MODAL ENVIRONMENT FOR BOOLEAN SPECULATIONS

(preliminary report)

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ABSTRACT The common form of a mathematical theorem consists in that "the truth of some properties for some objects is necessary and/or sufficient condition for other properties to hold for other objects". To formalize this, one happens to resort to Kripke modal logic K which, having in the syntax the notions of 'property' and 'necessity', appears to provide a reliable metamathematical fundament. In this paper we challenge this reliability. We propose two different approaches each claiming better formal treatment of the state of affairs. The first approach is in formalizing the notion of 'sufficiency' (which remains beyond the capacities of K), and consequently of 'sufficiency' and 'necessity' in a joint context. The second is our older idea to formalize the notion of 'object' in the same modal spirit. Having 'property, object, sufficiency, necessity', we establish some basic results and profess to properly formalize the everyday metamathematical reason.

1. INTRODUCTION: sufficiency

Modal logic extends syntactically the ordinary propositional language with new, as a rule unary, operators known as modalities, a typical one being the necessity modal operator $\Box$. On the semantical side one has the Kripke "possible worlds" interpretation of the extended language: frames $F = (W, R)$ with $W \neq \emptyset$ and $R \subseteq W^2$; and models based on them, i.e. (W, R, V) where V, denoted as $\models$, assignes to each formula A a (truth) set $V(A)$, or $\{s/\ s \models A\}$, of possible worlds. The truth set of a Boolean junction is the respective set-theoretic junction of the truth set(s). The truth set for the modality, i.e. $V(\Box A)$, is usually given as a particular first-order condition on the relationship between $V(A)$ and R.

Consider the simplest case when just $\Box$ has been added, and denote this modal language by $\mathcal{ML}(\Box)$. To each $\mathcal{ML}(\Box)$-formula A, a formula $St(A)$ corresponds in the language $\mathcal{L}_1$, cf. van Benthem (1977): a first-order language with one binary predicate Rxy, and infinitely many unary predicates P_ix.

<u>Definition</u> $St(A)$ is defined inductively as follows:

1. $St(p_i) = P_ix$, for propositional variables p_i
2. $St(0) = 0$ (0 is the falsity)
3. $St(A \to B) = St(A) \to St(B)$
4. $St(\Box A) = \forall y(Rxy \to [y/x]St(A))$, where y does not occur in $St(A)$. #

Since $St(A)$ reflects the semantics of the modal formula A, a model $\mathcal{M}$ is a

D. G. Skordev (ed.), *Mathematical Logic and Its Applications*

first-order structure for the language $\mathfrak{L}_1$, and

$\mathcal{M}, s \models A$ iff $\mathcal{M} \models St(A)[s]$.

Focus now on clause 4, which is the point of Kripke's approach to necessity. The truth of $\Box A$ depends on the truth set of A and the relation R, in a way given by a formula of one bound variable. In this paper we study modalities $\underline{M}$ for which the corresponding truth condition on x is given by an arbitrary formula of one bound variable y, containing Rxx, Rxy, Ryx, Ryy and $[y/x]St(A)$, and find a basis for these modalities, i.e. a "small" subset of them sufficient for defining each of the remaining.

Kripke's mathematical interpretation of "p is necessary (true) in x", $R(x) \subseteq V(p)$, only sharpens but does not satisfy one's desire to formally handle the "sufficiency" phenomena as well. The first and trivial attempt is to grammatically reduce the "sufficiency" to "necessity" saying that "x is sufficient for p" iff "p is necessary in x", and this surely will not enrich our knowledge. The first non-trivial suggestion is to interpret the sufficiency more Kripkely: "p is sufficient for (accessibility from) x" iff $V(p) \subseteq R(x)$. This leads to an "alternative" modal logic K*, Tehlikeli (1985), which formally at least, is equal in rights with Kripke's K.

Language of K* is $\mathcal{M}\mathfrak{L}(\boxplus)$, i.e., $\Box$ and $\Diamond$ are replaced by $\boxplus$ and Φ (named by Slavjan Radev as "window" and "kite").

Semantics of K*. Kripke models with: $x \models \boxplus A$ iff $\forall y(y \models A \rightarrow Rxy)$.

Axiomatics of K*: Besides the Boolean tautologies, we have also the scheme

$\vdash \boxplus A \wedge \boxplus(\neg A \wedge B) \rightarrow \boxplus B$, and the inference rule

If $\vdash A$, then $\vdash \boxplus\neg A$.

Common validity notions (possibly in a frame or model) will be freely used. Mathematically, the equivalence between K and K* is justified by the

Correspondence Theorem (Tehlikeli, 1985) Take the bijective translation * from $\mathcal{M}\mathfrak{L}(\Box)$ onto $\mathcal{M}\mathfrak{L}(\boxplus)$, which uniformly replaces $\Box$ by $\boxplus\neg$. For a K-model $\mathcal{M} = (W, R, V)$, let $\mathcal{M}^*$ denote the K*-model $(W, W^2\backslash R, V)$.

Then: (a) $K \vdash A$ iff $K^* \vdash A^*$.

(b) $\mathcal{M}, s \models A$ iff $\mathcal{M}^*, s \models A^*$, whence

(c) $K \models A$ iff $K^* \models A^*$.

Proof (a): straightforward induction on $\vdash$.

(b): straightforward induction on $\models$. #

Informally, the window $\boxplus$ may be pretty well interpreted as 'sufficiency', to the same extent at least to which 'necessity' is $\Box$ and 'possibility' is $\Diamond$.

Question (L. Ivanov) Is there in this line a natural (or, at least philosophical) interpretation of kite Φ: $x \models \Phi A$ iff $\exists y(\neg Rxy \ \& \ y \models \neg A)$? #

Metaphysically, R ($\neg$R) is the (in)accessibility, whence $\{A/ \ x \models \Box A\}$ captures the eternities for x, while $\{A/ \ x \models \boxplus A\}$ subsums the falsities of the eternity beyond x. Computationally, window-α-A, $[\![\alpha]\!]A$, states A's sufficiency as post-condition in a state, for program α's termination this last in.

By the Correspondence and the respective theorems for K, one gets:

Theorem K* is sound, (finitely) complete, decidable, compact, etc. #

Short historical notes on K* are left to a large discussion in the Epilogue.

So K* shares all traditional virtues of K, and foreseeably, cf. the Correspondence, all its deficiencies as well. For illustration, take the poly-modal case, where questions about modal-axiomatizability of relations between binary relations are of traditional interest. The class of three-relational K-frames determined by the property $R = S \cup T$ is modal-axiomatic (over K), by the scheme $[R]p \leftrightarrow [S]p \wedge [T]p$, whereas the properties $R = S \cap T$, or $R = \neg S$ do not determine such classes, after Theorem 8 of Goldblatt & Thomason (1975). Such a discrimination between intersection $\cap$ and union $\cup$ in K is a bit shocking for the democratic spirit of a Boolean consciousness. As expected, K* supports similar partiality, just reversing colours: $R = S \cap T$ is modally-axiomatizable over K*, via the scheme $[\![R]\!]p \leftrightarrow [\![S]\!]p \wedge [\![T]\!]p$, whereas union and complement turn out not to be.

This observation can be generalized. For a class $\mathfrak{F}$ of K-frames, let $\mathfrak{F}^* = \{F^*/\ F \in \mathfrak{F}\}$, where $(W, R)^* = (W, W^2 \backslash R)$ is regarded as a K^*-frame. Again by the Correspondence, we have:
<u>Theorem</u> $\mathfrak{F}^*$ is modal axiomatic over K^* iff $\mathfrak{F}$ is modal-axiomatic over K. #

This and Goldblatt & Thomason's Theorem 8 give necessary & sufficient conditions for a class of K^*-frames to be modal-axiomatic. Due to lack of space we do not explicitly mention these conditions here; just say that, not surprizingly, these last when compared to Goldblatt & Thomason's only transpose the relation R and its complement $\neg R$.

Thus necessity and sufficiency split the modal theory into two dual branches each of which spreads over less than a half of the Boolean realm. The complement $\neg R$, remaining outside the scope of both branches cannot be framed before uniting them: $[\neg R]A \leftrightarrow [\![R]\!]\neg A$, and $[\![\neg R]\!]A \leftrightarrow [R]\neg A$. Consequently, in this environment the "iff" modality is definable: "p is necessary & sufficient in x", i.e. $V(p) = R(x)$, iff $x \models \Box p \wedge \boxplus p$.

So the union of K and K^* appears to suggest a reliable base for governing the Boolean kingdom. This will be established in the next section.

2. THE BOOLEAN MODAL LOGIC $K^\sim$

The language of $K^\sim$ is $\mathcal{ML}(\Box, \boxplus)$, i.e. with $\{\Box, \boxplus\}$ as a modal fragment.
Semantics. Models for $K^\sim$ are Kripke models $\mathcal{M} = (W, R, V)$ with

$x \models \Box A$ iff $\forall y(Rxy \rightarrow y \models A)$, and
$x \models \boxplus A$ iff $\forall y(y \models A \rightarrow Rxy)$.

Abbreviations: $N(A, B) =_{DF} \Box A \wedge \boxplus\neg B$, $[U]A =_{DF} N(A, A)$, $\langle U\rangle A =_{DF} \Diamond A \vee \Diamond\!\!\!\!\cdot\ \neg A$.
Alternatively, in the language $\mathcal{ML}(N)$, one has $\Box A = N(A, 1)$ and $\boxplus B = N(1, \neg B)$.
Axiomatics (the non-Boolean part):

1. $N(A, B) \wedge N(A \rightarrow A', B \rightarrow B') \rightarrow N(A', B')$
2. $N(1, 1)$
3. $[U]A \rightarrow A$
4. $[U]A \rightarrow [U][U]A$
5. $A \rightarrow [U]\langle U\rangle A$

i.e. $[U]$ is an S5-modality, and the rule

(RN) If $\vdash A \rightarrow A'$ & $\vdash B \rightarrow B'$, then $\vdash N(A, B) \rightarrow N(A', B')$.

Omitting the trivial proof, we note that the axiomatics of $K^\sim$ is sound, i.e. all theorems of $K^\sim$ are valid. Such a system has been discussed earlier by van Benthem (1979): see Epilogue.

<u>Definition</u> A <u>generalized model</u> for $K^\sim$ is a quadruple (W, R, S, V), where

$R \cup S = W^2$, and
$x \models \Box A$ iff $\forall y(Rxy \rightarrow y \models A)$, and
$x \models \boxplus A$ iff $\forall y(y \models A \rightarrow \neg Sxy)$. #

In generalized models we still have: $x \models [U]A$ iff $\forall y(y \models A)$; though in general $R \cap S \neq \emptyset$, what makes them "generalized". The axiomatics of $K^\sim$ is sound with respect to generalized models as well.

<u>"Generalized" Completeness Theorem</u> If $K^\sim \not\vdash A$, then A is refuted in a generalized model.
<u>Proof</u> By the familiar, at least since Segerberg, canonical model techniques, consider the set W of maximal $K^\sim$ theories, and for $x \in W$ define:
$\Box x = \{B/\ \Box B \in x\}$, $\boxplus\neg x = \{B/\ \boxplus\neg B \in x\}$, and $[U]x = \{B/\ [U]B \in x\}$.
Note that $[U]x = \Box x \cap \boxplus\neg x$. Define also three relations R, S, T on W by:
Rxy iff $\Box x \subseteq y$; Sxy iff $\boxplus\neg x \subseteq y$; Txy iff $[U]x \subseteq y$.
By the axioms, one immediately obtains:

1. $T = R \cup S$, and 2. T is reflexive, transitive, symmetric.

Take now a maximal theory x such that $A \neg\in x$ and consider the <u>generated</u>

model $\mathcal{M}_x = (W_x, R_x, S_x, V_x)$, where: $W_x = \{y \in W / Txy\}$, $R_x = R \cap W_x^2$, $S_x = S \cap W_x^2$, $V_x(B) = \{y / y \in W_x \;\&\; B \in y\}$. Clearly, $R_x \cup S_x = W_x^2$, so $\mathcal{M}_x$ is a generalized model, refuting A at x. #

<u>Important Lemma</u> Each generalized model $\mathcal{M} = (W, R, S, V)$ is modally equivalent to some (generalized) model $\underline{\mathcal{M}} = (\underline{W}, \underline{R}, \underline{S}, \underline{V})$ with $\underline{R} \cap \underline{S} = \emptyset$.
<u>Proof</u> Take a disjoint copy $\mathcal{M}' = (W', R', S', V')$ of the initial model $\mathcal{M}$, where $x' \in W'$ is the image of $x \in W$, and construct $\underline{\mathcal{M}}$ as follows:

$\underline{W} =_{DF} W \cup W'$, $\underline{V}(p) =_{DF} V(p) \cup V'(p)$, for propositional variable p.

And $\underline{R}$, $\underline{S}$ are defined by cases, according to the following "important" construction (if no $\underline{R}st$ or $\underline{S}st$ is specified, then $\neg\underline{R}st$ or $\neg\underline{S}st$ is assumed):

If $Rxy \;\&\; Sxy$, then: $\underline{R}xy'$, $\underline{R}x'y$, $\underline{S}xy$, $\underline{S}x'y'$.
If $Rxy \;\&\; \neg Sxy$, then: $\underline{R}xy$, $\underline{R}xy'$, $\underline{R}x'y$, $\underline{R}x'y'$.
If $\neg Rxy \;\&\; Sxy$, then: $\underline{S}xy$, $\underline{S}xy'$, $\underline{S}x'y$, $\underline{S}x'y'$.
If $\neg Rxy \;\&\; \neg Sxy$, ..., but this cannot be the case, since $R \cup S = W^2$.

So we obtain a generalized model $\underline{\mathcal{M}} = (\underline{W}, \underline{R}, \underline{S}, \underline{V})$. The construction also gives: $\underline{R} \cup \underline{S} = \underline{W}^2$, $\underline{R} \cap \underline{S} = \emptyset$, i.e. $\underline{R} = \underline{W}^2 \setminus \underline{S}$. On the other hand, inducting on the complexity of B, using the "construction" on the modal step, we obtain: $V(B) \cup V'(B) = \underline{V}(B)$. Since $\mathcal{M}'$ copies $\mathcal{M}$, $\mathcal{M}$ and $\underline{\mathcal{M}}$ are modally equivalent. #
<u>Note</u> We owe this Important construction to Dimiter Vakarelov (see Epilogue).

<u>Completeness Theorem</u> $K^{\sim}$ is complete.
<u>Proof</u> By the "Generalized" Completeness, and the Important lemma. #

<u>Theorem</u> $K^{\sim}$ has the finite model property, and consequently is decidable.
<u>Proof</u> It is a routine task, cf. Segerberg (1971). Take the generated generalized model $\mathcal{M}_x$ (where x is the filter which does not contain the disprovable formula A) from the "Generalized" Completeness theorem. By the minimal filtration on $\mathcal{M}_x$ one obtains a finite generalized model refuting A. Then the Important construction leads to the finite countermodel desired. #
<u>Consequence</u> $K^{\sim}$ is conservative over K, and over K^*. #

Now we come to the point. Let π_0 be some index set, with $\vartheta \in \pi_0$, and let π be π_0's inductive closure under $\cup$, $\cap$, $-$. Let, for each $\alpha \in \pi$, N_α be a modality respecting the $K^{\sim}$ axioms. Let also the uniform substitution rule be assumed: $\vdash A(p)$ only if $\vdash A(B)$, for each formulae A, B, and proposition p.
<u>Theorem</u> The 5 axiom schemes $[\alpha\cup\beta]A \leftrightarrow [\alpha]A \wedge [\beta]A$, $N_{\bar\alpha}(A, B) \leftrightarrow N_\alpha(B, A)$, $[\![\alpha\cap\beta]\!]A \leftrightarrow [\![\alpha]\!]A \wedge [\![\beta]\!]A$, $[\vartheta]A \leftrightarrow [U_\alpha]A$, $[\![\vartheta]\!]1$, in addition to the rule:

$\vdash [\beta]p \rightarrow ([\alpha]p \rightarrow [\tau]p)$ only if $\vdash [\beta]p \rightarrow ([\![\tau]\!]\neg p \rightarrow [\![\alpha]\!]\neg p)$

yield a complete axiomatization for set-theoretic union, complement, intersection, and universe ($R(\vartheta) = W^2$), respectively.
<u>Proof</u> Via important constructions. (We thank to D. Vakarelov for pointining an error in a previous version of this theorem.) #

While the systems K and K^* share mirror-image advantages and drawbacks, the above theorem shows that $K^{\sim}$ enjoys the advantages of both avoiding the typical shortcomings of either, thus presenting a necessary and sufficient basis for Boolean speculations. So from the Boolean point of view, a bi-modal language, e.g. $\mathcal{ML}(\Box, \boxdot)$ or $\mathcal{ML}(N)$, seems more natural to deal with, at least as natural as, say, to develop arithmetics of all natural numbers, and not the odd one of solely the odd ones. We do not specify here the expressiveness capacities of the language of $K^{\sim}$, leaving this job to the next section.

3. THE PREDICATE <u>loop</u> IN $K^{\sim}$

We extend the language of $K^{\sim}$ to $\mathcal{ML}(\Box, \boxdot, \underline{loop})$ adding a propositional constant (or, possibly, a null-ary modal operator) <u>loop</u> with the semantics:

$x \models \underline{loop}$ iff Rxx .

Axioms for <u>loop</u> over $K^{\sim}$: $\vdash \underline{loop} \longrightarrow (\Box A \longrightarrow A)$
$\vdash \neg\underline{loop} \longrightarrow (\boxdot\neg A \longrightarrow A)$.

Theorem The axiomatics for $K^{\sim}_{LOOP}$ is sound and complete.
Proof Soundness is obvious. For the completeness we repeat the generalized canonical model construction from the "generalized" completeness theorem in the notation of which one has:

$\underline{loop} \in x$ implies Rxx, and $\neg\underline{loop} \in x$ implies Sxx.

We repeat now the important construction, modifying it only in the case when:

$x = y$ & Rxx & Sxx & $x \models \underline{loop}$.

In this case we exchange the places of $\underline{R}$ and $\underline{S}$ obtaining:

$\underline{R}xx$, $\underline{R}x'x'$, $\neg\underline{R}xx'$, $\neg\underline{R}x'x$, and
$\underline{S}xx'$, $\underline{S}x'x$, $\neg\underline{S}xx$, $\neg\underline{S}x'x'$. #

Theorem $K^{\sim}_{LOOP}$ has the fmp, and is decidable.
Proof Via the minimal filtration. #

Having $\Box$, $\boxminus$ and $\underline{loop}$ around, we reach a reliable base to express arbitrary "Boolean" modalities, which last are defined through a suitable sublanguage of $\mathcal{L}_1$. We first take four examples, representative enough, for such modalities $St(\underline{M})$ and their expressions $\underline{M}$ in $\mathcal{ML}(\Box, \boxminus, \underline{loop})$.

$St(\underline{M}A) =$	$\underline{M}A =$
Rxx	$\underline{loop}$
$\forall y(Rxy \vee Ryy \vee \neg[y/x]St(A))$	$\boxminus\neg(\underline{loop} \vee \neg A)$
$\forall y(\neg Rxy \vee Ryy \vee \neg[y/x]St(A))$	$\Box(\underline{loop} \vee \neg A)$
$\forall y(\neg Ryy \vee [y/x]St(A))$	$[U](\neg\underline{loop} \vee A)$

$\mathcal{L}_1$-Sublanguage Definition $\forall y\text{-}\mathcal{L}_1(Rxx, Rxy, Ryy) =_{DF} \{\varphi(x) \in \mathcal{L}_1 / \varphi$ is in prenex form with one free variable, x, at the most and one bound variable, y, at the most, the quantifier being $\forall$, and R occurs in the matrix only in Rxx, Rxy, Ryy (and not in Ryx)$\}$. #
Variations of this definition will reasonably reflect on the denotation.

Expressiveness theorem for $\mathcal{ML}(\Box, \boxminus, \underline{loop})$ Let $\varphi(x) \in Qy\text{-}\mathcal{L}_1(Rxx, Rxy, Ryy)$, where $Q \in \{\forall, \exists\}$. Then there exists a modal formula $\varphi^{\sim} \in \mathcal{ML}(\Box, \boxminus, \underline{loop})$ with

$St(\varphi^{\sim}) = \varphi$, hence with
$\mathcal{M}, s \models \varphi^{\sim}$ iff $\mathcal{M} \models \varphi[s]$, for each model $\mathcal{M}$ and state $s \in \mathcal{M}$.

Proof We shall only construct $\varphi^{\sim}$, the remaining being left to the reader. Let $\varphi(x) \in \forall y\text{-}\mathcal{L}_1(Rxx, Rxy, Ryy)$ and φ's matrix be in conjunctive normal form. (For an existential formula ψ we shall have $\psi^{\sim} = \neg(\neg\psi)^{\sim}$.) Now, distributing the quantifier $\forall y$ over the conjuncts, we obtain φ as a conjunction of $\forall$-quantified elementary disjunctions: $\varphi = \varphi_1 \wedge \ldots \wedge \varphi_k$.
Then we define $\varphi_i^{\sim}$ taking the sample of the four examples above and, finally, set $\varphi^{\sim}$ to be $\varphi_1^{\sim} \wedge \ldots \wedge \varphi_k^{\sim}$. #

This proof leads to the above-promissed expressiveness of $K^{\sim}$.
Expressiveness theorem for $\mathcal{ML}(\Box, \boxminus)$ Drop Rxx, Ryy, $\underline{loop}$ from last theorem. #

Another small demonstration of the capacities of our language, provided the identity, or dummy, relation δ with $x\delta y$ iff $x=y$ is present, is in expressing Bull's (1968) operator Q:

$\mathcal{M} \models QA$ iff $\forall y \forall z(y \models A$ & $z \models A \longrightarrow y = z)$, and consequently
$\mathcal{M} \models Q'A$ iff $\exists! y(y \models A)$, i.e. iff $\exists y \forall z(y \models A$ & $(z \models A \rightarrow z = y))$.

We have: $QA \leftrightarrow \langle U\rangle[\delta]A$, and
$Q'A \leftrightarrow \langle U\rangle(A \wedge [\delta]A)$.

Questions What is the expressiveness of $\mathcal{ML}(\langle U\rangle, \delta)$, and of $\mathcal{ML}(\Box, \boxminus, \delta)$? #

Concluding Remarks. We reached in these sections a kind of universal language, in which all Boolean operations are axiomatizable, and almost all modalities are definable. Out of "almost all" remain the cases in which Ryx (y the bound variable) occurs in φ. Here we touch the converse relation $R^{\smile}$, $R^{\smile}xy$ iff Ryx, which leads out of the Boolean realm into purely relational considerations, and consequently to the familiar tense logic. Therefore, take a new modality $\Box^{\smile}$ (over K or $K^{\sim}$) with the tense semantics

$x \models \Box^{\smile}A$ iff $\forall y(Ryx \longrightarrow y \models A)$, and the usual converse axioms

$A \to \Box\Diamond^{\cup}A$,

$A \to \Box^{\cup}\Diamond A$, and

$[U]A \to \Box^{\cup}A$, if over $K^{\sim}$.

This, although not so necessary, is sufficient for axiomatization of the non-Boolean relational operations composition and converse:

$\langle\alpha \circ \beta\rangle A \leftrightarrow \langle\alpha\rangle\langle\beta\rangle A$,

$\langle\alpha^{\cup}\rangle A \leftrightarrow \langle\alpha\rangle^{\cup}A$, and $\langle\alpha^{\cup}\rangle^{\cup}A \leftrightarrow \langle\alpha\rangle A$.

For expressibility, however, although necessary, this means does not suffice: $\Box^{\cup}$ covers only the negative occurrences of Ryx in φ, the positive ones being manageable by the remaining, the fourth possible, modality $\blacksquare^{\cup}$: $x \models \blacksquare^{\cup}A$ iff $\forall y(y \models A \to Ryx)$. This exhausts modal Q-$\mathcal{L}_1$-definability:

<u>Q-$\mathcal{L}_1$ Expressiveness Theorem</u> For every $\varphi(x) \in Qy$-$\mathcal{L}_1$(Rxx, Ryy, Rxy, Ryx) there is a modal formula $\varphi^{\sim} \in \mathcal{ML}(\Box, \blacksquare, \underline{\text{loop}}, \Box^{\cup}, \blacksquare^{\cup})$ with $St(\varphi^{\sim}) = \varphi$, hence with $\mathcal{M},x \models \varphi^{\sim}$ iff $\mathcal{M} \models \varphi[x]$. #

In the long run, all this is aimed at explicit description of the sets: $\{St(\varphi) \in \mathcal{L}_1 / \varphi \in \text{some } \mathcal{ML}\}$ and $\{St^{-1}(\Psi) \in \text{some } \mathcal{ML} / \Psi \in \text{some portion of } \mathcal{L}_1\}$, i.e., at first-order-definability and modal-expressibility results. And this can be embedded in, call it, "general modal program" which asks: what model condition under what truth definition responds to what modal axiom, where "responds to" means "guarantees" or "is guaranteed by", or both. This general modal program is in the spirit of van Benthem's (1984) "perhaps most basic question" concerning the interplay of the two 'degrees of freedom' in semantic explanation: truth definition and model condition, leaving a third parameter free - the modal axioms to be satisfied.

4. MODELS WITH NAMES FOR POSSIBLE WORLDS

The above expressiveness results state a relationship between one unmovable predicate language $\mathcal{L}_1$ and several flexile modal languages. On the modal side we examine also <u>loop</u> which at first glance appears to be a modal counterpart of the usual first-order equality =, and so it appeals to the "modal program" for the source language $\mathcal{L}_1$ to be replaced by its "equalized" version $\mathcal{L}_1^{=}$. Revised like this, the modal program however immediately fails: <u>loop</u> contains only an equality relevant to R. Indeed, take the simplest $\mathcal{L}_1^{=}$ formula $\Psi = \forall y(x=y)$, whose truth in a (state of) model or frame is equivalent with the universe's cardinality to be 1. There is no modal formula $\varphi \in \mathcal{ML}(\Box, \blacksquare, \underline{\text{loop}}, \Box^{\cup}, \blacksquare^{\cup})$ with $\forall\mathcal{M}(\mathcal{M} \models \varphi$ iff $card(\mathcal{M}) = 1)$. (For, e.g., the models $(\{x\}, \emptyset, V)$ and $(\{y,z\}, \emptyset, U)$, where $V^{-1}(x) = U^{-1}(y) = U^{-1}(z)$, are modally indiscernible in that language.)

In fact, it can hardly be expected to express the equality of states while no special means are available identifying the states themselves. All we have at our disposal in $\mathcal{ML}$'s are propositional letters interpreted as subsets of the universe, $V(p) \subseteq W$, and not even a syntactical hint is there for particular individuals the equality of which is the target. In this section we enhance the expressive power of $\mathcal{ML}$'s by adding to the syntax names for the states (or constants) with the natural for "name" semantics, and appropriate axiomatics. Such move is, modulo traditional virtues, quite a natural one: the traditional modal theory of anonymous worlds looks as unnatural as, say, an arithmetics in the language of which predicates are only available (e.g. 'even', 'odd', 'prime', '=0', etc) instead of individual variables. Thus we continue our works initiated in Passy & Tinchev (1985a,b) and settle in a modal background an idea (called "combinatory") which proved curious, if not even useful, in the ambience of the dynamic logic.

<u>Definition</u> The language $\mathcal{ML}_N(\Box)$ of named models contains two sorts of propositional variables: ordinary ones $p_1, p_2, \ldots$ and names (or, constants) $c_1, c_2, \ldots$ Formulae are built starting from variables and names applying the Boolean connectives and the modalities $\Box$ and $\Diamond$. #

Semantics. Models for $\mathcal{ML}_N$ are triples (W, R, V) where (W, R) is a frame, and the valuation V, besides the ordinary truth conditions, satisfies also: $V(c) \subseteq W$ is either empty or a singleton, for each name c. #

In modal-axiomatic capacities $\mathcal{ML}_N(\Box)$ is closer to $\mathcal{ML}(\Box, \boxdot)$:
1. R is irreflexive iff $(W, R) \models (c \to \Box\neg c)$.
2. In three-relational case,
$R = S \cap T$ iff $(W, R) \models \langle R\rangle c \leftrightarrow \langle S\rangle c \wedge \langle T\rangle c$.

However, we leave open the following
<u>Questions</u> Describe, in the spirit of Goldblatt & Thomason's theorems, the classes of frames modal-axiomatic over the respective $\mathcal{ML}$'s and $\mathcal{ML}_N$'s. #

<u>Definition</u> A model is <u>total</u>, if $\forall c(V(c) \neq \emptyset)$, i.e. when each name names some world. A model is <u>surjective</u>, if $\forall x_{x \in W}\exists c(V(c) = \{x\})$, i.e. if all worlds have names. A model is <u>standard</u>, if it is both total and surjective. These notions yield respectively total, surjective and standard validity, denoted by $\models_{TOT}$, $\models_{SUR}$, $\models_{STAND}$. #

<u>Notes</u> 1. The original "combinatory" models from our previous papers are standard, even <u>very standard</u> having an extra S5 modality [U] interpreted as the Cartesian square of the universe. The very standard language suffices for modal-axiomatization of one-world universes - fixing a name c, one has: $\mathrm{card}(\mathcal{M}) = 1$ iff $\mathcal{M} \models [U]c$.
2. Surjective (hence also standard) models are based on frames which are at most countable. On finite and countable frames validity and surjective validity coincide. On uncountable frames surjective validity is trivially fulfiled: all formulae are of course valid. #

We extend the translation St: $\mathcal{ML}(\Box) \longrightarrow \mathcal{L}_1$ to ST: $\mathcal{ML}_N(\Box) \longrightarrow \mathcal{L}_1^-$ defining $ST(c_i) = (x = y_i)$, for each name c_i, where $y_1, y_2, \ldots$ are, say, "half" of the individual variables of $\mathcal{L}_1^-$, and other than x.
<u>Fact</u> $(W, R) \models_{TOT} A$ iff $(W, R) \models \forall P_1 \ldots \forall y_1 \ldots \forall x ST(A)$. #

In order to axiomatize the set K_N of valid formulae we introduce, following Goldblatt (1982), the notions of necessity form ($\Box$-form) and possibility form ($\Diamond$-form).

<u>Definition</u> 1. \$ is a $\Box$-form.	1'. \$ is a $\Diamond$-form.
If L is a $\Box$-form, A - a formula,	If M is a $\Diamond$-form, A - a formula,
then 2. $A \longrightarrow L$ is a $\Box$-form, and	then 2'. $A \wedge M$ is a $\Diamond$-form, and
3. $\Box L$ is a $\Box$-form.	3'. $\Diamond M$ is a $\Diamond$-form. #

Each form L or M has a unique occurrence of the symbol \$; if it is replaced by a formula A of $\mathcal{ML}_N(\Box)$ a formula results, denoted by L(A) or M(A).

Axiomatics of K_N. We add to the deductive system of K over $\mathcal{ML}_N(\Box)$, also:
Ax_N. $M(c \wedge A) \longrightarrow L(c \longrightarrow A)$, for each name c, $\Diamond$-form M, and $\Box$-form L, which reflects the behaviour of V(c).
<u>Fact</u> If $\vdash A \longrightarrow B$, them $\vdash L(A) \longrightarrow L(B)$ and $\vdash M(A) \longrightarrow M(B)$. #
<u>Theorem</u> The axiomatics for K_N is sound and complete.
<u>Proof</u> Let, for the completeness, A be disprovable. Consider the standard canonical model construction and let x be a maximal theory such that $\neg A \in x$. Take the submodel (W_x, R_x, V_x) generated by x. We have:
<u>Lemma</u> For a name c, and states $y, z \in W_x$, $c \in y$ & $c \in z$ imply $y = z$.
<u>Proof of the lemma</u> Assume the contrary, i.e. $y \neq z$. Then there is B such that: $B \in y$ & $\neg B \in z$. So $c \wedge B \in y$ and since $y \in W_x$, for some $\Diamond$-form M, $M(c \wedge B) \in x$. By Ax_N $L(c \longrightarrow B) \in x$, for any $\Box$-form L. But then clearly $c \longrightarrow B \in z$. Thus $B \in z$ - a contradiction.
By the above, (W_x, R_x, V_x) is a model where A is refuted. #

The notion of surjective validity can be captured by the following axiomatic system K_N^{SUR}: add to K_N the (infinitary) inference rule
COV If $\vdash L(\neg c)$, for all names c, then $\vdash L(0)$ (here L is any $\Box$-form).

Soundness theorem for K_N^{sur}. If $K_N^{sur} \vdash A$, then $\models_{sur} A$.
Proof The rule COV preserves truth in surjective models. #

Notes 1. In fact, the rule COV is not infinitary; it is interchangeable with COV*: If $\vdash L(\neg c)$, for some c not occurring syntactically in L, then $\vdash L(0)$.
2. Both Ax_N and COV look more neat in the context of dynamic logic, where no explicit reference to $\Box$- and $\Diamond$-forms is needed (α, β are PDL programs):
(Ax_N') $\langle\alpha\rangle(c \wedge A) \longrightarrow [\beta](c \rightarrow A)$
COV' If $\vdash [\alpha]\neg c$, for all c, then $\vdash [\alpha]0$. #

Next we give completeness proof for K_N^{sur} which proceeds in two steps. On the first step, as usual, we place the disprovable formula in a maximal theory, cf. e.g. Goldblatt (1982), or Rasiowa-Sikorski's (1963) lemma on Q-filters. Secondly, instead of taking other maximal theories, we use the names in a typical Henkin way to build the counter-model.
Definition A theory is any set of formulae containing all K_N^{sur} theorems which is closed under MP and COV. Unless otherwise specified, theories will be consistent. For a set of formulae X let Th(X) be the smallest theory containing X, and let $Th(X, A) =_{DF} Th(X \cup \{A\})$. #
Deduction lemma $B \in Th(T, A)$ iff $A \longrightarrow B \in T$, for each theory T. #
Lindenbaum lemma Any theory T can be extended to a maximal one.
Proof Enumerate all formulae $A_0, A_1, \ldots$ Let T_0 be T. Assume T_n defined and consistent. If $Th(T_n, A_n)$ is consistent, then $T_{n+1} =_{DF} Th(T_n, A_n)$.
If no, then study the graphical form of A_n: a) if $A_n = L(0)$ for some $\Box$-form L, then for at least one name c, $L(\neg c) \neg\in T_n$ - otherwise, by the COV-closeness of T_n we would get $A_n \in T_n$. (By the Deduction lemma this leads to a contradiction.) In this case let $T_{n+1} = Th(T_n, \neg L(\neg c))$.
b) if A has any other graphical form, then let $T_{n+1} = T_n$.
This construction produces an infinite chain of growing theories. Their union T' is a theory, too, and moreover T' is maximal theory containing T. #

For a maximal theory T, let $N_T =_{DF} \{c / M(c) \in T$ for some $\Diamond$-form $M\}$.
For $c,d \in N_T$, let $c \sim d$ iff for some M, $M(c \wedge d) \in T$.
Lemma (maximal theory forms) Where M is a $\Diamond$-form, L - a $\Box$-form, and $c \in N_T$.
(*) $M(c \wedge A) \in T$, for some M iff $L(c \longrightarrow A) \in T$, for all L.
(*) $M(c \wedge A) \neg\in T$, for all M iff $M'(c \wedge \neg A) \in T$, for some M'.
(*) $M(c \wedge \Diamond A) \in T$ iff $M(c \wedge \Diamond d) \in T$ & $M'(d \wedge A) \in T$, for some M' and $d \in N_T$.
Proof By Ax_N and COV-closeness of T. #
Lemma $\sim$ is an equivalence relation on N_T .
Proof Use the above lemma. #

Definition For a maximal theory T, let $\mathcal{M}_T = (W_T, R_T, V_T)$, where $W_T = N_T/\sim$, $R_T = \{(|c|, |d|) / \exists M(M(c \wedge \Diamond d) \in T)\}$, and $V_T = \{|c| / \exists M(M(c \wedge A) \in T)\}$. #
Henkin model lemma $\mathcal{M}_T$ is a surjective model.
Proof The truth conditions have to be checked, i.e. $|c| \in V_T(\Box A)$ iff $\forall |d| (R_T|c||d|$ implies $|d| \in V_T(A))$ etc, and they follow from the lemma about forms. Surjectivity is clear: $V_T(c) = \{|c|\}$, for $|c| \in W_T$. #
Completeness theorem for K_N^{sur} If $\models_{sur} A$, then $K^{sur} \vdash A$.
Proof For a disprovable formula A, there is a maximal theory T with $\neg A \in T$, and it can be easily verified that A is refuted in the model $\mathcal{M}_T$. #

The next proves the rule COV redundant for this basic system (but not for the extensions).
Lemma K_N has the finite model property.
Proof Standard filtration. #

Note now that the finite model refuting A (obtained by the above filtration) can be transformed into a surjective model (refuting A) by redefining the valuations of names not occurring in A. Moreover, the model can be "totalized" (hence "standardized"), by adding, if necessary, one new world in order to ensure totality. Thus we have

Theorem K_N and K_N^{SUR} coincide as sets of theorems. #

Corollaries 1. Both K_N and K_N^{SUR} are decidable.

2. For a fixed formula, all four kinds of validity coincide, and the problem whether it is valid is decidable. #

Adding names with standard interpretation to the language of $K^{\sim}$, one obtains a very standard language $\mathcal{ML}_N(\Box,\boxplus)^{STAND}$ in which the equality can be spoken of. We have, for a model $\mathcal{M}$ with $V(c) = s$ and $V(d) = t$, that $s = t$ iff $\mathcal{M} \models \langle U\rangle(c\wedge d)$. Indeed, identifying individuals, the language $\mathcal{ML}_N(\Box,\boxplus)^{STAND}$ serves as the modal analogue sought, of the first-order language with equality $\mathcal{L}_1^=$, and one has:

Expressiveness theorems for named modal languages Let us replace, in the expressiveness theorems, section 3, $\mathcal{L}_1$ by $\mathcal{L}_1^=$, and the $\mathcal{ML}$'s - by the respective very standard $\mathcal{ML}_N$'s. These are still the cases. #

Such a nice language deserves to be axiomatized. We propose the following axioms and rules for $K_N^{\sim\ STAND}$.

1. Axioms and rules of $K^{\sim}$ (over the new language)
2. $\langle U\rangle(c\wedge A) \longrightarrow [U](c \longrightarrow A)$ (instead of Ax_N)
3. $\langle U\rangle c$ (guaranteeing totality)
4. $\Diamond c \leftrightarrow \boxplus c$ (the implication $\leftarrow$ is in fact a theorem)
5. The rule COV for the suitably extended notion of a modal form.

Standard-completeness theorem $\vdash_{STAND} A$ iff $\models_{STAND} A$.

Proof Soundness is clear. If A is not a "standard" theorem, then the Henkin model construction from above will extract a model (W_T, R_T, V_T) out of a maximal theory not containing A. Here W_T is the set of all names factorized by the relation $\langle U\rangle(c\wedge d) \in T$. Further $R_T =_{DF} \{(|c|,|d|)/ \langle U\rangle(c\wedge\Diamond d) \in T\}$, and $V_T(B) =_{DF} \{|c|/ \langle U\rangle(c\wedge B) \in T\}$. Axiom 4 guarantees the correct relationship between $\Box$ and $\boxplus$. Axiom 3 yields totality of the model. So we have a standard model where A is refuted. #

Concluding remarks. The names on the modal soil provide an effective tool for a first-order quantification, cf. Passy & Tinchev (1985b). Let the Quantified $\mathcal{ML}_N$ extend the respective $\mathcal{ML}_N$, allowing, on the inductive step, formulae of the type $\forall cA$, where c is a name, and A is a formula with the semantics: $\mathcal{M}, s \models \forall cA$ iff for each d, $\mathcal{M}, s \models [d/c]A$. Thus at long last we reach on a modal level the expressibility of the entire $\mathcal{L}_1^=$ language.

The $\mathcal{L}_1^=$-Expressiveness Theorem For every formula φ in $\mathcal{L}_1^=$ of one free variable there is a closed formula $\varphi^{\sim}$ in Quantified $\mathcal{ML}_N(\Box,\boxplus)^{STAND}$ with: $\mathcal{M} \models \varphi[s]$ iff $\mathcal{M}, s \models \varphi^{\sim}$.

Proof Let φ be in prenex form with bound variables $y_1, y_2 \ldots$, and free variable x. Each atomic subformula of φ's matrix has one of the forms $z=y$, Rzy, $P(z)$ or the negations of these, where $z, y \in \{x, y_1, y_2, \ldots\}$. Let Ψ be the quantified modal formula obtained by uniform replacement of all variables z in φ's prefix by a name c_z, and all occurrences of $z=y$, Rzy, $P(z)$ in φ's matrix by $\langle U\rangle(c_z\wedge c_y)$, $\langle U\rangle(c_z\wedge\langle R\rangle c_y)$, $\langle U\rangle(c_z\wedge p)$, respectively. Finally, define $\varphi^{\sim}$ to be $\exists c_x(c_x\wedge\Psi)$. #

Another impact of the names is in first-order definability. Call an $\mathcal{ML}_N$-formula pure if it does not contain propositional variables, i.e. consists only of names, 0 and connectives. Clearly, in terms of van Benthem (1977), pure formulae are first-order definable. In particular, the "pure" instant $\Box\Diamond c \rightarrow \Diamond\Box c$ of the famous first-order undefinable formula $\Box\Diamond p \rightarrow \Diamond\Box p$, is already $\mathcal{L}_1^=$-definable, via the sentence: $\forall z(\forall y(Rxy \rightarrow Ryz) \rightarrow \exists y(Rxy \,\&\, \forall t(Ryt \rightarrow t=z)))$.

Theorem If A is a pure formula, then A is complete.

Sketch of the proof A defines a first-order condition φ true in every frame where A is standardly valid (frames are at most countable). Now translating φ into $\varphi^{m} \in$ Quantified $\mathcal{ML}_N(\Box, \boxplus)^{STAND}$ we can obtain a quantified theory containing φ^{m} which is conservative over standard $K_N^{\sim} + \{A\}$. Now the Henkin model of the theory is based on a frame for which we can check φ. #

Conjecture If A is first-order definable, then A is complete. #

In the long run, all this aims at a "general named modal program", which is a matter of another, probably longer, discussion.

EPILOGUE: The Ghost of the Modality vs. the Spirit of Kripke

The present paper gives another flavour to the series of "combinatory" investigations of the three of us, initiated in Passy (1984), Passy & Tinchev (1985a,b), Tinchev (1986). The names in the modal logic give a satisfactory solution to some problems, and supplying the deficiences of Kripke nature, they as if put a (first-) order in the modal atonality. The combinatory solutions given, however, do not explain the reason and the entity of this last. Such an explanation in the person of K^*, together with some other solutions among which $K^\sim$, $K^\sim_{LOOP}$, $K^\sim_N$ is probably what is gained here.

In particular, we claim that K^* presents itself as an equipollent counterpart of K, and being such it gives (a partial, at least) answer to van Benthem's (1984, p. 385) query about truth definitions alternative to Kripke's and working equally well (hence, equally bad). So it is not to be expected the idea of K^* to be a novelty in the modal field, and the referees agree on this: the semantics of $\boxminus$ repeats the semantics for negation in quantum logic given in Goldblatt (1974); moreover, our translation * and the Correspondence theorem have their similitudes in Goldblatt's definition 4.2 and lemma 4.3. Reportedly, Humberstone has also axiomatized similar modalities. D. Vakarelov (1974) has a semantics for the negation which is the semantics of $\Box\neg$ from K. The essay however closest to our own also belongs to van Benthem (1979), where he studies some modal operators in a deontic context. Modulo philosophical background, our K^* and $K^\sim$ turn to be van Benthem's K_a - logic of permissions and K_D - permissions & obligations, his mixing principle being a theorem of $K^\sim$. So the completeness theorems for K^* and $K^\sim$, up to the proof strategies, can be attributed to van Benthem.

Concerning the completeness proof strategy presented for $K^\sim$ and $K^\sim_{LOOP}$, we use the construction called Important, which was invented by Dimiter Vakarelov (for some other completeness results). This construction replaces one of our own, which - although of smaller size - is less transparent. And all these constructions originate probably from Sahlqvist. Concluding the first part of the paper, we frame the largely discussed in modal logic "reflexivity" by adding the <u>loop</u> predicate. Such a step can be thought of as inspired by the familiar in dynamic logic constructs δ (identity program), or by cycle-predicate or by iteration, and when so it also appears not to be something very original. Moreover, the natural extensions of $K^\sim_{LOOP}$ from section 3 lead to the ordinary tense logic, as notes one of the referees. Thus the first part of the paper suggests just a new arrangement of common notions and is only a step towards fulfilling the "general modal program".

In the second part, we fuse together these and the idea of the names, which, cf. the Postscriptum of Passy (1984), on its part can also be thought of as rearrangement of folklore speculations. Hence we join here some roving notions proving some theorems for them, some of which one may even find inadmissibly simple. However, our aim is not at all notions' introduction or theorems' proving: such are quite abundant in modal logic. Our goal is to propose another viewpoint towards the latter.

The viewpoint sought is, as opposed to Kripke's, not discriminating first-order phenomena when regarded thereof. Consequently introducing several $\mathcal{ML}$'s and their named versions, we are approaching, and hopefully encircling, Hermann Weyl's 1940 "ghost of the modality". The question unfortunately still remains open: which of all these logics is better, and is there a best one, with "better" and "best" referring to some aesthetical order predicted by the pure mathematical reason.

ACKNOWLEDGEMENTS

Van Benthem, Goldblatt, Segerberg, Vakarelov, the two referees: these six, or at least four, persons, incidentally members of the same Program Committee, contributed - probably unknown to them - in some way to our understanding of modality. Rumi Draganova, Christo Kazasov, Ivan Soskov and, of course, Kuzyo Kuzev were very helpful, and made our work on the computer more than a pleasure. Special thanks are due to Iliana Sherkova, for polishing the language and encouraging the style of this paper.

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ADDED IN PROOF

On November 14, 1986, after having prepared the above - then hoped to be final - version of this exhaustible typescript, we received offprints of the papers of Lloyd Humberstone mentioned, kindly sent to us by the author.
In the former paper, essentially, $K^{\sim}$ is considered: by curious coincidence, on one hand, the axioms suggested there are exactly as van Benthem's mixing principles, and, on the other, the completeness proof goes through Beth's semantic tableaus (as this also is claimed in van Benthem's (1979) abstract); some other valuable observations on "complementary" modalities are stated.
In the latter paper, the "intersection" of modalities is mentioned, in an epistemic context. We do regret for not having these worthy papers earlier.