A Tableau Calculus for Multimodal Logics and Some (Un)Decidability Results

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Abstract. In this paper we present a *prefixed analytic tableau calculus* for a class of *normal multimodal logics* and we present some results about decidability and undecidability of this class. The class is characterized by axioms of the form $[t_1] \ldots [t_n]\varphi \supset [s_1] \ldots [s_m]\varphi$, called *inclusion axioms*, where the t_i's and s_j's are constants. This class of logics, called *grammar logics*, was introduced for the first time by Fariñas del Cerro and Penttonen to simulate the behaviour of grammars in modal logics, and includes some well-known modal systems. The prefixed tableau method is used to prove the *undecidability* of modal systems based on *unrestricted*, *context sensitive*, and *context free* grammars. Moreover, we show that the class of modal logics, based on *right-regular* grammars, are *decidable* by means of the *filtration methods*, by defining an extension of the Fischer-Ladner closure.

Keywords: Multimodal logics, Prefixed Tableaux methods, Decidability, Formal Grammars.

1 Introduction and Motivations

Modal logics are widely used in artificial intelligence for representing *knowledge* and *beliefs* [19] together with other attitudes in *agent systems* like, for instance, *goals*, *intentions* and *obligations* [33]. Moreover, modal logics are well suited for representing *dynamic* aspects in *agent systems* and, in particular, to formalize reasoning about *actions* and *time*. Last but not least, modal logics are shown useful to extend logic programming languages with new features [31,13,4].

In this paper we focus on a class of *normal multimodal logics*, called *grammar logics*, which are characterized by a set of logical axioms of the form:

$$[t_1] \ldots [t_n]\varphi \supset [s_1] \ldots [s_m]\varphi \quad (n > 0; m \geq 0) \tag{1}$$

that we call *inclusion axiom*, where the t_i's and s_j's are modalities. This class includes some well-known modal systems such as K, $K4$, $S4$ and their multimodal versions. Differently from other logics, such as those studied in [19], these systems can be *non-homogeneous* (i.e., every modal operator is not restricted to

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belong to the same system) and can contain some *interaction axioms* (i.e., every modal operator is not restricted to be independent from the others).

This class of logics has been introduced by Fariñas del Cerro and Penttonen in [11], where a method to define multimodal logics from *formal grammars* is presented, in such a way to simulate the behaviour of grammars. Given a formal grammar, a modality is associated to each terminal and nonterminal symbol, while, for each production rule of the form $t_1 \cdots t_n \to s_1 \cdots s_m$, an associated inclusion axiom $[t_1] \ldots [t_n]\varphi \supset [s_1] \ldots [s_m]\varphi$ is defined. In [11], it is shown that testing whether a word is generated by the formal grammar is equivalent to proving a theorem in the logic. Moreover, relying on this relation with formal grammars, an *undecidability* result for this class of multimodal logics is proved. However, in [11], *neither* a *proof method* is presented to deal with the class of grammar logics *nor (un)decidability* of restricted subclasses is studied.

In this paper, we develop an *analytic tableau calculus* for the class of *grammar logics*. The calculus is parametric with respect to each modal system in this class. In particular, it deals with *non-homogeneous* multimodal systems with arbitrary *interaction axioms* of the form (1).

The calculus is an extension of the one proposed in [26], which is closely related to the systems of prefixed tableaux presented in [14]. As a difference with [14], worlds are not represented by prefixes (which describe paths in the model from the initial world), but they are given an atomic name and the accessibility relationships among them are explicitly represented in a graph. The method is based on the idea of using the characterizing axioms of the logic as "*rewrite rules*" which create new paths among worlds in the counter-model construction.

Making use of the tableau calculus we prove the *undecidability* of the modal systems based on context sensitive and context-free grammars. Moreover, we show that the class of modal logics based on right regular grammars is *decidable*. We use the well-known *filtration methods* by defining an extension of the Fischer-Ladner closure for modal logics. This result is close to those that have been established for *propositional dynamic logic* [12,20].

2 Grammar Modal Logics

Let us define a propositional multimodal language $\mathcal{L}$, containing the logical connectives $\wedge$, $\vee$, $\supset$, and $\neg$, a set of modal operators of the form $[t]$ and $\langle t \rangle$, where t belongs to a nonempty countable set MOD (the *alphabet of modalities*) and a nonempty countable set VAR of *propositional variables*. MOD and VAR are disjoint. The set of formulae of the languages are constructed as usual by means of the propositional variables, the connectives, and the modal operators.

We only consider *normal* modal logics, that is those ones whose axiomatization at least contains the axiom schemas for the classical propositional calculus, *modus ponens* and *necessitation* rules, and the axiom schema $K(t)$: $[t](\varphi \supset \psi) \supset ([t]\varphi \supset [t]\psi)$ for all modal operators. In particular, we focus on normal multimodal logics that are characterized by a set of axiom schemas of the form (1). We call these logics *grammar logics*. Let $\mathcal{A}$ be a set of inclusion axioms, we denote by $\mathcal{I}_{\mathcal{L}}^{\mathcal{A}}$ the grammar logic determined by the set $\mathcal{A}$ with $\mathcal{L}$ as underlying language, while we use $\mathcal{S}_{\mathcal{L}}^{\mathcal{A}}$ to denote its characterizing axiom systems (containing the axioms for normal modalities plus $\mathcal{A}$). As we will see, the inclusion axioms determine *inclusion properties* on the accessibility relations.

Some examples of grammar logics are the well-known modal systems K, T, $K4$, $S4$ [23], their multimodal versions K_n, T_n, $K4_n$, $S4_n$ [19], extensions of K_n and $S4_n$ with interaction axioms or with agent "any fool" in [16,10,3].

Example 1. (The friends puzzle) Peter is a friend of John, so if Peter knows that John knows something, then John knows that Peter knows that thing. That is, A_1: $[p][j]\varphi \supset [j][p]\varphi$, where $[p]$ and $[j]$ are modal operators of type $S4$ (i.e., A_2: $[p]\varphi \supset \varphi$, A_3: $[p]\varphi \supset [p][p]\varphi$, A_4: $[j]\varphi \supset \varphi$, and A_5: $[j]\varphi \supset [j][j]\varphi$) and they are used to denote what is known by Peter and John, respectively. Peter is married, so if Peter's wife knows something, then Peter knows the same thing, that is, A_6: $[wp]\varphi \supset [p]\varphi$ holds, where $[wp]$ is a modality of type $S4$ representing the knowledge of Peter's wife. John and Peter have an appointment, let us consider the following situation:

(1)	$[p]time$	(3)	$[wp]([p]time \supset [j]time)$
(2)	$[p][j]place$	(4)	$[p][j](place \wedge time \supset place)$

That is, (1) Peter knows the time of their appointment; (2) Peter also knows that John knows the place of their appointment. Moreover, (3) Peter's wife knows that if Peter knows the time of their appointment, then John knows that too; (4) Peter knows that if John knows the place and the time of their appointment, then John knows that he has an appointment. From this situation we will be able to prove $[j][p]appointment \wedge [p][j]appointment$, that is, each of the two friends knows that the other one knows that he has an appointment.

In order to define the meaning of a formula, we introduce the notion of *Kripke interpretation*. Formally, a Kripke interpretation M is a triple $(W, \{\mathcal{R}_t \mid t \in \text{MOD}\}, V)$, consisting of a non-empty set W of "*possible worlds*" and a set of *binary relations* $\mathcal{R}_t$ (one for each $t \in \text{MOD}$) on W, and a *valuation function* V, that is a mapping from $W \times \text{VAR}$ to the set $\{\mathbf{T}, \mathbf{F}\}$. We say that $\mathcal{R}_t$ is the *accessibility relation* of the modality $[t]$ and w' is *accessible* from w *by means of* $\mathcal{R}_t$ if $(w, w') \in \mathcal{R}_t$ (or $w\mathcal{R}_t w'$).

The meaning of a formula is given by means of a *satisfiability relation*, denoted by $\models$. Let $M = \langle W, \{\mathcal{R}_t \mid t \in \text{MOD}\}, V\rangle$ be a Kripke interpretation, w a world in W and φ a formula, then, we say that φ is *satisfiable in the Kripke interpretation* M *at* w, denoted by $M, w \models \varphi$, if the following conditions hold:

- $M, w \models \varphi$ and $\varphi \in \text{VAR}$ iff $V(w, \varphi) = \mathbf{T}$;
- $M, w \models \neg\varphi$ iff $M, w \not\models \varphi$;
- $M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$;
- $M, w \models \varphi \vee \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$;
- $M, w \models \varphi \supset \psi$ iff $M, w \not\models \varphi$ or $M, w \models \psi$;
- $M, w \models [t]\varphi$ iff for all $w' \in W$ such that $(w, w') \in \mathcal{R}_t$, $M, w' \models \varphi$;
- $M, w \models \langle t\rangle\varphi$ iff there exists a $w' \in W$ such that $(w, w') \in \mathcal{R}_t$ and $M, w' \models \varphi$.

Let $\mathcal{M}_{\mathcal{L}}$ be the set of all Kripke interpretations, as defined above. For each grammar logic $\mathcal{I}_{\mathcal{L}}^{\mathcal{A}}$ we introduce a suitable notion of Kripke $\mathcal{A}$-interpretation, by adding some restriction on the accessibility relations. More precisely, let $M = (W, \{\mathcal{R}_t \mid t \in \text{MOD}\}, V)$ be a Kripke interpretation and let $\mathcal{A}$ be a set of inclusion axioms, we say M is a *Kripke $\mathcal{A}$-interpretation* if and only if for each axiom schema $[t_1][t_2]\ldots[t_n]\varphi \supset [s_1][s_2]\ldots[s_m]\varphi \in \mathcal{A}$, the following *inclusion property* on the accessibility relation holds:

$$\mathcal{R}_{t_1} \circ \mathcal{R}_{t_2} \circ \ldots \circ \mathcal{R}_{t_n} \supseteq \mathcal{R}_{s_1} \circ \mathcal{R}_{s_2} \circ \ldots \circ \mathcal{R}_{s_m} \tag{2}$$

where "$\circ$" means the relation composition $\mathcal{R}_t \circ \mathcal{R}_{t'} = \{(w, w'') \in W \times W \mid \exists w' \in W$ such that $(w, w') \in \mathcal{R}_t$ and $(w', w'') \in R_{t'}\}$[1].

The set of all Kripke $\mathcal{A}$-interpretations is denoted by $\mathcal{M}_{\mathcal{L}}^{\mathcal{A}}$ and it is a subset of $\mathcal{M}_{\mathcal{L}}$. Given a Kripke $\mathcal{A}$-intepretation $M = \langle W, \{\mathcal{R}_t \mid t \in \text{MOD}\}, V\rangle$ in $\mathcal{M}_{\mathcal{L}}^{\mathcal{A}}$, we say that a formula φ of $\mathcal{I}_{\mathcal{L}}^{\mathcal{A}}$ is *satisfiable in M* if $M, w \models_{\mathcal{A}} \varphi$ for some world $w \in W$. We say that φ is *valid in M* if $\neg\varphi$ is not satisfiable in M. Moreover, a formula φ is *satisfiable* if φ is $\mathcal{A}$-satisfiable in some Kripke $\mathcal{A}$-interpretation in $\mathcal{M}_{\mathcal{L}}^{\mathcal{A}}$ and *$\mathcal{A}$-valid* f it is valid in all Kripke $\mathcal{A}$-interpretations in $\mathcal{M}_{\mathcal{L}}^{\mathcal{A}}$ (in this case, we write $\models_{\mathcal{A}} \varphi$).

The axiom system $\mathcal{S}_{\mathcal{L}}^{\mathcal{A}}$ is *sound* and *complete* axiomatization with respect to $\mathcal{M}_{\mathcal{L}}^{\mathcal{A}}$ [2] (see also [11] for a subclass).

Due to the similarity between inclusion axioms and production rules in a grammar, we can associate to a given grammar a corresponding grammar logic.

A *grammar* is a quadruple $G = (V, T, P, S)$, where V and T are disjoint finite sets of *variables* and *terminals*, respectively. P is a finite set of *productions*, each production is of the form $\alpha \to \beta$, where the form of α and β depends on the *type* of grammar as follows[2]:

Production grammar form for different classes of languages

type-0	*type-1*	*type-2*	*type-3*
$\alpha \in (V \cup T)^* V (V \cup T)^*$	$\alpha \in (V \cup T)^* V (V \cup T)^*$	$\alpha \in V$	$\alpha \in V$
$\beta \in (V \cup T)^*$	$\beta \in (V \cup T)^+$	$\beta \in (V \cup T)^*$	$\beta = \sigma A$ or $\beta = \sigma$
	$\lvert\beta\rvert \leq \lvert\alpha\rvert$		$\sigma \in T^*$, $A \in V$

Finally, $S \in V$ is a special variable called the *start symbol* [21]. We say that the production $\alpha \to \beta$ is applied to the string $\gamma\alpha\delta$ to *directly derive* $\alpha\beta\delta$ in grammar G (written $\gamma\alpha\delta \Rightarrow_G \gamma\beta\delta$). The relation *derives*, $\Rightarrow_G^*$, is the reflexive, transitive closure of $\Rightarrow_G$. The *language generated* by a grammar G, denoted by $L(G)$ is the set of *words* $\{w \in T^* \mid S \Rightarrow_G^* w\}$.

Given a formal grammar $G = (V, T, P, S)$, we can associate to it a *grammar logic* (*based on* G) containing the modalities $\text{MOD} = V \cup T$ and characterized

[1] If $m = 0$ then we assume $\mathcal{R}_{s_1} \circ \mathcal{R}_{s_2} \circ \ldots \circ \mathcal{R}_{s_m} = I$, where I is the identity relation on W.

[2] We denote by "L^*" the Kleene closure of the language L (i.e. it denotes zero or more concatenation of L) and by "+" the positive closure of L (i.e. it denotes one or more concatenation of L) [21].

by the a axiom schema $[t_1]\ldots[t_n]\varphi \supset [s_1]\ldots[s_m]\varphi$, one for each production rule $t_1\cdots t_n \to s_1\cdots s_m \in P$, where the t_i's and s_j's are either in V or in T.

We will call *unrestricted*, *context sensitive*, *context-free*, and *right-regular* modal logic a grammar logic based on a type-0, type-1, type-2, and type-3 grammar, respectively.

3 A Tableau Calculus for Grammar Logics

Before introducing our tableau calculus, we need to define some notions. We define a *signed formula* Z as a formula prefixed by one of the two symbols **T** and **F** (*signs*). For instance, if φ is a formula then, $\mathbf{T}\varphi$ and $\mathbf{F}\varphi$ are signed formulae.

Definition 1. *Let $\mathcal{L}$ be a propositional modal language and let $\mathcal{W}_C$ be a countable non-empty set of* constant world symbols *(or* prefixes*). A* prefixed signed formula*, $w : Z$, is a prefix $w \in \mathcal{W}_C$ followed by a signed formula Z.*

Intuitively, prefixes are used to name worlds, and a formula $w : \mathbf{T}\varphi$ ($w : \mathbf{F}\varphi$) on a branch of a tableau means that the formula φ is *true* (*false*) at the world w in the Kripke interpretation associated with that branch. We assume that $\mathcal{W}_C$ contains always at least the prefix i, that is interpreted as the *initial world.*

Definition 2. *Let $\mathcal{L}$ be a propositional modal language, an* accessibility relation formula *$w\ \rho_t\ w'$, where $t \in \mathrm{MOD}$, is a binary relation between prefixes of $\mathcal{W}_C$.*

We say that an accessibility relation formula $w\ \rho_t\ w'$ is *true* in a tableau branch if it belongs to that branch and, intuitively, this means that in the Kripke interpretation associated with that branch $(w, w') \in \mathcal{R}_t$ holds.

Remark 1. Using prefixed formulae is very common in modal theorem proving (see [17] for an historical introduction on the topic). We would like to mention the well-known *prefixed tableau systems* in [14] and the TABLEAUX system in [8]. In [14], differently than here and [26,8], a prefix is a sequence of integers which represents a world as a *path* from the initial world to it. As a result, instead of representing *explicitly* worlds and accessibility relations of a Kripke interpretation in a *graph*, by means of the accessibility relation formulae, [14] represents them by a set of paths, which can be considered as a *spanning tree* of the graph. Similar ideas are also used by other authors, such as the proposals in [25,18,32,9].

In order to simplify the presentation of the calculus we use the well-known *uniform notation* for signed formulae [14] (see Fig. 1). In the following, we will often use α, β, ν^t, and π^t as formulae of the corresponding type.

A *tableau* is a *labeled tree* where each node consists of a *prefixed signed formula* or an *accessibility relation formula.* It is an attempt to build an interpretation in which a given formula is satisfiable. Starting from a formula φ, the interpretation is progressively constructed applying a set of *extension rules*, which reflect the semantics of the considered logic. At any stage, a branch of a tableau is a partial

α	α_1	α_2
$\mathbf{T}(\varphi \wedge \psi)$	$\mathbf{T}\varphi$	$\mathbf{T}\psi$
$\mathbf{F}(\varphi \vee \psi)$	$\mathbf{F}\varphi$	$\mathbf{F}\psi$
$\mathbf{F}(\varphi \supset \psi)$	$\mathbf{T}\varphi$	$\mathbf{F}\psi$
$\mathbf{F}(\neg\varphi)$	$\mathbf{T}\varphi$	$\mathbf{T}\varphi$

β	β_1	β_2
$\mathbf{F}(\varphi \wedge \psi)$	$\mathbf{F}\varphi$	$\mathbf{F}\psi$
$\mathbf{T}(\varphi \vee \psi)$	$\mathbf{T}\varphi$	$\mathbf{T}\psi$
$\mathbf{T}(\varphi \supset \psi)$	$\mathbf{F}\varphi$	$\mathbf{T}\psi$
$\mathbf{T}(\neg\varphi)$	$\mathbf{F}\varphi$	$\mathbf{F}\varphi$

ν^t	ν_0^t
$\mathbf{T}([t]\varphi)$	$\mathbf{T}\varphi$
$\mathbf{F}(\langle t\rangle\varphi)$	$\mathbf{F}\varphi$

π^t	π_0^t
$\mathbf{F}([t]\varphi)$	$\mathbf{F}\varphi$
$\mathbf{T}(\langle t\rangle\varphi)$	$\mathbf{T}\varphi$

Fig. 1. Uniform notation for propositional signed modal formulae.

description of an interpretation. In our case, the tableau method tries to build Kripke interpretations, one for each branch: the worlds are formed by the prefixes that appear on the branch, the accessibility relations for the modalities are given by means of the accessibility relation formulae, and the valuation function is given by means of the prefixed signed atomic formulae.

Now, we can present the set of extension rules. We say that a prefix w is *used* on a tableau branch if it occurs on the branch in some accessibility relation formula, otherwise we say that the prefix w is *new*.

Definition 3 ((Extension rules)). *Let $\mathcal{L}$ be a modal language and let $\mathcal{A}$ be a set of inclusion axioms, the extension rules for $\mathcal{I}_{\mathcal{L}}^{\mathcal{A}}$ are given in Fig. 2.*

$$\frac{w:\alpha}{\begin{array}{c} w:\alpha_1 \\ w:\alpha_2 \end{array}}\ \alpha\text{-rule} \qquad\qquad \frac{w:\beta}{w:\beta_1 \mid w:\beta_2}\ \beta\text{-rule}$$

$$\frac{w:\nu^t \quad w\ \rho_t\ w'}{w':\nu_0^t}\ \nu\text{-rule} \qquad\qquad \frac{w:\pi^t}{\begin{array}{c} w':\pi_0^t \\ w\ \rho_t\ w' \end{array}}\ \pi\text{-rule}$$

where w' is *new* on the branch

$$\frac{w\ \rho_{s_1}\ w_1 \quad \cdots \quad w_{m-1}\ \rho_{s_m}\ w'}{\begin{array}{c} w\ \rho_{t_1}\ w_1' \\ \vdots \\ w_{n-1}'\ \rho_{t_n}\ w' \end{array}}\ \rho\text{-rule}$$

where $w_1', \ldots, w_{n-1}'$ are *new* on the branch
and $[t_1]\ldots[t_n]\varphi \supset [s_1]\ldots[s_m]\varphi \in \mathcal{A}$ ($n > 0$ and $m \geq 0$)

Fig. 2. Tableau rules for propositional inclusion modal logics.

The interpretation of the different kinds of extension rules is rather easy taking into account the possible-worlds semantics. The rules for the formula of type α and β are the usual ones of the classical calculus.

A formula of type ν^t is true at world w if ν_0^t is true in all worlds w' accessible from w by means of t. Therefore, if $w : \nu^t$ occurs on an open branch, we can add $w' : \nu_0^t$ to the end of that branch for any w' which is accessible from w by means of $\mathcal{R}_t$ (such that $w\ \rho_t\ w'$ is true on that branch).

A formula of type π^t is true at the world w if there exists a world w' accessible from w at which π_0^t is true. Therefore, if $w : \pi^t$ occurs on an open branch, we can add $w' : \pi_0^t$ to the end of that branch, provided w' is new and $w\ \rho_t\ w'$ is true on it.

The intuition behind ρ-rule is quite simple. Let us suppose, for instance, that $[t_1]\dots[t_n]\varphi \supset [s_1]\dots[s_m]\varphi \in \mathcal{A}$ is an axiom of our grammar logic $\mathcal{I}_{\mathcal{L}}^{\mathcal{A}}$. If $w\ \rho_{s_1}\ w_1, \dots, w_{m-1}\ \rho_{s_m}\ w'$ are on a branch, then $(w, w_1) \in \mathcal{R}_{s_1}, \dots, (w_{m-1}, w') \in \mathcal{R}_{s_m}$ in the Kripke interpretation associated with that branch. Since $[t_1]\dots[t_n]\varphi \supset [s_1]\dots[s_m]\varphi \in \mathcal{A}$ then, the corresponding inclusion property (2) must holds. Thus, we can add the formulae $w\ \rho_{t_1}\ w'_1, \dots, w'_{n-1}\ \rho_{t_n}\ w'$ to that branch. Moreover, in the case of $m = 0$ we can always add the formulae $w\ \rho_{t_1}\ w'_1, \dots, w'_{n-1}\ \rho_{t_n}\ w$, for every world constant w, provided that $w'_1, \dots, w'_{n-1}$ are new on the branch.

Remark 2. It is worth noting that the ρ-rule works for the whole class of grammar logics. Nevertheless, the proposed tableau could be easily extended in order to deal with modal logics which are different than those we have considered. By introducing new rules, which operate on accessibility relation formulae, one could also deal with multimodal logics characterized by *serial*, *symmetric*, and *Euclidean* accessibility relations [2].

We say that a tableau branch is *closed* if it contains $w : \mathbf{T}\varphi$ and $w : \mathbf{F}\varphi$ for some formula φ. A tableau is *closed* if *every* branch in it is closed. Finally, let $\mathcal{L}$ be a modal language, $\mathcal{A}$ a set of inclusion axioms, and φ a formula. Then a closed tableau for $i : \mathbf{F}\varphi$ obtained by using the tableau rules of Fig. 2, is said to be a *proof* of φ.

Theorem 1. *Let $\mathcal{I}_{\mathcal{L}}^{\mathcal{A}}$ be grammar logic then, a formula φ of $\mathcal{L}$ has a tableau proof if and only if it is $\mathcal{A}$-valid.*

Due to space limitation we do not present here the proof of Theorem 1 but it follows the well-known guideline of [14,25,17] and it can be found in [2].

Example 2. In Figure 3 we have reported the proof of the first conjunct of the formula $[j][p]$*appointment* $\wedge$ $[p][j]$*appointment* of Example 1. We denote with “a” and “b” the two branches which are created by the β-rule at step 13., “c” and “d” the two ones created by the β-rule at step 14b., “e” and “f” the two ones created by the β-rule at step 17d. Moreover, to save space, we use “*ap*” instead of *appointment*, “*tm*” instead of *time*, and “*pl*” istead of *place*. The explanation: *1., 2., 3., and 4.*: formula (1), (2), (3), and (4); *5.*: goal, formula (5); *6. and 7.*: from 5., by π-rule; *8. and 9.*: from 6., by π-rule; *10. and 11.*: from 7. and 9., by A_1 and ρ-rule; *12.*: from 4. and 10., by ν-rule; *13.*: from 12. and 11., by ν-rule; *14a. and 14b*: from 13, by β-rule, branch “a” closes; *15c. and 15d.*: from 14b., by

β-rule; *16c.*: from 3. and 10., by ν-rule; *17c.*: from 16c. and 11., by ν-rule, branch "c" closes; *16d.*: from 10., by axiom A_6 and π-rule; *17d.*: from 2. and 16d., by ν-rule; *18e. and 18f*: from 17d., by β-rule; *19e.*: from 18e. and 11., by ν-rule, branch "e" closes; *19f. and 20f.*: from 18f., by π-rule; *21f.*: from 10. and 10f., by axiom A_3 and ρ-rule; *22f.*: from 1. and 21f., by ν-rule, branch "f" closes.

1. $i : \mathbf{T}[p]tm$
2. $i : \mathbf{T}[w(p)]([p]tm \supset [j]tm)$
3. $i : \mathbf{T}[p][j]pl$
4. $i : \mathbf{T}[p][j](pl \wedge tm \supset ap)$
5. $i : \mathbf{F}[j][p]ap$
6. $w_1 : \mathbf{F}[p]ap$
7. $i\ \rho_j\ w_1$
8. $w_2 : \mathbf{F}ap$
9. $w_1\ \rho_p\ w_2$
10. $i\ \rho_p\ w_3$
11. $w_3\ \rho_j\ w_2$
12. $w_3 : \mathbf{T}[j](pl \wedge tm \supset ap)$
13. $w_2 : \mathbf{T}(pl \wedge tm \supset ap)$

14a. $w_2 : \mathbf{T}ap$	14b. $w_2 : \mathbf{F}(pl \wedge tm)$		
$\times$	15c. $w_2 : \mathbf{F}pl$	15d. $w_2 : \mathbf{F}tm$	
	16c. $w_3 : \mathbf{T}[j]pl$	16d. $i\ \rho_{w(p)}\ w_3$	
	17c. $w_2 : \mathbf{T}pl$	17d. $w_3 : \mathbf{T}([p]tm \supset [j]tm)$	
	$\times$	18e. $w_3 : \mathbf{T}[j]tm$	18f. $w_3 : \mathbf{F}[p]tm$
		19e. $w_2 : \mathbf{T}tm$	19f. $w_4 : \mathbf{F}tm$
		$\times$	20f. $w_3\ \rho_p\ w_4$
			21f. $i\ \rho_p\ w_4$
			22f. $w_4 : \mathbf{T}tm$
			$\times$

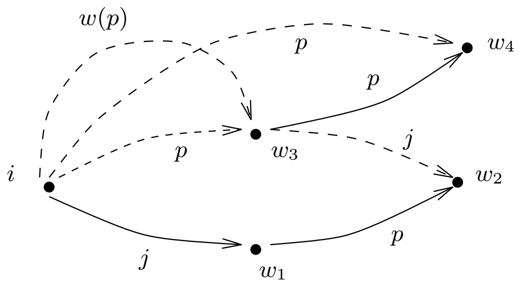

Fig. 3. ρ-rule as rewriting rule: counter-model construction of Example 1.

The ρ-rule can be regarded as a *rewriting* rule which creates new paths among worlds according to the inclusion properties of the grammar logic. In fact, given a tableau branch S, let w_0 and w_n two prefixes used on S, a *path* $\xi(w_0, w_n)$ is a collection $\{w_0\ \rho_{t_1}\ w_1,\ w_1\ \rho_{t_2}\ w_2,\ \ldots,\ w_{n-1}\ \rho_{t_n}\ w_n\}$ of accessibility relation formulae in S. We say that the path $\xi(w_0, w_n)$ *directly ρ-derives* the path $\xi'(w_0, w_n)$ if the path $\xi'(w_0, w_m)$ is obtained from $\xi(w_0, w_m)$ by means of the application of a ρ-rule to a subpath of $\xi(w_0, w_n)$. The relation *ρ-derive* is the reflexive, transitive closure of the relation *directly ρ-derive*. For instance, let us consider Fig. 3. Then, the path $\xi_1(i, w_2) = \{i\ \rho_j\ w_1,\ w_1\ \rho_p\ w_2\}$ directly ρ-derives the path $\xi_2(i, w_2) = \{i\ \rho_p\ w_3,\ w_3\ \rho_j\ w_2\}$, and ρ-derives the path $\xi_3(i, w_2) = \{i\ \rho_{wp}\ w_3,$ $w_3\ \rho_j\ w_2\}$.

For a path $\xi(w_0, w_n) = \{w_0\ \rho_{t_1}\ w_1,\ \ldots,\ w_{n-1}\ \rho_{t_n}\ w_n\}$, we denote by $\overline{\xi}(w_0, w_n)$ the *word* $t_1 \cdots t_n$. It is worth noting that for a grammar logic $\mathcal{I}_{\mathcal{L}}^{\mathcal{A}}$ based on a grammar G, if $\xi(w_0, w_n)$ is a path occurring in a tableau branch, then, $\xi(w_0, w_n)$ ρ-derives a path $\xi'(w_0, w_n)$ *if and only if* $\overline{\xi'}(w_0, w_n) \Rightarrow_G^* \overline{\xi}(w_0, w_n)$.

4 Undecidability Results for Grammar Logics

The tableau method developed in the previous section allows to generalize the correspondence between the membership problem for a given grammar and the validity problem in the corresponding grammar logic established by Fariñas del Cerro and Penttonen in [11].

Theorem 2. *Given a grammar $G = (V, T, P, S)$, let $\mathcal{I}_{\mathcal{L}}^{\mathcal{A}}$ be the grammar logic based on G. Then, for any propositional variable p of $\mathcal{L}$, $\models_{\mathcal{A}} [S]p \supset [s_1] \dots [s_m]p$ if and only if $S \Rightarrow_G^* s_1 \cdots s_m$, where the s_i's are in $V \cup T$.*

Proof. (*If*) Let us suppose that $\models_{\mathcal{A}} [S]p \supset [s_1] \dots [s_m]p$, then, the tableau starting from $i : \mathbf{F}([S]p \supset [s_1] \dots [s_m]p)$ closes. Now, by applying the β-rule we obtain: $i : \mathbf{T}[S]p$, $i : \mathbf{F}[s_1] \dots [s_m]p$, and m times the π-rule: $w_1 : \mathbf{F}[s_2] \dots [s_m]p$, $i\ \rho_{s_1}\ w_1, \dots, w_m : \mathbf{F}p$, and $w_{m-1}\ \rho_{s_m}\ w_m$. Since, by hypothesis, the above tableau closes, the only way for this to happen is that after a finite number of applications of the ρ-rule we have the prefixed signed formula $w_m : \mathbf{T}p$ in the branch. This happens if the path $\xi(i, w_m) = \{i\ \rho_{s_1}\ w_1, \dots, w_{m-1}\ \rho_{s_m}\ w_m\}$ ρ-derives the path $\xi'(i, w_m) = \{i\ \rho_S\ w_m\}$, that is, if there exits a derivation $\overline{\xi'}(i, w_m) \Rightarrow_G^* \overline{\xi}(i, w_m)$. (*Only if*) Assume $S \Rightarrow_G^* s_1 \cdots s_m$. Since a systematic attempt to prove $i : \mathbf{F}([S]p \supset [s_1] \dots [s_m]p)$ generates a path $\xi(i, w_m) = \{i\ \rho_{s_1}\ w_1, \dots, w_{m-1}\ \rho_{s_m}\ w_m\}$ and $\xi(i, w_m)$ ρ-derives the path $\xi'(i, w_m) = \{i\ \rho_S\ w_m\}$, after a finite number of steps the only branch of the tableau closes by $w_m : \mathbf{T}p$ and $w_m : \mathbf{F}p$.

It is well known that the problem of establishing if a word belongs to the language generated by an arbitrary type-0 grammar is undecidable [21]. Hence, we have the following corollary.

Corollary 1. *The validity problem for the class of grammar logics is undecidable.*

Indeed, this result has already been shown in [11]. However, Fariñas del Cerro and Penttonen do not prove Theorem 2 for the type-0 grammars but for a more restricted class of the grammar logics, that they call *Thue logics* because they are based on the *Thue systems* [6]. A Thue system is a type-0 grammars whose productions are *symmetric* and, thus, the Thue logics are grammar logics characterized by axiom schemas where the implication is replaced by the biimplication. In [11] the undecidability of grammar logics is proved by showing that the Thue logics are undecidable. In fact, since the membership problem for the Thue systems is *undecidable*, proving that a formula is a theorem of a Thue logic is also undecidable.[3]

[3] The Thue systems have also been used in [24] to define logics similar to those studied in [11], which, however, are not in the class on grammar logics since modalities enjoy some further properties like seriality and determinism. In [24] undecidability results are proved for this class of logics.

In [11] some problems are left open. In particular, it is not established whether more restricted classes of grammar logics, such as context sensitive, context-free, regular modal logics are decidable. In the following, we show that also the class of context sensitive and context-free modal logics are undecidable by reducing the solvability of the problem $L_1 \cap L_2 \neq \emptyset$ (where L_1 and L_2 are languages) to the satisfiability of formulas of context sensitive and context-free modal logics.

Theorem 3. *Let $G_1 = (V_1, T_1, P_1, S_1)$ and $G_2 = (V_2, T_2, P_2, S_2)$ be two grammars such that $V_1 \cap V_2 = \emptyset$ and $T_1 = T_2 \neq \emptyset$. Then, there exists a grammar logic $\mathcal{I}^{\mathcal{A}}_{\mathcal{L}}$ and a formula φ of $\mathcal{L}$ such that $\models_{\mathcal{A}} \varphi$ if and only if $L(G_1) \cap L(G_2) \neq \emptyset$.*

Proof. Let us define a grammar $G = (V, T, P, S)$, where $V = V_1 \cup V_2 \cup \{S\}$, $T = T_1 = T_2$, $P = P_1 \cup P_2 \cup \{S \to t, S \to S\ t \mid t \in T\}$, and $S \notin V_1$ and $S \notin V_2$. Then, we assume as $\mathcal{I}^{\mathcal{A}}_{\mathcal{L}}$ the inclusion modal logic based on G and we consider the formula $\varphi_T(q) = \bigwedge_{t \in T}(\langle t \rangle q \wedge [S]\langle t \rangle q)$ where $q \in$ VAR. A tableau starting from $i : \mathbf{T}\varphi_T(q)$ is formed by only one branch that goes on forever. It is easy to see that for each word $x \in T^*$ the tableau branch contains a path $\xi(i, w)$ such that $\overline{\xi}(i, w) = x$. Now, let us define $\varphi = \varphi_T(q) \supset ([S_1]p \supset \langle S_2 \rangle p)$, where $p, q \in$ VAR and $p \neq q$. (*If*) Suppose that $\models_{\mathcal{A}} \varphi$ then, the tableau starting from 1. $i : \mathbf{F}(\varphi_T(q) \supset ([S_1]p \supset \langle S_2 \rangle p))$ closes. Now, by applying twice the β-rule we obtain: 2. $i : \mathbf{T}\varphi_T(q)$, 3. $i : \mathbf{T}[S_1]p$, and 4. $i : \mathbf{F}\langle S_2 \rangle p$. Since the above tableau must close, the only way for this to happen is that after a finite number of steps we must have a pair of prefixed signed formulae $w : \mathbf{T}p$ and $w : \mathbf{F}p$, for some prefix w and, therefore, a path $\xi(i, w)$ that ρ-derives both the path $\xi_1(i, w) = \{i\ \rho_{S_1}\ w\}$ and the path $\xi_2(i, w) = \{i\ \rho_{S_2}\ w\}$. Thus, there is a derivation of $\overline{\xi}(i, w)$ both from $\overline{\xi_1}(i, w) = S_1$ and from $\overline{\xi_2}(i, w) = S_2$ ($S_1 \Rightarrow^*_G \overline{\xi}(i, w)$ and $(S_2 \Rightarrow^*_G \overline{\xi}(i, w))$, i.e. $\overline{\xi}(i, w) \in L(G_1) \cap L(G_2)$. (*Only if*) Assume that $S_1 \Rightarrow^*_{G_1} x$ and $S_2 \Rightarrow^*_{G_2} x$, for some $x \in T^*$. Since a systematic attempt to prove the formula $i : \mathbf{T}\varphi_T(q)$ can generate a path $\xi(i, w)$, for some prefix w, such that $\overline{\xi}(i, w) = y$, for any $y \in T^*$, after a finite number of steps we have a path $\xi'(i, w')$ such that $\overline{\xi'}(i, w') = x$. Thus, we have also the paths $\xi'_1(i, w') = \{i\ \rho_{S_1}\ w'\}$ and $\xi'_2(i, w') = \{i\ \rho_{S_2}\ w'\}$ by application of the ρ-rule for a finite number of times. This is enough to close the only branch of the tableau by $w' : \mathbf{T}p$ and $w' : \mathbf{F}p$.

It is well known that, given two arbitrary type-1 (type-2) grammars G_1 and G_2, it is undecidable if $L(G_1) \cap L(G_2) \neq \emptyset$ [21]. Hence, we have the following corollary.

Corollary 2. *The validity problem for the class of context sensitive and context-free modal logic is undecidable.*

5 A Decidability Result for Grammar Logics

In the previous section we have shown that it is not possible to supply a general decision procedure for the class of unrestricted, context sensitive and context-free modal logics. In this section, instead, we give a *decidability* result for *right* *regular grammar* logics, that is, those ones whose productions are of the form $A \to \sigma\ A'$, where A, A' are variables and σ a string of terminals.

Definition 4. *Let $G = (V, T, P, S)$ be a right type-3 grammar and let A be a variable. Then, a derivation of a sentential form σX from A*[4] *is said to be* non-recursive *if and only if each variable of V appears in the derivation, apart from σX, at most once.*

Proposition 1. *Let $G = (V, T, P, S)$ be a right type-3 grammar, let A_0 be a variable and let $A_0 \Rightarrow_G^* \sigma_1 \cdots \sigma_n A_n \Rightarrow_G \sigma_1 \cdots \sigma_n \sigma_{n+1} A_{n+1}$ be a derivation, where either $A_{n+1} \in V$ or $A_{n+1} \in T$ and $A_i \to \sigma_{i+1} A_{i+1} \in P$, for $i = 0, \ldots, n$. Then, there exists a non-recursive derivation $A_0 \Rightarrow_G^* \sigma\sigma_{n+1} A_{n+1}$, for some $\sigma \in T^*$.*

Proposition 2. *Let $G = (V, T, P, S)$ be a right type-3 grammar. Then, the number of different non-recursive derivation by means of G is bounded by $\mathrm{der}_G = |V| \cdot \sum_{i=1}^{|V|} n^i$, where n is the maximum number of production associated to a same variable of V.*

The proofs of the proposition above are simple and they can be found in [2].

Let $G = (V, T, P, S)$ be a right type-3 grammar and $\mathcal{I}_{\mathcal{L}}^{\mathcal{A}}$ the regular inclusion modal logic based on G. Then, we define the *Fischer-Ladner closure* $FL(\varphi)$ of a formula φ of $\mathcal{L}$ (that only uses *existential modal operators*, *or*, and *negation*[5]) as follows:

- if $\psi \vee \psi' \in FL(\varphi)$ then $\psi \in FL(\varphi)$ and $\psi' \in FL(\varphi)$;
- if $\neg\psi \in FL(\varphi)$ then $\psi \in FL(\varphi)$;
- if $\langle t \rangle \psi \in FL(\varphi)$ and $t \in T$ then $\psi \in FL(\varphi)$;
- if $\langle A \rangle \psi \in FL(\varphi)$, $A \in V$, and there is a non-recursive derivation $A \Rightarrow_G^* t_1 \cdots t_n X$, where $t_1, \ldots, t_n \in T$ and either $X \in T \cup V$, then $\langle t_1 \rangle \ldots \langle t_n \rangle \langle X \rangle \psi \in FL(\varphi)$.

By Proposition 2 and the fact that φ has finite length, the Fischer-Ladner closure is finite for any formula of a right regular modal logic. Consider a Kirpke $\mathcal{A}$-interpretation $M = \langle W, \{R_t \mid t \in \mathrm{MOD}\}, V \rangle$ and a formula φ of $\mathcal{L}$, we define an *equivalence* relation $\equiv$ on state of W by: $w \equiv w'$ *if and only if for all* $\psi \in FL(\varphi)$ *we have* $M, w \models_{\mathcal{A}} \psi$ *iff* $M, w' \models_{\mathcal{A}} \psi$ (we use the notation $\overline{w}$ for this equivalence class). The *quotient* Kripke $\mathcal{A}$-interpretation $M^{FL(\varphi)} = \langle W^{FL(\varphi)}, \{\mathcal{R}_t^{FL(\varphi)} \mid t \in \mathrm{MOD}\}, V^{FL(\varphi)} \rangle$ (the *filtration of M through $FL(\varphi)$*) is defined as follows:

- $W^{FL(\varphi)} = \{\overline{w} \mid w \in W\}$;

[4] Note that, every sentential form derived from A has the form σX, where $\sigma \in T^*$ and either $X \in T$ or $X \in V$.

[5] Since all other connectives can be defined in terms of these, this is not a restrictive condition.

– $V^{FL(\varphi)}(\overline{w}, p) = V(w, p)$, for any $p \in \text{VAR}$ and $\overline{w} \in W^{FL(\varphi)}$;
– $\mathcal{R}_t^{FL(\varphi)} \supseteq \{(\overline{w}, \overline{w'}) \in W^{FL(\varphi)} \times W^{FL(\varphi)} \mid (w, w') \in \mathcal{R}_t\}$.
Moreover, $\mathcal{R}_t^{FL(\varphi)}$ is closed with respect to the inclusion axioms, that is, for each inclusion axiom $[t]\alpha \supset [s_1] \ldots [s_m]\alpha$ if $(\overline{w_0}, \overline{w_1}) \in \mathcal{R}_{s_1}^{FL(\varphi)}$, ..., $(\overline{w_{m-1}}, \overline{w_m}) \in \mathcal{R}_{s_m}^{FL(\varphi)}$ then the pair $(\overline{w_0}, \overline{w_m})$ belongs to $\mathcal{R}_t^{FL(\varphi)}$.

The following lemma states that when we insert any extra binary relation between $\overline{w}$ and $\overline{w'}$ in a accessibility relation $\mathcal{R}_t^{FL(\varphi)}$ of $M^{FL(\varphi)}$, in order to satisfy the relative set of inclusion properties, it is not the case that there was any $\langle t\rangle\psi \in FL(\varphi)$ which was true at w while ψ itself was false at w' [22].

Lemma 1. *For all $\psi = \langle t\rangle\psi' \in FL(\varphi)$, if $(\overline{w}, \overline{w'}) \in \mathcal{R}_t^{FL(\varphi)}$ and $M, w' \models_{\mathcal{A}} \psi'$ then $M, w \models_{\mathcal{A}} \langle t\rangle\psi'$.*

Proof. Assume that $\psi = \langle t\rangle\psi' \in FL(\varphi)$ then $\psi' \in FL(\varphi)$ by definition of the closure. Now, there are two cases which depend on whether $(\overline{w}, \overline{w'}) \in \mathcal{R}_t^{FL(\varphi)}$ has been added to originary definition of filtration because an inclusion axiom of the form $[t]\alpha \supset [s_1] \ldots [s_m]\alpha \in \mathcal{A}$ or not.

Assume that it has not been added. Since by definition of $\mathcal{R}_t^{FL(\varphi)}$, there exist $w_1, w_1' \in W$ such that $(w_1, w_1') \in \mathcal{R}_t$, $w_1 \equiv w$, and $w_1' \equiv w'$. Since $M, w' \models_{\mathcal{A}} \psi'$, $M, w_1' \models_{\mathcal{A}} \psi'$ because $\psi' \in FL(\varphi)$ and $w' \equiv w_1'$. Hence, $M, w_1 \models_{\mathcal{A}} \langle t\rangle\psi'$ because $(w_1, w_1') \in \mathcal{R}_t$. Finally, $M, w \models_{\mathcal{A}} \langle t\rangle\psi'$ since $\langle t\rangle\psi' \in FL(\varphi)$ and $w \equiv w'$.

Assume that $(\overline{w}, \overline{w'}) \in \mathcal{R}_t^{FL(\varphi)}$ but $(w, w') \notin \mathcal{R}_t$. The pair $(\overline{w}, \overline{w'})$ has been added in $\mathcal{R}_t^{FL(\varphi)}$ by the closure operation in order to satisfy an inclusion property of an inclusion axiom of the form $[t]\alpha \supset [s_1] \ldots [s_m]\alpha \in \mathcal{A}$. Then, there exist $\overline{w_1}$, ..., $\overline{w_{m-1}}$ such that $(\overline{w_0}, \overline{w_1}) \in \mathcal{R}_{s_1}^{FL(\varphi)}$, ..., $(\overline{w_{m-1}}, \overline{w_m}) \in \mathcal{R}_{s_m}^{FL(\varphi)}$, where w_0 is w and w_m is w'. Now, in turn, for each $(\overline{w_{i-1}}, \overline{w_i}) \in \mathcal{R}_{s_i}^{FL(\varphi)}$, for $i = 1, \ldots, n$, either the pair $(\overline{w_{i-1}}, \overline{w_i})$, has been added by the closure operation or not. Going on this way, we have $(\overline{v_0}, \overline{v_1}) \in \mathcal{R}_{t_1}^{FL(\varphi)}$, ..., $(\overline{v_{h-1}}, \overline{v_h}) \in \mathcal{R}_{t_h}^{FL(\varphi)}$ such that the corresponding pairs belong to $\mathcal{R}_t$ and $t \Rightarrow_G^* t_1 \cdots t_h$, v_0 is w_0 (that, in turn, is w), and v_h is w_m (that, in turn, is w'). By construction, there exist $v_{i-1}', v_i'' \in W$ such that $(v_{i-1}', v_i'') \in \mathcal{R}_{t_i}^{FL(\varphi)}$ and $v_{i-1} \equiv v_{i-1}'$ and $v_i \equiv v_i''$, for $i = 1, \ldots, h$.

Assume that $t \Rightarrow_G^* t_1 \cdots t_h$ is the derivation $A_0 \Rightarrow_G \sigma_1 A_1 \Rightarrow_G \ldots \Rightarrow_G \sigma_1 \cdots \sigma_n A_n \Rightarrow_G \sigma_1 \cdots \sigma_n \sigma_{n+1}$, where A_0 is t and $A_n \to \sigma_{n+1}$ and $A_{i-1} \to \sigma_i A_i$, for $i = 1, \ldots, n$, are in P, and that σ_{n+1} is $d_1 \cdots d_r$ $(= t_{h-r+1} \cdots t_h)$. We know $M, v_h \models_{\mathcal{A}} \psi'$ and we have to prove that $M, v_{h-r+1} \models_{\mathcal{A}} \langle d_1\rangle \ldots \langle d_r\rangle\psi'$. Assuming that $\langle d_1\rangle \ldots \langle d_r\rangle\psi' \in FL(\varphi)$ then, we have that $M, v_h'' \models_{\mathcal{A}} \psi'$ since $v_h \equiv v_h''$ and $\psi' \in FL(\varphi)$. Since $(v_{h-1}', v_h'') \in \mathcal{R}_{t_h}$ and $M, v_h'' \models_{\mathcal{A}} \psi'$ then, $M, v_{h-1}' \models_{\mathcal{A}} \langle d_r\rangle\psi'$ and, since $\langle d_r\rangle\psi' \in FL(\varphi)$ and $v_{h-1}' \equiv v_{h-1}''$, we have that $M, v_{h-1}'' \models_{\mathcal{A}} \langle d_r\rangle\psi'$. We can proceed so on until we have $M, v_{h-r+1}'' \models_{\mathcal{A}} \langle d_1\rangle \ldots \langle d_r\rangle\psi'$ and $M, v_{h-r+1} \models_{\mathcal{A}} \langle d_1\rangle \ldots \langle d_r\rangle\psi'$ since $v_{h-r+1} \equiv v_{h-r+1}''$. Now, since the inclusion axiom $[A_n]\alpha \supset [d_1] \ldots [d_r]\alpha$ belongs to $\mathcal{A}$, $M, v_{h-r+1} \models_{\mathcal{A}} \langle A_n\rangle\psi'$. We can repeat the above argumentation for all derivation steps from A_0 obtaining $M, w \models_{\mathcal{A}} \langle A_0\rangle\psi'$.

We have now to prove that $\langle d_1\rangle \ldots \langle d_r\rangle\psi' \in FL(\varphi)$. By hypothesis $\langle A_0\rangle\psi' \in FL(\varphi)$ (A_0 is t) and $A_0 \Rightarrow_G^* \sigma_1 \cdots \sigma_n \sigma_{n+1}$. Then, by Proposition 1, there exists a non-recursive derivation $A_0 \Rightarrow_G^* \sigma\sigma_{n+1}$, for some $\sigma \in T^*$. By definition of Fischer-Ladner closure, since $\langle A_0\rangle\psi' \in FL(\varphi)$, we have $\langle t'_1\rangle \dots \langle t'_{n'}\rangle\langle d_1\rangle \dots \langle d_r\rangle\psi' \in FL(\varphi)$, where σ is $t'_1 \cdots t'_{n'}$ and σ_{n+1} is $d_1 \cdots d_r$, and, hence, $\langle d_1\rangle \dots \langle d_r\rangle\psi' \in FL(\varphi)$.

Lemma 2 (Filtration Lemma). *For all $\psi \in FL(\varphi)$, $M, w \models_{\mathcal{A}} \psi$ if and only if $M^{FL(\varphi)}, \overline{w}, \models_{\mathcal{A}} \psi$.*

Proof. The proof is by induction on the structure of ψ. (*Base* step) For $\psi \in$ VAR the thesis holds trivially. (*Induction* step) The cases $\psi = \psi' \vee \psi''$ and $\psi = \neg\psi'$ are immediate from the definitions. Assume that $\psi = \langle t\rangle\psi'$. (*If*) If $M, w \models_{\mathcal{A}} \langle t\rangle\psi'$ then there exists w' such that $M, w' \models_{\mathcal{A}} \psi'$ and $(w, w') \in \mathcal{R}_t$. By definition, we have $(\overline{w}, \overline{w'}) \in \mathcal{R}_t^{FL(\varphi)}$ and, by induction hypothesis, $M^{FL(\varphi)}, \overline{w'} \models_{\mathcal{A}} \psi'$. Hence $M^{FL(\varphi)}, \overline{w} \models_{\mathcal{A}} \langle t\rangle\psi'$. (*Only if*) If $M^{FL(\varphi)}, \overline{w} \models_{\mathcal{A}} \langle t\rangle\psi'$ then, there exists $\overline{w'} \in W^{FL(\varphi)}$ such that $M^{FL(\varphi)}, \overline{w'} \models_{\mathcal{A}} \psi'$ and $(\overline{w}, \overline{w'}) \in \mathcal{R}_t^{FL(\varphi)}$. By inductive hypothesis, we have that $M, w' \models_{\mathcal{A}} \psi'$ and, by Lemma 1, since $(\overline{w}, \overline{w'}) \in \mathcal{R}_t^{FL(\varphi)}$, $M, w \models_{\mathcal{A}} \langle t\rangle\psi'$.

Theorem 4 (Small Model Theorem). *Let φ be a satisfiable formula of a grammar logic $\mathcal{I}_{\mathcal{L}}^{\mathcal{A}}$ based on a type-3 grammar G. Then, φ is satisfied in a Kripke $\mathcal{A}$-interpretation with no more that $2^{|FL(\varphi)|}$ states.*

Proof. If φ is satisfiable, then there is a Kripke $\mathcal{A}$-interpretation M and a state w in M such that $M, w \models_{\mathcal{A}} \varphi$. Let $FL(\varphi)$ be the Fischer-Ladner closure of φ. By Lemma 2, $M^{FL(\varphi)}, \overline{w} \models_{\mathcal{A}} \varphi$. Moreover, since, by Proposition 2, $|FL(\varphi)|$ is bounded, the filtration through $FL(\varphi)$ is a finite Kripke interpretation having at most $2^{|FL(\varphi)|}$ worlds, that being the maximum number of ways that worlds can disagree on sentences in $FL(\varphi)$.

Each right regular modal logics, by Theorem 4, is determined by a class of finite standard Kripke interpretations and, hence, it has the *finite model property* [22]. Then, we have the following corollary.

Corollary 3. *The validity problem for the class of right regular modal logics is decidable.*

6 Discussion and Related Work

In this paper we have established some undecidability results for multimodal logics, reducing well-known unsolvable problems of formal languages to satisfiability problems of multimodal systems by means of a tableau calculus based on prefixed formulas. Moreover, the decidability of the class of multimodal logics based on right regular grammars has been proved using the filtration method introduced by Fischer and Ladner in [12].

In order to have a general framework able to cope with any kind of grammar logics, we have chosen the simplest way of representing models: prefixes are worlds, and relations between them are built step by step by the rules of the calculus. In particular, axioms are used as rewrite rules which create new paths among worlds.

This approach is closely related to the approaches based on prefixes used by Fitting and other authors for classical modal systems (non-multimodal) [14,25, 9]. There, prefixes are sequences of integers which represent a world as a path in the model that goes from the initial world to it. Thus, instead of representing a model as a graph, as in this paper, a model is represented as a set of paths, which can be considered as a spanning tree of the graph. Although this representation may be more efficient, it requires a specific ν-rule for each logic. Properties of accessibility relations are coded in these rules, and thus, depending on the logic, the ν-rules may express complex relations between prefixes, which instead in our case are explicitly available from the representation. Massacci [25] has proposed a "single step calculus", where ν-rules make use only of immediately accessible prefixes. His approach works for many logics, but it still requires the definition of specific ν-rules.

Besides the disadvantage of requiring specific ν-rules and the fact that they do not work with multimodal systems, we think that though the approach based on prefixes as sequences might be adapted for some subclasses of grammar logics it is difficult to extend it to the whole class. In particular, it can be shown that, for some grammar logic, a "generation lemma" like those used in [25,17], does not hold, i.e. it is not true that, for any prefix occurring on a branch, all intermediate prefixes occur too. Let us consider, for instance, the derivation of Example 1. We can image to use the prefix $1.1_j.1_p$ to represent the world w_2. Now, by applying axiom A_1, the same world can also be represented with the sequence $1.1_p.1_j$, whose subprefix 1.1_p does not occur on the branch. On the other hand, this subprefix is needed in order to conclude with success the proof. Moreover, adding exsplicitly the subprefixes, as the one above, is not enough to solve the problem, since all prefixes representing the same world have to be identified. Similar consideration can be done for the proposals in [18,32].

The proposals in [18,32,5] address the problem of an efficient implementation of the tableau calculi for a wide class of modal logics. They generalize the prefixes by allowing occurrences of variables and they use unification to show that two prefixes are names for the same world. While a straightforward implementation of our calculus is unlikely to be efficient, the generality of the approach makes it suitable to study the properties of different classes of logics.

Instead of developing specific proof techniques for modal logics, some authors have proposed the alternative approach of translating modal logics into classical first order logic [29]. The translation methods are based on the idea of making explicit reference to the worlds by adding to all predicates an argument representing the world where the predicate holds, so that the modal operators can be transformed into quantifiers of classical logic. In particular, the *functional translation* [30,1] is based on the idea of representing paths in the possible worlds structure by means of compositions of functions which map worlds to accessible worlds. An advantage of this approach is that it keeps the structure of the original formula. However the approach is suitable mainly for *serial logics*, for which optimization technique have been studied [28,15], and it requires a different equational unification algorithm for each logic. A way to avoid equational reasoning while retaining the advantages of the functional translation has been developed by Nonnengart [27]. Gasquet in [15] deals with the same class of multimodal logics we have presented, where, however, the seriality is assumed for each modal operator.

Though in this paper we have focused on a propositional language, the tableau calculus we have proposed can be naturally extended to the first order case by introducing the usual rules for quantifiers. Moreover, it can be extended to deal with a wider class of logics. In particular, in [2] a tableau calculus is developed for the class of multimodal logics characterized by "a, b, c, d-incestuality" axioms (defined by Catach in [7]) and, then, as a special case, also for the multimodal logics characterized by *serial*, *symmetric*, and *Euclidean* accessibility relations.

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