme. So for any set A the class $Pow(A)$ of all subsets of A cannot generally be shown to be a set (it cannot, if A is non-empty), and nor can subclasses of A be generally taken to be sets. It is worth noting at this point that, even when A is a class it makes perfectly good sense to form the class $Pow(A)$ of all *subsets* of A. So the Pow operation can be iterated on classes. But we cannot take the collection of subclasses of a class to be itself a class.

The reader should refer to [4] for any unfamiliar details concerning **CZF**. Here we will only recall the two key axiom schemes of Strong Collection and Subset Collection that are used in this note. Strong Collection is a strengthening of the classical axiom scheme of Replacement that is a theorem of **ZF**. Given the other axioms and schemes of **CZF**, Subset Collection is equivalent to the more useful Fullness axiom and is used instead of the Powerset axiom.

For sets a, b let $\mathbf{mv}(b^a)$ be the class of 'multivalued functions' from a to

b, i.e. subsets r of $a \times b$ such that $(\forall x \in a)(\exists y \in b)\ (x, y) \in r$.

Strong Collection: Let a be a set and let B, R be classes with $R \subseteq a \times B$ such that $(\forall x \in a)(\exists y \in B)\ (x, y) \in R$. Then there is a set $b \subseteq B$ such that $(\forall x \in a)(\exists y \in b)\ (x, y) \in R$ and $(\forall y \in b)(\exists x \in a)\ (x, y) \in R$.

Fullness: Given sets a, b there is a subset c of $\mathbf{mv}(a^b)$ such that every element of $\mathbf{mv}(a^b)$ has a subset in c.

Even when A is just a singleton set the assumption that $Pow(A)$ is a set implies, in **CZF**, the full Powerset axiom. For this reason class-sized mathematical structures naturally arise. For example, a topological space (X, τ) consists of a set X of the points of the space together with a topology τ of the open sets of the space, that, when non-trivial, is a class that cannot be proved to be a set. So the collection of all topological spaces is not even a class. It follows that the category **Top** of topological spaces and continuous maps is not even a large category, according to the usual set-theoretic definition, where the collections of objects and maps of a large category are required to form classes. We will call such a category of class-sized objects a superlarge category. Another example is the superlarge category of classes and class functions between them. We note that superlarge entities are used in this context only for organizing the objects and morphisms under consideration in 'categories', and to relate them via functors.

Lacking the Powerset axiom and the full Separation scheme, it is often much harder than in fully impredicative settings to prove that a certain class is a set. A significant theorem of **CZF** is for instance the result that the class of Dedekind real numbers forms a set ([4]; note that the principle of (dependent and) countable choice are not part of the basic set of axioms of **CZF**). In some cases it is however possible to represent a certain class of objects as the class of points of a locale. Recent results have shown that if this locale is set-presented, (cf. Definition 11 below), and has sufficiently strong separations properties, then its class of points is a set. In particular these results generalise the mentioned theorem concerning the class of Dedekind real numbers. So it is natural to try to find the most general conditions under which the points of a locale do form a set. This paper presents (inter alia) a contribution to this task.

In section 1 we review how formal topology is a theory of formal spaces that gives a constructive approach to locale theory. We introduce the notion of a T_1 formal space in section 2 and show that every regular formal space is T_1. In section 3 we introduce the notion of a T_1^* formal space, a weakening of the notion of a T_1 formal space, and prove our main result that, for every weakly set-presentable T_1^* formal space, the class of its points form a set. This strengthens the earlier result that, for every set-presentable regular formal space, the class of its points form a set. In section 4 we explore separation properties T_i^* for ct-spaces. The notion of a ct-space was introduced in [2] as a generalisation of Bishop's notion of neighborhood system. The T_i^* separation properties for these spaces strengthen separation properties previously discussed in [3]. Finally, in section 5 we relate the T_i^* separation properties for ct-spaces to corresponding properties for formal spaces.

1 Formal topology as constructive locale theory

Review of some locale theory

We review the basic notion of a locale. We start with the standard definition in classical mathematics which we then need to modify slightly to the notion of an *sg*-locale so as to conform to our constructive setting in which the Powerset axiom is not assumed.

In classical mathematics a *frame* is a poset with a top element, binary meets and sups of subsets such that meets distribute over sups. Frames form a category whose maps between frames preserve the frame structure. The category of *locales* is the opposite category. So locales are just frames and a locale map is just a frame map going in the opposite direction.

When, as in constructive set theory, we do not assume the Powerset axiom we need to consider class frames such as the poclass $\Omega = Pow(1)$, where $1 = \{\emptyset\}$. A poclass is a class with a class relation on it that satisfies the standard requirements for a partial ordering. A *class frame* is a poclass that has a top element $\top$, binary meets $a_1 \wedge a_2$ of elements a_1, a_2 and sups $\bigvee X$ of sets X of elements, with binary meets distributing over sups; i.e. $a \wedge \bigvee X = \bigvee\{a \wedge x \mid x \in X\}$ for all elements a and subsets X. A class frame is a frame if it is small; i.e. its elements form a set. Note that it can be shown, [9], that no non-trivial class frame can be proved to be small in systems of constructive set theory such as **CZF**. So the notion of class frame is really needed.

But this notion of class frame is a little too general. A *set-generated frame* *(sg*-frame) is a class frame that comes equipped with a set-indexed family of generators, $\{\gamma(s)\}_{s\in S}$. So S is an index set, each $\gamma(s)$ is an element of the frame and for each element a of the frame the class $S_a = \{s \in S \mid \gamma(s) \leq a\}$ is a set such that $\bigvee\{\gamma(s) \mid s \in S_a\} = a$.

Note that any frame A can be trivially equipped with the set-indexed family of generators $\{a\}_{a\in A}$ to become an *sg*-frame and, if we assume the Powerset axiom any sg-frame is small and so is a frame. We define the category sgLoc of *sg-locales* to be the opposite of the (superlarge) category of *sg*-frames and frame maps.

Review of some formal topology theory

Formal topology has been developed as a version of locale theory in the setting of Martin-Löf's constructive type theory [16], but can also be developed in constructive set theory, see [2]. We review here a definition of the category of formal spaces and describe how it is equivalent to the category sgLoc.

Given a set S we define an operator $\mathcal{A} : Pow(S) \to Pow(S)$ to be a *cover operator on* S if, for all $U, V \in Pow(S)$

CO1 $U \subseteq \mathcal{A}U$,

CO2 $U \subseteq \mathcal{A}V$ implies $\mathcal{A}U \subseteq \mathcal{A}V$,

CO3 $\mathcal{A}U \cap \mathcal{A}V \subseteq \mathcal{A}(U \downarrow V)$,

where, if $s, t \in S$ then $s \downarrow t = \mathcal{A}\{s\} \cap \mathcal{A}\{t\}$, and $U \downarrow V = \bigcup_{s\in U,t\in V} s \downarrow t$.

Definition: 1 $\mathcal{S} = (S, \lhd)$ *is defined to be a* formal space (also sometimes called a formal topology) *if S is a set and $\lhd$ is a class relation between S and $Pow(S)$ such that, for $a \in S$ and $U \in Pow(S)$*

$$a \lhd U \iff a \in \mathcal{A}U.$$

for some (necessarily unique) cover operation $\mathcal{A}$ on S.

The set S is called the base of the formal space. Intuitively, the set $\mathcal{A}U$ is the set of basic neighbourhoods that are covered by U.

The above properties of the cover operation $\mathcal{A}$ can be rewritten as the following properties of the cover relation $\lhd$.

FS1 $s \in U$ implies $s \lhd U$,

FS2 $s \triangleleft U$ and $U \triangleleft V$ implies $s \triangleleft V$,

FS3 $s \triangleleft U$ and $s \triangleleft V$ implies $s \triangleleft (U \downarrow V)$,

for $s \in S, U, V \in Pow(S)$, where, $U \triangleleft V$ iff $\mathcal{A}U \subseteq \mathcal{A}V$.

Definition: 2 *Let $\mathcal{S} = (S, \triangleleft)$ and $\mathcal{S}' = (S', \triangleleft')$ be formal spaces with associated covers $\mathcal{A}$ and $\mathcal{A}'$ respectively. We define a class function $f : Pow(S) \to Pow(S')$ to be a* continuous map $\mathcal{S}' \to \mathcal{S}$ *if*

CTS1 $f(S) = S'$,

CTS2 $f(U) \cap f(V) = f(\mathcal{A}U \cap \mathcal{A}V)$ *for* $U, V \in Pow(S)$,

CTS3 $f(\cup\mathcal{U}) = \mathcal{A}'(\bigcup\{f(U) \mid U \in \mathcal{U}\})$ *for sets* $\mathcal{U} \subseteq Pow(S)$.

Note that, for continuous maps $f : \mathcal{S}' \to \mathcal{S}$,

$$f(U) = f(\mathcal{A}U) = \mathcal{A}'(f(U)) \text{ for } U \in Pow(S),$$

so that $f(U) \cap f(V) = f(U \downarrow V)$ for $U, V \in Pow(S)$. Also note that formal spaces and continuous maps form a category FSpace when the identity maps and composition of maps are defined in the obvious way.

Proposition: 3 *The categories* FSpace *and* sgLoc *are equivalent.*

The equivalence is obtained using functors $\mathrm{fs} : \mathsf{sgLoc} \to \mathsf{FSpace}$ and $Sat : \mathsf{FSpace} \to \mathsf{sgLoc}$.

The functor Sat is defined as follows. Let $\mathcal{S} = (S, \triangleleft)$ be a formal space with associated cover operator $\mathcal{A}$. Each $U \in Pow(S)$ is a set, so that $\mathcal{A}U$ is also a set in $Pow(S)$. We define $Sat(\mathcal{S})$ to be the subclass $\{\mathcal{A}U \mid U \in Pow(S)\}$ of $Pow(S)$, which is an sg-locale when partially ordered by the subset relation and equipped with the set-indexed family of generators $\{\mathcal{A}\{s\}\}_{s \in S}$. If $f : Pow(S) \to Pow(S')$ is a continuous map $\mathcal{S}' \to \mathcal{S}$ in FSpace then $Sat(f) : Sat(\mathcal{S}') \to Sat(\mathcal{S})$ is defined to be the restriction of f to $Sat(\mathcal{S})$.

For the functor fs, if A is an sg-locale equipped with the set-indexed family of generators $\{\gamma(s)\}_{s \in S}$ then $\mathrm{fs}(A)$ is defined to be the formal space $(S, \triangleleft)$ where

$$s \triangleleft U \iff \gamma(s) \leq \bigvee\{\gamma(t) \mid t \in U\}$$

for $s \in S, U \in Pow(S)$. If $F : A' \to A$ is a map in sgLoc then $\mathrm{fs}(F) : \mathrm{fs}(A') \to \mathrm{fs}(A)$ in FSpace is the class function $Pow(S) \to Pow(S')$ where, for $U \in Pow(S)$,

$$\mathrm{fs}(F)(U) = \mathcal{A}'\{F(\gamma(s)) \mid s \in U\}.$$

Here $\mathcal{A}'$ is the cover operator associated with the formal space $\mathrm{fs}(A')$. It is a routine matter to check that Sat and fs are indeed adjoint functors forming an equivalence between the two categories. Indeed $\mathrm{fs}(Sat(\mathcal{S})) = \mathcal{S}$ for each formal space $\mathcal{S}$ and $\eta_A : A \cong Sat(\mathrm{fs}(A))$ for each sg-locale A, where, for $a \in A$,

$$\eta_A(a) = S_a = \{s \in S \mid \gamma(s) \leq a\}.$$

Note that the formal topologies are class-sized as are the continuous maps between them. So the categories FSpace and sgLoc are superlarge, as are the functors between them.

2 The notion of a T_1 formal space

We want to carry over to formal topology the classical notion of T_1 locale as a locale whose points, considered as sublocales, are closed sublocales. We start by formulating the notion of T_1 sg-locale.

An *sg-sublocale* of an sg-locale A is just a regular subobject of A in the category sgLoc; i.e. a subobject represented by a surjective frame map $f : A \to A'$, for some sg-frame A'. The sg-sublocale is a *closed sg-sublocale* if it is represented by the surjective frame map $f_a : A \to A_a$, for some $a \in A$, where $f_a(x) = a \vee x$ for $x \in A$ and $A_a = \{x \in A \mid a \leq x\}$. A *point* of an sg-locale A is a locale map $\Omega \to A$, where Ω is the locale of subsets of $1 = \{\emptyset\}$ partially ordered by the subset relation, which is an sg-locale when equipped with the set-indexed family $\{\gamma(s)\}_{s \in S}$, where $S = \{1\}$ and $\gamma(1) = 1$. This sg-locale Ω is a terminal object in the category sgLoc and each point of A represents an sg- sublocale of A. An sg-locale is defined to be a T_1 *sg-locale* if every point represents a closed sg-sublocale.

We are now ready to review the definitions of subpace, closed subspace and formal point of a formal space and show how they correspond to the notions of sg-sublocale, closed sg-sublocale and point of an sg-locale via the equivalence between the categories of formal spaces and sg-locales.

Definition: 4 *A* subspace *of a formal space* $\mathcal{S} = (S, \lhd)$ *is defined to be a formal space* $\mathcal{S}' = (S, \lhd')$ *on the same base, with* $\lhd'$ *satisfying the following conditions.*

1. $a \lhd U$ *implies* $a \lhd' U$ *for all* $U \in Pow(S)$,
2. $(a \downarrow' b) \lhd' (a \downarrow b)$ *for all* $a, b \in S$.

The subspace $\mathcal{S}'$ *is defined to be* closed *if it is of the form* $\mathcal{S}_V = (S, \lhd_V)$, *with* $a \lhd_V U \iff a \lhd (U \cup V)$, *for* V *a given (fixed) subset of* S *[8].*

The following result states that we have formulated correct notions of subspace and closed subspace.

Proposition: 5 *If* $\mathcal{S}' = (S, \lhd')$ *is a subspace of* $\mathcal{S} = (S, \lhd)$, *with associated cover operation* $\mathcal{A}'$, *then* $F_{\mathcal{S}'}$ *is a surjective frame map* $Sat(\mathcal{S}) \to Sat(\mathcal{S}')$ *and hence represents an sg-sublocale of* $Sat(\mathcal{S})$, *where*

$$F_{\mathcal{S}'}(U) = \mathcal{A}'U \text{ for } U \in Sat(\mathcal{S}).$$

Moreover, every sg-sublocale of $Sat(\mathcal{S})$ *is represented by* $F_{\mathcal{S}'}$ *for a unique subspace* $\mathcal{S}'$ *of* $\mathcal{S}$. *Also,* $F_{\mathcal{S}'}$ *represents a closed sg-sublocale of* $Sat(\mathcal{S})$ *iff* $\mathcal{S}'$ *is a closed subspace of* $\mathcal{S}$.

Definition: 6 *A (formal) point of a formal space* $\mathcal{S} = (S, \lhd)$ *is a subset* α *of* S *such that the following conditions hold.*

FP1 $S \between \alpha$,

FP2 $a, b \in \alpha$ *implies* $(a \downarrow b) \between \alpha$,

FP3 $a \in \alpha$ *and* $a \lhd U$ *implies* $U \between \alpha$,

where, for sets U, V *we write* $U \between V$ *if* $U \cap V$ *is inhabited.*

The class of points of $\mathcal{S}$ *is denoted by* $\mathcal{P}t(\mathcal{S})$. *If* $\alpha \in Pt(\mathcal{S})$ *let* $\mathcal{S}^\alpha = (S, \lhd^\alpha)$ *where, for* $s \in S, U \in Pow(S)$,

$$s \lhd^\alpha U \iff [s \in \alpha \Rightarrow U \between \alpha].$$

The next result makes explicit how the notion of a formal point of a formal space relates to the standard notion of point of a locale.

Proposition: 7 *Let α be a subset of S, where $\mathcal{S} = (S, \lhd)$ is a formal space and let $f_\alpha(U) = \{x \in 1 \mid U \between \alpha\}$ for $U \in Sat(\mathcal{S})$. Then*

$$\alpha \in Pt(\mathcal{S}) \iff f_\alpha \text{ is a frame map } Sat(\mathcal{S}) \to \Omega.$$

Also, if $\alpha \in Pt(\mathcal{S})$ then $\mathcal{S}^\alpha$ is a subspace of $\mathcal{S}$ and $F_{\mathcal{S}^\alpha}$ and f_α represent the same sg-sublocale of $\mathcal{S}$. Moreover every frame map $Sat(\mathcal{S}) \to \Omega$ is f_α for a unique $\alpha \in Pt(\mathcal{S})$.

Let us write $\mathcal{S}' \hookrightarrow \mathcal{S}$ if $\mathcal{S}'$ is a subspace of the formal space $\mathcal{S}$. It is an easy exercise to prove the following.

Proposition: 8 *Let $\mathcal{S}$ be a formal space with cover operator $\mathcal{A}$.*

1. *$\mathcal{S}_U \hookrightarrow \mathcal{S}_V \iff \mathcal{A}V \subseteq \mathcal{A}U$ for $U, V \in Pow(S)$,*
2. *$Pt(\mathcal{S}') = \{\alpha \in Pt(\mathcal{S}) \mid \mathcal{S}^\alpha \hookrightarrow \mathcal{S}'\}$, for $\mathcal{S}' \hookrightarrow \mathcal{S}$,*
3. *$\mathcal{S}^\alpha \hookrightarrow \mathcal{S}^\beta \iff \alpha \in Pt(\mathcal{S}^\beta)$ for $\alpha, \beta \in Pt(\mathcal{S})$,*
4. *$\mathcal{S}^\alpha \hookrightarrow \mathcal{S}_U \iff \alpha \subseteq \neg U$ for $\alpha \in Pt(\mathcal{S})$ and $U \in Pow(S)$, where $\neg U \equiv \{x \in S : x \notin U\}$,*
5. *$\alpha \in Pt(\mathcal{S}^\beta) \Rightarrow \beta \subseteq \alpha \subseteq \neg\neg\beta$ for $\beta \in Pt(\mathcal{S})$,*

Definition: 9 *A formal space $\mathcal{S}$ is a T_1* formal space *if, for every point α of the space, the subspace $\mathcal{S}^\alpha$ is closed.*

We end this section by showing that every regular formal space is T_1. Recall that a formal space $\mathcal{S} = (S, \lhd)$ is defined to be *regular* if $s \lhd wc(s)$ for all $s \in S$. Here $wc(s) = \{t \in S \mid S \lhd t^* \cup \{s\}\}$, where $t^* = \{r \in S \mid (t \downarrow r) \lhd \emptyset\}$.

Proposition: 10 *Every regular formal space is T_1.*

Proof: Let $\mathcal{S} = (S, \lhd)$ be a regular formal space and let $\alpha \in Pt(\mathcal{S})$. We must show that $\mathcal{S}^\alpha = \mathcal{S}_W$ for some $W \in Pow(S)$. Let $W = \bigcup_{t \in \alpha} t^*$. We show that, for $s \in S, U \in Pow(S)$,

$$s \lhd^\alpha U \iff s \lhd_W U.$$

The implication from right to left always holds. For if $s \lhd W \cup U$ and $s \in \alpha$ then, by **FP3**, there is $r \in \alpha$ such that $r \in W \cup U$. But if $r \in W$ then $r \in t^*$

for some $t \in \alpha$, and, as $r, t \in \alpha$, $(r \downarrow t) \between \alpha$, contradicting $r \in t^*$ (indeed, by **FP3**, for every $a \in \alpha$, $\neg a \lhd \emptyset$). So $r \in U$ and hence $U \between \alpha$, as desired.

For the implication from left to right let $s \lhd^{\alpha} U$; i.e.

$$s \in \alpha \Rightarrow U \between \alpha.$$

We must show that $s \lhd W \cup U$. As the space is regular $s \lhd wc(s)$ so that it suffices to show that

$$t \in wc(s) \Rightarrow t \lhd W \cup U.$$

So let $t \in wc(s)$; i.e $S \lhd\ t^* \cup \{s\}$. Then, by **FP1** and **FP3** there is $r \in \alpha$ such that $r \in t^* \cup \{s\}$. So either $r \in t^*$ or $r = s$. We show that $t \lhd W \cup U$.

If $r \in t^*$ then $t \in r^* \subseteq W$ so that $t \lhd W \cup U$. If $r = s$ then $s \in \alpha$ so that $r_1 \in U$ for some $r_1 \in \alpha$. As $r_1 \lhd wc(r_1)$ there is $r_2 \in \alpha$ such that

$$S \lhd (\{r_1\} \cup r_2^*) \subseteq W \cup U$$

and hence again $t \lhd W \cup U$.

■

Classically, locales/formal spaces that are T_1 but not regular are easy to find: the frame of any Hausdorff non-regular space is such a locale. Constructively, this may be more tricky since it is not even possible to show that every Hausdorff space is sober.

3 Set-presentable T_1 formal spaces

Definition: 11 *A formal space $\mathcal{S} = (S, \lhd)$ is defined to be* set-presented by $C : S \to Pow(Pow(S))$ *if,*

$$a \lhd U \iff (\exists V \in C(a))\ V \subseteq U,$$

and is set-presentable *if there is such a function C.*

We aim to show that, for every set-presentable T_1 formal space $\mathcal{S}$, the class $Pt(\mathcal{S})$ is a set. In fact we will prove a more general result by weakening both the conditions of being set-presentable and being T_1.

Definition: 12 *A formal space* $\mathcal{S} = (S, \lhd)$ *is defined to be* T_1^* *if, for every formal point* α *of* $\mathcal{S}$,

$$(\forall a \in \alpha) \;\; S \lhd (\{a\} \cup \neg\alpha)$$

where $\neg\alpha = \{a \in S \mid a \notin \alpha\}$.

Proposition: 13 *If* $\mathcal{S} = (S, \lhd)$ *is a* T_1 *formal space then it is* T_1^* *and, for every formal point* α *the subspace* $\mathcal{S}^\alpha$ *is the closed subspace* $\mathcal{S}_{\neg\alpha}$.

Proof: Let $\mathcal{S}$ be T_1. So if α is a formal point there is a subset W of S such that for all $c \in S$ and $U \in Pow(S)$

$$(*) \quad [c \in \alpha \Rightarrow U \between \alpha] \iff c \lhd (U \cup W).$$

In particular, if $a \in \alpha$, putting $U = \{a\}$ we get

$$[c \in \alpha \Rightarrow a \in \alpha] \iff c \lhd (\{a\} \cup W),$$

so that $c \lhd (\{a\} \cup W)$. Thus $S \lhd (\{a\} \cup W)$. If $c \in W$ then $c \lhd (\emptyset \cup W)$ so that $c \in \alpha \Rightarrow \emptyset \between \alpha$ and hence $c \notin \alpha$. Thus $W \subseteq \neg\alpha$ and hence $S \lhd (\{a\} \cup \neg\alpha)$.

Finally, we need to show that for every formal point α

$$[c \in \alpha \Rightarrow U \between \alpha] \iff c \lhd (U \cup \neg\alpha).$$

As $W \subseteq \neg\alpha$ the direction from left to right is a consequence of $(*)$. For the reverse direction let $c \lhd (U \cup \neg\alpha)$. Then, by **FP3**, if $c \in \alpha$ then $(U \cup \neg\alpha) \between \alpha$ so that $U \between \alpha$.
■

We have seen that every regular formal space is T_1 and hence T_1^*. In section 5 we will give an example of a T_1^* formal space that is not regular.

Let us call a formal space T_1^{max} if every formal point α is maximal; i.e. if β is also a formal point and $\alpha \subseteq \beta$ then $\alpha = \beta$. Note that, by Proposition 8, for every formal point α of a T_1^{max} formal space, $Pt(\mathcal{S}^\alpha) = \{\alpha\}$.

Proposition: 14 *Every* T_1^* *formal space is* T_1^{max}.

Proof: Let α, β be formal points such that $\alpha \subseteq \beta$. We show that also $\beta \subseteq \alpha$. So let $b \in \beta$. Choose some $a \in \alpha$. By T_1^*, as $b \in \beta$,

$$a \lhd (\{b\} \cup \neg\beta) \subseteq (\{b\} \cup \neg\alpha)$$

so that, as $a \in \alpha$, by **FP3**, $(\{b\} \cup \neg\alpha) \between \alpha$ so that $b \in \alpha$.
■

Remark. We cannot expect to constructively prove the converse to this result as the converse would imply the non-constructive principle **REM** that asserts that $\forall x, y[x = y \vee x \neq y]$. To see this, given x, y we may form the discrete formal space $\mathcal{S} = (S, \lhd)$, where $S = \{x, y\}$ and, for $a \in S$ and $U \in Pow(S)$, $a \lhd U \Rightarrow a \in U$. Clearly $\mathcal{S}$ is T_1^{max} and it is not hard to see that the assumption that $\mathcal{S}$ is T_1^* implies $[x = y \vee x \neq y]$.

Definition: 15 *If $\mathcal{S} = (S, \lhd)$ is a formal space a subset C of $Pow(S)$ is defined to* weakly set-present $\mathcal{S}$ *if, for all $U \in Pow(S)$,*

$$S \lhd U \iff (\exists V \in C)\ V \subseteq U.$$

If there is such a set C then $\mathcal{S}$ is weakly set-presentable*.*

Proposition: 16 *Every set-presentable formal space is weakly set-presentable.*

Proof: Let $C : S \to Pow(Pow(S))$ set-present $\mathcal{S}$ and let $C_0 = \bigcup_{a \in S} C(a)$. Then C_0 is a set and, by Subset Collection, there is a set $D_0 \subseteq \mathbf{mv}(C_0^S)$ such that

$$(\forall R \in \mathbf{mv}(C_0^S))(\exists R' \in D_0)\ R' \subseteq R.$$

Let $D = \{R \in D_0 \mid (\forall (a, V) \in R)\ V \in C(a)\}$. Then D is a set, by the Restricted Separation scheme. For each $R \in D$ let

$$V_R = \{b \in S \mid (\exists (a, V) \in R)\ b \in V\}$$

and let $C_1 = \{V_R \mid R \in D\}$. Then C_1 is a set by Replacement.

Let $U \in Pow(S)$. Note that, for $R \in D$

$$V_R \subseteq U \iff R \subseteq R_U,$$

where $R_U = \{(a, V) \in S \times C_0 \mid V \in C(a) \wedge V \subseteq U\}$. So

$$\begin{aligned} S \lhd U &\iff (\forall a \in S)(\exists V \in C(a))\ V \subseteq U \\ &\iff (\forall a \in S)(\exists V \in C_0)\ [V \in C(a) \wedge V \subseteq U] \\ &\iff R_U \in \mathbf{mv}(C_0^S) \\ &\iff (\exists R \in D)\ R \subseteq R_U \\ &\iff (\exists R \in D)\ V_R \subseteq U \\ &\iff (\exists V \in C_1)\ V \subseteq U \end{aligned}$$

Thus, C_1 weakly set-presents $\mathcal{S}$.

■

Proposition: 17 *Let $\mathcal{S} = (S, \lhd)$ be a formal space. If α is an inhabited subset of S such that*

1. *for all $U \in Pow(S)$, $[S \lhd (U \cup \neg\alpha) \;\Rightarrow\; U \between \alpha]$,*

2. *for all $a \in \alpha$, $S \lhd (\{a\} \cup \neg\alpha)$,*

then α is a formal point.

Proof: We have **FP1** by hypothesis. For **FP2** let $a, b \in \alpha$. Then, by 2, $S \lhd (\{a\} \cup \neg\alpha)$ and $S \lhd (\{b\} \cup \neg\alpha)$ so that $S \lhd ((a \downarrow b) \cup \neg\alpha)$ and hence, by 1, $(a \downarrow b) \between \alpha$. For **FP3** let $a \in \alpha$ and $a \lhd U$. Then, by 2, $S \lhd (\{a\} \cup \neg\alpha)$ and hence $S \lhd (U \cup \neg\alpha)$ so that, by 1, $U \between \alpha$.
■

Lemma: 18 *Let $\mathcal{S}$ be a T_1^* formal space and let $C \subseteq Pow(S)$ weakly set-present it. Then an inhabited subset α of S is a formal point iff*

FP′1. $(\forall V \in C)\ [V \between \alpha]$,

FP′2. $(\forall a \in \alpha)(\exists V \in C)\ [V \subseteq \{a\} \cup \neg\alpha]$.

Proof: Let α be a formal point. By **FP1** we may choose $a \in \alpha$. Then

$$\begin{aligned} V \in C \;&\Rightarrow S \lhd V \\ &\Rightarrow a \lhd V \\ &\Rightarrow V \between \alpha, \text{ by } \mathbf{FP3}, \text{ as } a \in \alpha. \end{aligned}$$

Thus **FP′1**. As $\mathcal{S}$ is T_1^*, $(\forall a \in \alpha)\ [S \lhd (\{a\} \cup \neg\alpha)]$, and so **FP′2**.

Conversely, assume that α is an inhabited subset of S such that **FP′1** and **FP′2** hold. It suffices to show that 1 and 2 of Proposition 17 hold. 2 is an immediate consequence of **FP′2**. For 1, let $S \lhd (U \cup \neg\alpha)$. Then, for some $V \in C$, $V \subseteq (U \cup \neg\alpha)$. By **FP1′** there is $b \in V \cap \alpha$. It follows that $b \in U$ so that $U \between \alpha$.
■

Theorem: 19 *If $\mathcal{S}$ is a weakly set-presented T_1^* formal space then $Pt(\mathcal{S})$ is a set.*

Proof: Let C weakly set-present $\mathcal{S}$. By Subset Collection there is a subset D_0 of $\mathbf{mv}(S^C)$ such that

$$(\forall R \in \mathbf{mv}(S^C))(\exists R' \in D_0)\ R' \subseteq R.$$

For each $R \in D_0$ let

$$\alpha_R = \{b \in S \mid (\exists V \in C)\ (V, b) \in R\}.$$

Let $D = \{R \in D_0 \mid \alpha_R \in Pt(\mathcal{S})\}$. Note that, by Lemma 18, for each $R \in D_0$, $\alpha_R \in Pt(\mathcal{S})$ iff the conjunction of the following three conditions hold.

FP″0 $(\exists a \in S)(\exists V \in C)\ (V, a) \in R$,

FP″1 $(\forall V \in C)(\exists a \in V)(\exists V' \in C)\ (V', a) \in R$,

FP″2 $(\forall a \in S)(\forall V \in C)[(V, a) \in R \Rightarrow Q(a, R)]$,

where

$$Q(a, R) \iff (\exists V' \in C)(\forall a' \in V')[a' = a \vee (\forall V'' \in C)\ (V'', a') \notin R].$$

As these conditions on R can be given by restricted formulae we may apply the Restricted Separation scheme to get that D is a set. Hence, by the Replacement scheme,

$$P = \{\alpha_R \mid R \in D\}$$

is a set. To prove the theorem we show that $Pt(\mathcal{S}) = P$.

If $\alpha \in P$ then $\alpha = \alpha_R$ for some $R \in D$ so that $\alpha \in Pt(\mathcal{S})$. Thus $P \subseteq Pt(\mathcal{S})$. To show that $Pt(\mathcal{S}) \subseteq P$ let α be a formal point. Then, by **FP′1**,

$$R_\alpha = \{(V, a) \in C \times \alpha \mid a \in V\} \in \mathbf{mv}(S^C)\},$$

and hence there is $R \in D_0$ such that $R \subseteq R_\alpha$. We show that $\alpha = \alpha_R$. It then follows that $\alpha_R \in Pt(\mathcal{S})$ and hence $R \in D$ so that $\alpha \in P$.

For $\alpha \subseteq \alpha_R$, let $a \in \alpha$ so that, by **FP′2**, there is $V \in C$ such that $V \subseteq \{a\} \cup \neg\alpha$. Since $R \in \mathbf{mv}(S^C)$ there exists $b \in S$ with $(V, b) \in R$. As $R \subseteq R_\alpha$,

$$b \in (V \cap \alpha)\ \subseteq\ (\{a\} \cup \neg\alpha) \cap \alpha\ =\ \{a\}.$$

This gives $a = b$ and $(V, a) \in R$ so that $a \in \alpha_R$.

For $\alpha_R \subseteq \alpha$, let $a \in \alpha_R$; i.e. $(V, a) \in R$ for some $V \in C$. As $R \subseteq R_\alpha$, $a \in \alpha$.

■

Corollary: 20 *For every set-presentable T_1 formal space $\mathcal{S}$, $Pt(\mathcal{S})$ is a set.*

Remark. The above theorem and corollary are also provable in constructive type theory [13], by exploiting the type-theoretic principle of choice. See e.g. [14] for the formalization of the formal topology notions in the type-theoretic setting. A formal space is defined to be *weakly set-presented* in that context if a family of subsets $C(i) \subseteq S$, for i in a set I, exists such that

$$S \lhd U \iff (\exists i \in I)\ C(i) \subseteq U.$$

If I_{Id} is I endowed with the equality given by the identity type, one proves that $\mathcal{P}t(S)$ can be identified with the subset D of $S^{I_{Id}}$ given by

$$D \equiv \{f \in S^{I_{Id}} : \alpha_f \in \mathcal{P}t(S)\},$$

where $\alpha_f \equiv \{a \in S : (\exists i \in I_{Id}) f(i) \lhd a\}$. To a formal point α one associates the mapping $f : I_{Id} \to S$ obtained by **FP′1** using type-theoretic choice (here **FP′1** reads as: $(\forall i \in I)(\exists a \in C(i))\ a \in \alpha$; I_{Id} is a projective cover for I).

Theorem 19 could be reformulated as follows: if a certain class A can be represented as the class of points of a weakly set-presented T_1^* formal space, then A is a set. In [4] the class of Dedekind reals is proved to form a set in **CZF**. This result is particularly meaningful in this context as the impredicative Powerset axiom and full Separation scheme are missing; furthermore, the principles of dependent and countable choice are not part of the basic formulation of **CZF**, so that Dedekind and Cauchy reals do not coincide (the latter are easily seen to form a set by Fullness). As the class of Dedekind real numbers can be represented as the class of points of a formal space satisfying the hypotheses of Theorem 19, we get a new proof of the smallness of this class.

Other applications of this kind of result can be found in [8]. They yield the smallness of classes of continuous functions, and therefore allow, for instance, for the construction of Tychonoff embedding and Stone-Čech compactification.

We note that, by Proposition 10, the above theorem generalises previous results asserting that the class of points of a locally compact regular formal space [7], and more generally, of a set-presented and regular formal space, is a set [2].

Let us call a topological space T_1^{max} if it is T_0 and each point has a maximal set of neighbourhoods. This is equivalent to the condition that each point is closed. Considered in point-free terms, in view of the remark after Proposition 14 the former is strictly weaker (at least constructively) than the latter. So the result just proved, considered for (fully) set-presented formal spaces, is less general than the one in [14], asserting that if a set-presented formal space is T_1^{max} then the formal points form a set. However, the proof of this result makes use of the type-theoretic axiom of choice, and its set-theoretical version [2] seems to require an extension of **CZF**. On the other hand, the simple proof of Theorem 19 has, as shown, a choice-free formulation in **CZF**.

4 Separation properties, T_i^*, for ct-spaces

Bishop, in [6], introduced the notion of a neighborhood system as a version of the classical notion of a topological space adapted to his approach to constructive mathematics by having an explicit indexed family of basic opens. The paper [2] generalised that notion to the notion of a ct-space by allowing the points of the space to be a class and so allowing the opens to be classes while keeping the family of basic opens to be indexed by a set. The advantage of this notion is that, although the points of a formal space do not form a set in general and so then do not form a neighborhood system they do form a ct-space.

Separation properties $T_i^{\#}, i = 0, 1, 2, 3$ for a ct-space, were discussed in [3]. Here we will formulate new separation properties $T_i^*, i = 0, 1, 2, 3$ for a ct-space and we will relate them to the $T_i^{\#}$ properties. In the next section we will also relate those separation properties to corresponding properties for formal spaces.

Definition: 21 *Let X be a class (of* points*), S a set, and $\Vdash \subseteq X \times S$ be a (class-) relation. Define $B_a = \{x \in X : x \Vdash a\}$ for each $a \in S$, $B_U = \bigcup_{a \in U} B_a$ for each $U \subseteq S$, and $\alpha_x = \{a \in S : x \Vdash a\}$ for each $x \in X$. The triple $\mathcal{X} = (X, S, \Vdash)$ is a* constructive topological space (ct-space) *if the following conditions are satisfied:*

CS1 *$X = B_S$,*

CS2 *If $a_1, a_2 \in S$ then $B_{a_1} \cap B_{a_2} \subseteq B_U$, where $U = \{a \in S : B_a \subseteq B_{a_1} \cap B_{a_2}\}$,*

CS3 *For $x \in X$ the classes α_x and $\{y \in X : \alpha_y = \alpha_x\}$ are sets.*

See [2, 3] for more on this notion. The class of points, $Pt(\mathcal{S})$ of a formal space $\mathcal{S}$ form the ct-space $\mathsf{Pt}(\mathcal{S}) = (Pt(\mathcal{S}), S, \Vdash)$, where

$$\alpha \Vdash s \iff s \in \alpha$$

for each point α of $\mathcal{S}$ and each $s \in S$. In the following definitions $\mathcal{X} = (X, S, \Vdash)$ is a ct-space.

Definition: 22 *For sets $\alpha, \beta \subseteq S$ we define $\alpha \parallel_i \beta$ for $i = 0, 1, 2$ as follows.*

1. $\alpha \parallel_0 \beta \iff \alpha \between \neg\beta$ *or* $\beta \between \neg\alpha$.
2. $\alpha \parallel_1 \beta \iff \alpha \between \neg\beta$.
3. $\alpha \parallel_2 \beta \iff (\exists a \in \alpha)(\exists b \in \beta)[B_a \cap B_b = \emptyset]$.

Definition: 23 *For $i = 0, 1, 2$,*

$$\mathcal{X} \textit{ is } S_i^{\#} \iff (\forall x, y \in X)[\alpha_x \subseteq (\alpha_y \cup \{a \in S \mid \alpha_y \parallel_i \alpha_x\})],$$

and

$$\mathcal{X} \textit{ is } S_3^{\#} \iff \forall x \in X \forall a \in \alpha_x \exists b \in \alpha_x [X \subseteq B_b^* \cup B_a].$$

Here, for any class $Z \subseteq X$, Z^* is the largest open class disjoint from Z. Also, for $i = 0, 1, 2, 3$,

$$\mathcal{X} \textit{ is } T_i^{\#} \iff \mathcal{X} \textit{ is } T_0 \textit{ and } S_i^{\#},$$

where

$$\mathcal{X} \textit{ is } T_0 \iff (\forall x, y \in X)[\alpha_x = \alpha_y \to x = y].$$

Definition: 24 *An* ideal point *of $\mathcal{X}$ is a subset α of S such that:*

IP1 $S \between \alpha$

IP2 $(\forall a, b \in \alpha)\ \{c \in S \mid B_c \subseteq B_a \cap B_b\} \between \alpha$,

IP3 $a \in \alpha$ *and* $B_a \subseteq B_U$ *imply* $U \between \alpha$.

Note that each α_x is always an ideal point of $\mathcal{X}$ and a ct-space is defined to be *sober* if every ideal point is α_x for some $x \in X$. The class $sob(\mathcal{X})$ of all ideal points of a ct-space $\mathcal{X}$ itself forms a sober ct-space $\mathsf{sob}(\mathcal{X}) = (sob(X), S, \Vdash)$ where

$$\alpha \Vdash s \iff s \in \alpha.$$

Definition: 25 *For $i = 0, 1, 2$ we define $\mathcal{X}$ to be S_i^* if, for every ideal point α of $\mathcal{X}$,*

$$(\forall y \in X)[\alpha \subseteq (\alpha_y \cup \{a \in S \mid \alpha_y \parallel_i \alpha\})]$$

and define $\mathcal{X}$ to be S_3^* if, for each ideal point α of $\mathcal{X}$,

$$(\forall a \in \alpha)(\exists b \in \alpha)[X \subseteq B_b^* \cup B_a].$$

For $i = 0, 1, 2, 3$, the ct*-space is T_i^* if it is also T_0.*

Note that if $\mathcal{X}$ is sober then $\mathcal{X}$ is $S_i^{\#}$ iff $\mathcal{X}$ is S_i^*. Clearly if $\mathcal{X}$ is T_i^* then $\mathcal{X}$ is $T_i^{\#}$, with the converse also holding for sober X. In fact we have the following characterisation.

Proposition: 26 *A* ct*-space $\mathcal{X}$ is S_i^* iff* $\mathsf{sob}(X)$ *is $S_i^{\#}$ ($T_i^{\#}$),*

Proof. That $\mathcal{X}$ is S_i^* implies $\mathsf{sob}(\mathcal{X})$ is $S_i^{\#}$, for $i = 1, 2, 3$, can easily be proved directly, or follows by composing the proof of 1 and 2 of theorem 28 below (the requirement that $\mathcal{X}$ be standard plays no role here). The converse is trivial, we consider only the case $i = 3$: if α is an ideal point of $\mathcal{X}$, and $a \in \alpha$, the hypothesis gives $b \in \alpha$ such that $sob(X) \subseteq \bar{B}_b^* \cup \bar{B}_a$, with $\bar{B}_a \equiv \{\alpha \in sob(X) \mid a \in \alpha\}$. Then, given $x \in X$, $\alpha_x \in \bar{B}_b^* \cup \bar{B}_a$, that means that either $\alpha_x \in \bar{B}_a$, i.e. $x \in B_a$, or that there is $c \in S$ such that $\bar{B}_b \cap \bar{B}_c = \emptyset$ and $\alpha_x \in \bar{B}_c$. But $\bar{B}_b \cap \bar{B}_c = \emptyset$ implies $B_b \cap B_c = \emptyset$, and $\alpha_x \in \bar{B}_c$ gives $x \in B_c$. Thus, $X \subseteq B_b^* \cup B_a$.
■

We have the following implications.

$$\begin{array}{ccccccc}
T_3^* & \Longrightarrow & T_2^* & \Longrightarrow & T_1^* & \Longrightarrow & T_0^* \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
T_3^{\#} & \Longrightarrow & T_2^{\#} & \Longrightarrow & T_1^{\#} & \Longrightarrow & T_0^{\#}
\end{array}$$

It is an easy exercise to check that S_3^* is in fact equivalent to $S_3^{\#}$, so that T_3^* and $T_3^{\#}$ define the same property.

As the following example shows, the T_1^* property for topological spaces is classically strictly stronger than the classical T_1 property, so that it should probably be re-baptized $T_{1\frac{1}{2}}^*$: consider the "cofinite" topology on the natural numbers (example 8 of [3]); this is the (small) ct-space $(\mathbb{N}, S, \Vdash)$, with $S \equiv \mathbb{N} \times \mathbb{N}$ and $n \Vdash (a, b) \iff (n = a) \vee (b \leq n)$. This space is classically T_1 (and constructively $T_1^{\#}$). Since $\alpha = \mathbb{N} \times \mathbb{N}$ is an ideal point, condition S_1^* is not satisfied for this space. (On the other hand, T_1^* is strictly weaker than T_2, at least classically: any Hausdorff space is classically sober, so T_2^* coincides with $T_2^{\#}$, that in turn coincides with T_2. Moreover, there are sober T_1 spaces which are not T_2).

5 Separation properties, T_i^*, for formal spaces

It is well known that, in classical mathematics and even in topos mathematics, topological spaces and locales are connected via an adjunction that restricts to an equivalence between the full subcategories of sober topological spaces and spatial locales. A constructive predicative version of this result appears in [2]. There a ct-space $\mathcal{X}$ is defined to be *standard* if the class $\mathcal{A}U = \{s \in S \mid B_s \subseteq B_U\}$ is a set for all $U \in Pow(S)$. When this is the case $\mathcal{A}$ is a cover operation on S and so gives rise to a formal space $\mathsf{ft}(\mathcal{X}) = (S, \lhd)$ where, for $s \in S, U \in Pow(S)$,

$$s \lhd U \iff s \in \mathcal{A}U.$$

Let $\mathcal{S} = (S, \lhd)$ be a formal space. We have already defined the T_1^* separation property for $\mathcal{S}$. We repeat it here along with definitions of the separation properties T_2^*, T_3^* for $\mathcal{S}$. We define the T_i^* separation properties for $\mathcal{S}$ as follows.

Definition: 27

1. *$\mathcal{S}$ is T_1^* if, for every formal point α*

$$(\forall a \in \alpha)[S \lhd \{a\} \cup \neg\alpha].$$

2. *$\mathcal{S}$ is T_2^* if, for every formal point α,*

$$(\forall a \in \alpha)[S \lhd \{a\} \cup !\alpha]$$

where $!\alpha = \{b \in S \mid (\exists a \in \alpha)\ B_a \cap B_b = \emptyset\}$.

3. *$\mathcal{S}$ is T_3^* if it is regular.*

For a formal space $\mathcal{S}$ we have the following implications

$$T_3^* \Rightarrow T_2^* \Rightarrow T_1^*$$

We also have the following results.

Theorem: 28 *For $i = 1, 2, 3$,*

1. *If $\mathcal{X}$ is a standard* ct*-space then,*

$$\mathcal{X} \text{ is } S_i^* \iff \mathsf{ft}(\mathcal{X}) \text{ is } T_i^*.$$

2. *If $\mathcal{S}$ is a formal space then*

$$\mathcal{S} \text{ is } T_i^* \Rightarrow \mathsf{Pt}(\mathcal{S}) \text{ is } S_i^*.$$

Proof. 1. One has that $\mathcal{X}$ is S_1^* if and only if given any ideal point α of $\mathcal{X}$ and any $a \in \alpha$, and $y \in X$, one has $y \in B_a$ or $\exists b \in \alpha_y$ with $b \notin \alpha$, namely $B_S \subseteq B_a \cup B_{\neg\alpha}$. This precisely means $\mathsf{ft}(\mathcal{X})$ is T_1^*. Similarly one proves that $\mathcal{X}$ is S_2^* iff $\mathsf{ft}(\mathcal{X})$ is T_2^*. Finally, it is easy to check that $\mathcal{X}$ is S_3^* if and only if, for all $a \in S$, $B_a \subseteq \bigcup_{b \in U} B_b$, where $U = \{c \in S : X \subseteq B_c^* \cup B_a\}$; but this is exactly the same as saying that $a \lhd wc(a)$ in $\mathsf{ft}(\mathcal{X})$, so that $\mathcal{X}$ is S_3^* iff $\mathsf{ft}(\mathcal{X})$ is T_3^*.

2. Assume that $\mathcal{S}$ is T_1^*, let α, β be formal points of $\mathcal{S}$, and let $a \in \alpha$. As $S \lhd \{a\} \cup \neg\alpha$ and β is formal point, one has that $a \in \beta$ or $\exists b \in \beta$ such that $b \notin \alpha$, i.e. $\mathsf{Pt}(\mathcal{S})$ is S_1^*. An analogous argument shows that $\mathsf{Pt}(\mathcal{S})$ is S_2^* whenever $\mathcal{S}$ is T_2. Finally, assume that $\mathcal{S}$ is regular. It has been proved in [3, Th. 21] that $\mathsf{Pt}(\mathcal{S})$ is $S_3^{\#}$. But $\mathsf{Pt}(\mathcal{S})$ is sober, so it is also S_3^*.
■

By this theorem, Theorem 19, and [2, Proposition 19] we have that if $\mathcal{X}$ is small (has a set of points) and S_1^* then $\mathsf{sob}(\mathcal{X})$ is small too: indeed, if $\mathcal{X}$ is small, by [2, Proposition 19], $\mathsf{ft}(\mathcal{X})$ is set-presentable. Thus, for instance, the soberification of the set of rational numbers with their standard topology (that is not constructively a sober space), will have a set of ideal points.

We now use Theorem 28 to show that there is a formal space which is T_2^* but not regular: consider the topological space of positive integers with the relatively prime integer topology [18, Example 60]. This has the set $X = \mathbb{Z}^+$ of points and the family $\{U_a(b)\}_{(a,b)\in S}$ of basic open sets where

$$S = \{(a, b) \in X \times X \mid a, b \text{ are relatively prime}\},$$

with $U_a(b) \equiv \{b + na \in \mathbb{Z}^+ \mid n \in \mathbb{Z}\}$. To see that this family gives a base, assume $q \in (U_a(b) \cap U_c(d))$. Let $[a, c]$ denote the least common multiple of a and c. Then q and $[a, c]$ are relatively prime: let the greatest common divisor of q and $[a, c]$ be $k > 1$, and let $p > 1$ be a prime that divides k. Then p divides q and a or q and c. From this it easily follows that a, b or c, d are not relatively prime, so that it must be $k = 1$. Now it is immediate to check that $U_{[a,c]}(q) \subseteq (U_a(b) \cap U_c(d))$ (in fact the equality holds). Thus we have a standard ct-space $\mathcal{X} = (X, S, \Vdash)$ where $x \Vdash (a, b) \iff (\exists n \in \mathbb{Z})\ x = b+na$ and we now show that this space is S_2^*. Let α be an ideal point, $y \in \mathbb{Z}^+$, and $(a, b) \in \alpha$. Then, either $y \in U_a(b)$ or not. In the latter case, one observes that $U_a(b) \subseteq \bigcup_{x \in U_a(b)} U_{p_x}(x)$, with p_x prime and $p_x > x + y$. Since α is an ideal point, there is $x \in U_a(b)$ such that $(p_x, x) \in \alpha$. Then, $U_{p_x}(x) \cap U_{p_x}(y) = \emptyset$, as, having assumed $p_x > x + y$, there is no n with $(y - x) = np_x$, i.e. $x \not\equiv y\ mod\ (p_x)$. Thus $y \in U_a(b)$ or there is a neighbourhood of y disjoint from a neighbourhood of α, as wished.

By the above proposition, the formal space $\mathsf{ft}(\mathcal{X})$ is T_2^*. But $\mathcal{X}$ is not regular, so that $\mathsf{ft}(\mathcal{X})$ cannot be regular either.

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