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On Expressive Description Logics
with Composition of Roles
in Number Restrictions

Fabio Grandi

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Fabio Grandi

CSITE-CNR and DEIS,

Alma Mater Studiorum – Università di Bologna,

Viale Risorgimento 2, I-40136 Bologna, Italy

Email: `fgrandi@deis.unibo.it`

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Abstract

Description Logics are knowledge representation formalisms which have been used in a wide range of application domains. Owing to their appealing expressiveness, we consider in this paper extensions of the well-known concept language $\mathcal{ALC}$ allowing for *number restrictions* on complex role expressions. These have been first introduced by Baader and Sattler as $\mathcal{ALCN}(M)$ languages, with the adoption of role constructors $M \subseteq \{\circ, {}^-, \sqcup, \sqcap\}$. In particular, they showed in 1999 that, although $\mathcal{ALCN}(\circ)$ is decidable, the addition of other operators may easily lead to undecidability: in fact, $\mathcal{ALCN}(\circ, \sqcap)$ and $\mathcal{ALCN}(\circ, {}^-, \sqcup)$ were proved undecidable.

In this work, we further investigate the computational properties of the $\mathcal{ALCN}$ family, aiming at narrowing the decidability gap left open by Baader and Sattler's results. In particular, we will show that $\mathcal{ALCN}(\circ)$ extended with inverse roles both in number and in value restrictions becomes undecidable, whereas it can be safely extended with qualified number restrictions without losing decidability.

1   Introduction

Description Logics are a family of first-order formalisms that have been found useful for domain knowledge representation in several application fields, from database design —including conceptual, object-oriented, temporal, multimedia and semistructured data modeling— to software engineering and ontology management (e.g. [1, 2, 8, 10, 11, 12, 15, 20, 3]). Different Description Logics provide for *constructors* which can be used to combine atomic *concepts* (unary predicates) and *roles* (binary predicates) to build *complex* concepts and roles. The available constructors characterize the description language as to *expressiveness* and *computational behaviour* (decidability and complexity) of the basic reasoning tasks like concept satisfiability and subsumption.

Well-known Description Logics are $\mathcal{ALC}$ [21], which allows for Boolean propositional constructors on concepts and (universal and existential) value restrictions on atomic roles, and its extension $\mathcal{ALCN}$ [13, 19] introducing (non-qualified) *number restrictions* on atomic roles. Basic inference problems for both these Description Logics are PSPACE-complete [13, 19]. However, in order to better fulfil requirements of real-world application domains, more expressive extensions of the basic concept languages have been investigated. One direction along which useful extensions have been sought is the introduction of *complex* roles under number restrictions. In fact, considering role composition ($\circ$), inversion ($^-$), union ($\sqcup$) and intersection ($\sqcap$), expressive extensions of $\mathcal{ALCN}$ can be defined as $\mathcal{ALCN}(M)$ with the adoption of role constructors $M \subseteq \{\circ, ^-, \sqcup, \sqcap\}$ [4]. By allowing (different kinds of) complex roles also in value restrictions, different families of Logics can also be defined: for example $\mathcal{ALC}_+\mathcal{N}$ (or $\mathcal{ALC}_{\mathrm{reg}}\mathcal{N}$) allows the transitive closure of atomic roles (or regular roles) under value restrictions [4, 9]. Also Logics $\mathcal{ALC}\bar{\mathcal{N}}(M)$, allowing for the same role constructors either in value *and* in number restrictions, can be considered [14]. Further extensions involve the introduction of *qualified* number restrictions [18] on complex roles, giving rise to $\mathcal{ALCQ}(M)$ Logics. Since qualified number restrictions also allow to express value restrictions, we have the inclusions $\mathcal{ALCN}(M) \subseteq \mathcal{ALC}\bar{\mathcal{N}}(M) \subseteq \mathcal{ALCQ}(M)$ as far as expressiveness (and compexity) are concerned. Therefore, for instance, undecidability of $\mathcal{ALCN}(M)$ directly extends to $\mathcal{ALC}\bar{\mathcal{N}}(M)$ and $\mathcal{ALCQ}(M)$, whereas decidability of $\mathcal{ALCQ}(M)$ implies decidability of $\mathcal{ALCN}(M)$ and $\mathcal{ALC}\bar{\mathcal{N}}(M)$.

Our present investigation is aimed at improving the (un)decidability results presented by Baader and Sattler in [4] for $\mathcal{ALCN}$ extensions including composition of roles ($\circ$). In particular, they proved that concept satisfiability in $\mathcal{ALCN}(\circ, \sqcap)$ and $\mathcal{ALCN}(\circ, ^-, \sqcup)$ is undecidable via reduction of a domino problem, and provided a sound and complete Tableau algorithm for deciding satisfiability of $\mathcal{ALCN}(\circ)$-concepts. They also observed that $\mathcal{ALCN}(^-, \sqcup, \sqcap)$ is decidable since $\mathcal{ALCN}(^-, \sqcup, \sqcap)$-concepts can easily be translated into a formula in $\mathcal{C}^2$ [7], that is the two-variable FOL fragment with counting quantifiers, which has proved to be decidable [17]. In fact, satisfiability of $\mathcal{C}^2$ formulae can be decided in NEXPTIME [22] if unary coding of numbers is used (which is a common assumption in the field of Description Logics; if binary coding is adopted we have a 2-NEXPTIME upper bound). We can further observe that a similar translation is still possible when *qualified* number restrictions are considered and, thus, also $\mathcal{ALCQ}(^-, \sqcup, \sqcap)$ and $\mathcal{ALC}\bar{\mathcal{N}}(^-, \sqcup, \sqcap)$ are decidable.

In this paper, we consider extensions of $\mathcal{ALCN}(\circ)$ with role inversion ($\mathcal{I}$) or qualified number restrictions ($\mathcal{Q}$), whose decidability status, to the best of our knowledge, is still

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$$
\begin{array}{llll}
C, D \rightarrow & A \mid & A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} & \text{atomic concept} \\
& \top \mid & \top^{\mathcal{I}} = \Delta^{\mathcal{I}} \\
& \bot \mid & \bot^{\mathcal{I}} = \emptyset \\
& \neg C \mid & (\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
& C \sqcap D \mid & (C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
& C \sqcup D \mid & (C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
& \forall R.C \mid & (\forall R.C)^{\mathcal{I}} = \{ i \in \Delta^{\mathcal{I}} \mid \forall j.\ R^{\mathcal{I}}(i,j) \Rightarrow C^{\mathcal{I}}(j) \} \\
& \exists R.C \mid & (\exists R.C)^{\mathcal{I}} = \{ i \in \Delta^{\mathcal{I}} \mid \exists j.\ R^{\mathcal{I}}(i,j) \wedge C^{\mathcal{I}}(j) \} \\
& \exists^{\geq n} R \mid & (\exists^{\geq n} R)^{\mathcal{I}} = \{ i \in \Delta^{\mathcal{I}} \mid \sharp\{ j \in \Delta^{\mathcal{I}} \mid R^{\mathcal{I}}(i,j) \} \geq n \} \\
& \exists^{\leq n} R \mid & (\exists^{\leq n} R)^{\mathcal{I}} = \{ i \in \Delta^{\mathcal{I}} \mid \sharp\{ j \in \Delta^{\mathcal{I}} \mid R^{\mathcal{I}}(i,j) \} \leq n \} \\
* & \exists^{\geq n} R.C \mid & (\exists^{\geq n} R.C)^{\mathcal{I}} = \{ i \in \Delta^{\mathcal{I}} \mid \sharp\{ j \in \Delta^{\mathcal{I}} \mid R^{\mathcal{I}}(i,j) \wedge C^{\mathcal{I}}(j) \} \geq n \} \\
* & \exists^{\leq n} R.C \mid & (\exists^{\leq n} R.C)^{\mathcal{I}} = \{ i \in \Delta^{\mathcal{I}} \mid \sharp\{ j \in \Delta^{\mathcal{I}} \mid R^{\mathcal{I}}(i,j) \wedge C^{\mathcal{I}}(j) \} \leq n \} \\
R, S \rightarrow & P \mid & P^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} & \text{atomic role} \\
* & R^{-} \mid & (R^{-})^{\mathcal{I}} = \{ (i,j) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid R^{\mathcal{I}}(j,i) \} \\
* & R \circ S \mid & (R \circ S)^{\mathcal{I}} = \{ (i,j) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \exists k.\ R^{\mathcal{I}}(i,k) \wedge S^{\mathcal{I}}(k,j) \}
\end{array}
$$

Figure 1: Syntax and model-theoretic semantics of $\mathcal{ALCN}$ and its extensions (marked with $*$) considered in this paper.

unknown. In particular, we will show in Sec. 2 (via reduction of a domino problem) undecidability of $\mathcal{ALCN}(\circ)$ extended with inverse roles both in value and in number restrictions (which we can call $\mathcal{ALCN}(\circ)\mathcal{I}$, but we also show in Sec. 2 that it is a syntactic variant of $\mathcal{AL\bar{C}N}(\circ,^{-})$) is undecidable. This result implies undecidability of $\mathcal{ALCQ}(\circ,^{-})$, whereas decidability of "pure" $\mathcal{ALCN}(\circ,^{-})$ remains an open question. On the other hand, we will show how the decidability results of [4] lift up to $\mathcal{ALCQ}(\circ)$. In particular, we will show in Sec. 3 that $\mathcal{ALCQ}(\circ)$-concept satisfiability is decidable and provide an effective decision procedure in the form of a tableau-based algorithm, which extends the $\mathcal{ALCN}(\circ)$ Tableau proposed by Baader and Sattler [4]. In a similar way as done in [4], we will also show that the decision algorithm can be extended to cope with qualified number restrictions on union and/or intersections of role chains of the same length.

Preliminaries on Description Logics

Description Logics expressiveness is based on the definition of complex concepts and roles, which can be built with the help of available constructors, starting from a set af (atomic) concept names NC and a set of (atomic) role names NR. A DL system, enabling concept descriptions to be interrelated, allows the derivation of implicit knowledge from explicitly represented knowledge by means of inference services. For a full account of Description Logics, the reader is referred, for example, to [3].

In the DL $\mathcal{ALC}$ [21], concept descriptions are formed using the constructors negation, conjunction and disjunction, value (and existential) restrictions. The DL $\mathcal{ALCN}$ [13, 19] additionally allows for unqualified (at-least and at-most) number restrictions on atomic roles. The syntax rules at the left hand side of Fig. 1 inductively define valid concept and role expressions for $\mathcal{ALCN}$ and its extensions considered in this paper. As far as semantics is concerned, concepts are interpreted as sets of individuals and roles as sets of pairs of individuals. Formally, an *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty set of individuals (the *domain* of $\mathcal{I}$) and $\mathcal{I}$ is a function (the *interpretation*

function) which maps each concept to a subset of $\Delta^{\mathcal{I}}$ and each role to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, such that the equation at the right hand side of Fig. 1 are satisfied. One of the most important inference services of DL systems used in knowledge-representation and conceptual modeling applications is computing the subsumption hierarchy of a given finite set of concept descriptions.

Definition 1 The concept description C is *satisfiable* iff there exist an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$; in this case, we say that $\mathcal{I}$ is a model for C. The concept description D *subsumes* the concept description C (written $C \sqsubseteq D$) iff $I^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for all interpretations $\mathcal{I}$; concept descriptions C and D are *equivalent* iff $C \sqsubseteq D$ and $D \sqsubseteq C$.

Since $\mathcal{ALC}$ is propositionally complete, subsumption can be reduced to concept satisfiability and *vice versa*: $C \sqsubseteq D$ iff $C \sqcap \neg D$ is unsatisfiable and C is satisfiable iff not $C \sqsubseteq A \sqcap \neg A$, where A is an arbitrary concept name.

In $\mathcal{ALCN}$, number restrictions can be used to restrict the cardinality of the set of fillers of roles (role successors). For instance, the concept description:

$$\exists^{\leq 3} \texttt{child} \sqcap \forall \texttt{child}.\texttt{Female}$$

defines individuals who have at most three daughters and no sons. $\mathcal{ALCN}(\circ)$ [4] allows counting successors of role chains, which can be used to express interesting cardinality constraints on the interrelationships some individuals hold with other objects of the domain. Moreover, $\mathcal{ALCN}(\circ)$ [4] allows counting successors of role chains in concept descriptions, which can be used to express interesting cardinality constraints on the interrelationships some individuals hold with other objects of the domain. For example, the $\mathcal{ALCN}(\circ)$-concept:

$$\texttt{Man} \sqcap \exists^{\geq 50}(\texttt{friend} \circ \texttt{tel\_number})$$

allows us to define men for which the count of different telephone numbers of their friends amounts at least to fifty. Notice that such description does not impose further constraints (disregarding obvious ones) either on the number of friends one may have, or on the number of telephone numbers each friend may have (e.g. some friends might have no telephone at all), or even on the fact that some numbers may be shared by more than one friends (e.g. if husband and wife). It only gives, for example, a constraint on the minimum size of a phonebook such men need.

The additional role constructs we consider in this paper further improve the expressiveness of the resulting DLs and, thus, make them very appealing from an application viewpoint. For instance, we may use the $\mathcal{ALC\bar{N}}(\circ,^-)$-concept:

$$\texttt{Person} \sqcap \exists \texttt{child}^- \sqcap \exists^{\leq 1}(\texttt{child}^- \circ \texttt{child})$$

to define persons who are a only child, or the $\mathcal{ALCQ}(\circ)$ concept:

$$\texttt{Woman} \sqcap \exists^{\geq 3}(\texttt{husband} \circ \texttt{brother}).\texttt{Lawyer}$$

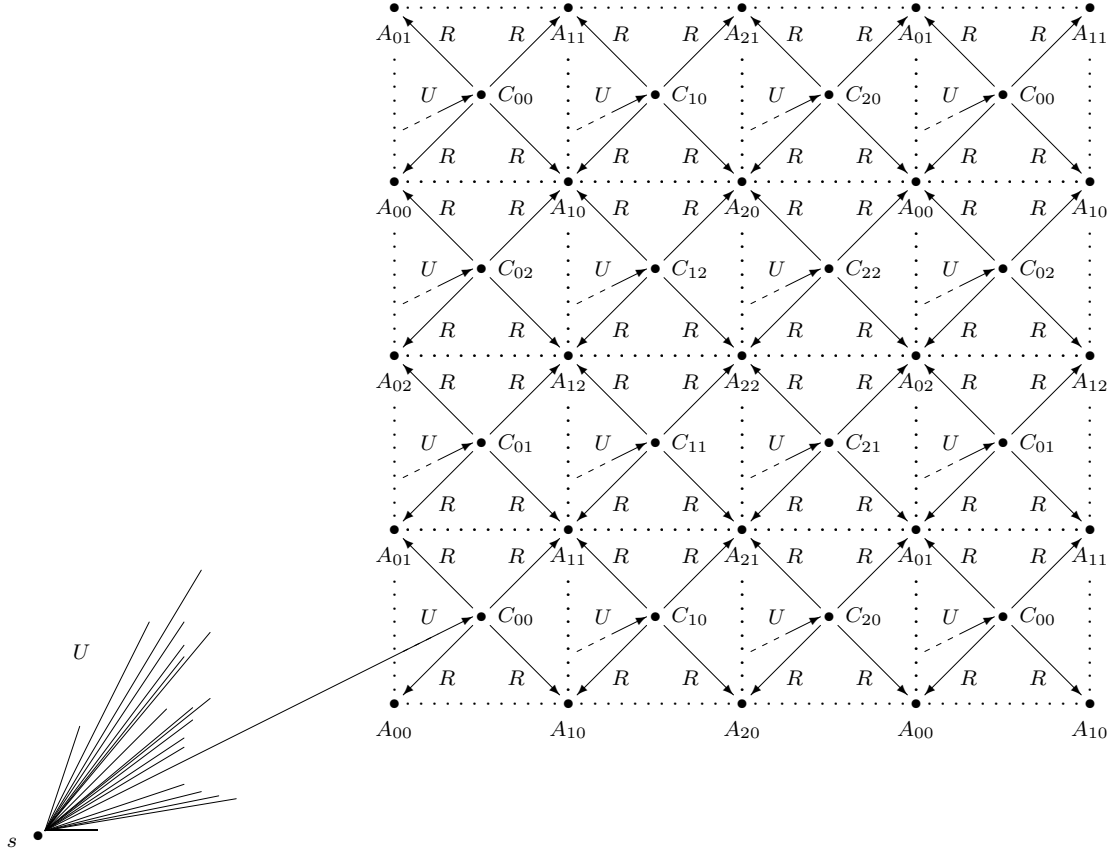to describe women having at least three lawyers as brother-in-law.

Figure 2: The grid structure used in the $\mathcal{ALCN}(\circ, ^-)$ undecidability proof.

2 Undecidability of $\mathcal{ALC\bar{N}}(\circ, ^-)$

We consider in this Section the extension of $\mathcal{ALCN}(\circ)$ by inverse roles ($\mathcal{I}$). Notice that allowing the use of role inversion both in number and in value restrictions, we obtain a Logic which is a syntactic variant of $\mathcal{ALC\bar{N}}(\circ, ^-)$. Obviously, $\mathcal{ALCN}(\circ)\mathcal{I}$ concept descriptions are also $\mathcal{ALC\bar{N}}(\circ, ^-)$ concept descriptions. Conversely, by recursively applying rules $(R \circ S)^- = S^- \circ R^-$ (pushing inverses inwards and eliminating parentheses) and $(R^-)^- = R$, we can put any $\mathcal{ALC\bar{N}}(\circ, ^-)$ complex role expression in the form $\bar{R}_1 \circ \bar{R}_2 \circ \cdots \circ \bar{R}_n$, where each $\bar{R}_i$ is either an atomic role or the inverse of an atomic role ($\bar{R}_i \in \{R_i, R_i^-\}$). Then we can get rid of role composition in value restrictions thanks to the following equivalences:

$$\exists(\bar{R}_1 \circ \bar{R}_2 \circ \cdots \circ \bar{R}_n).C \equiv \exists\bar{R}_1.\exists\bar{R}_2.\cdots\exists\bar{R}_n.C$$
$$\forall(\bar{R}_1 \circ \bar{R}_2 \circ \cdots \circ \bar{R}_n).C \equiv \forall\bar{R}_1.\forall\bar{R}_2.\cdots\forall\bar{R}_n.C$$

This procedure gives a translation of concept descriptions from $\mathcal{ALC\bar{N}}(\circ, ^-)$ to $\mathcal{ALCN}(\circ)$ $\mathcal{I}$.

To show undecidability of $\mathcal{ALC\bar{N}}(\circ, ^-)$, borrowing the proof procedure from [4], we use a reduction of the well-known undecidable domino problem [6]:

Definition 2 *A tiling system $\mathcal{D} = (D, H, V)$ is given by a non-empty set $D = \{D_1, \ldots, D_m\}$ of domino types, and by horizontal and vertical matching pairs $H \subseteq D \times D$, $V \subseteq D \times D$.*

The (unrestricted) domino problem asks for a compatible tiling of the plane, i.e. a mapping $t : \mathbb{Z} \times \mathbb{Z} \to D$ such that, for all $m, n \in \mathbb{Z}$,

$$\langle\, t(m,n), t(m+1,n) \,\rangle \in H \quad and \quad \langle\, t(m,n), t(m,n+1) \,\rangle \in V$$

We will show reducibility of the domino problem to *concept satisfiability* in $\mathcal{ALCN}(\circ, ^-)$. In particular, we show how a given tiling system $\mathcal{D}$ can be translated into a concept $E_\mathcal{D}$ which is satisfiable iff $\mathcal{D}$ allows for a compatible tiling. Following the same lines of undecidability proofs in [4], such translation can be split into three subtasks which can be described as follows:

Grid specification It must be possible to represent a "square" of $\mathbb{Z} \times \mathbb{Z}$, which consists of points $(m, n), (m+1, n), (m, n+1)$ and $(m+1, n+1)$, in order to yield a complete covering of the plane via a repeating regular grid structure. The idea is to introduce concepts to represent the grid points and roles to represent the x- and y-successor relations.

Local compatibility It must be possible to express that a tiling is locally compatible, that is that the x-successor and the y-successor of a point have an admissible domino type. The idea is to associate each domino type D_i with an atomic concept D_i, and to express the horizontal and vertical matching conditions via value restrictions.

Total reachability It must be possible to impose the above local conditions on all points in $\mathbb{Z} \times \mathbb{Z}$. This can be achieved by constructing a "universal" role and a "start" individual such that every grid point can be reached from the start individual. The local compatibility conditions can then be globally imposed via value restrictions.

The grid structure that we will use to tile the plane is shown in Fig.2. In particular, in addition to grid points, we also consider "centers" of grid squares, which are connected to grid square vertices by means of a role named R. All grid cell centers are instances of the C concept, whereas grid points are instances of the A concept. We introduce nine different (disjoint) types of grid centers via the concepts C_{ij} $(0 \le i, j \le 2)$ and nine different types of (disjoint) grid points via the concepts A_{ij} $(0 \le i, j \le 2)$, as follows:

$$C \;\; := \;\; \bigsqcup_{0 \le i,j \le 2} \left(C_{ij} \sqcap \left(\bigsqcap_{\substack{0 \le k, \ell \le 2 \\ (i,j) \ne (k,\ell)}} \neg C_{k\ell} \right) \right)$$

$$A \;\; := \;\; \bigsqcup_{0 \le i,j \le 2} \left(A_{ij} \sqcap \left(\bigsqcap_{\substack{0 \le k, \ell \le 2 \\ (i,j) \ne (k,\ell)}} \neg A_{k\ell} \right) \right) \sqcap \neg C$$

Grid specification can then be accomplished by means of the $C_\boxplus$ and $A_\boxplus$ concepts which follow:

$$C_\boxplus \;\; := \;\; C \sqcap \exists^{\le 4} R \sqcap \forall R.A_\boxplus \sqcap \exists^{\le 9} R \circ R^- \sqcap$$
$$\bigsqcap_{0 \le i,j \le 2} \left(C_{ij} \Rightarrow (\exists R.A_{ij} \sqcap \exists R.A_{i \oplus 1, j} \sqcap \exists R.A_{i, j \oplus 1} \sqcap \exists R.A_{i \oplus 1, j \oplus 1}) \right)$$

$$A_\boxplus \;\; := \;\; A \sqcap \bigsqcap_{0 \le i,j \le 2} \left(A_{ij} \Rightarrow (\exists R^-.C_{ij} \sqcap \exists R^-.C_{i \oplus 2, j} \sqcap \exists R^-.C_{i, j \oplus 2} \sqcap \exists R^-.C_{i \oplus 2, j \oplus 2}) \right)$$

where $a \oplus b = (a + b) \bmod 3$.

Some relevant constraints that are imposed by these concept descriptions on their models are studied in the Lemmata and Corollaries which follow.

R-successors	$(R \circ R^-)$-successors	C_{ij}-types
A_{ij}	C_{ij}	❶
	$C_{i\oplus2,j}$	❷
	$C_{i,j\oplus2}$	❸
	$C_{i\oplus2,j\oplus2}$	❹
$A_{i\oplus1,j}$	$C_{i\oplus1,j}$	❺
	$C_{i\oplus1\oplus2,j} = C_{ij}$	①
	$C_{i\oplus1,j\oplus2}$	❻
	$C_{i\oplus1\oplus2,j\oplus2} = C_{i,j\oplus2}$	③
$A_{i,j\oplus1}$	$C_{i,j\oplus1}$	❼
	$C_{i\oplus2,j\oplus1}$	❽
	$C_{i,j\oplus1\oplus2} = C_{ij}$	①
	$C_{i\oplus2,j\oplus1\oplus2} = C_{i\oplus2,j}$	②
$A_{i\oplus1,j\oplus1}$	$C_{i\oplus1,j\oplus1}$	❾
	$C_{i\oplus1\oplus2,j\oplus1} = C_{i,j\oplus1}$	⑦
	$C_{i\oplus1,j\oplus1\oplus2} = C_{i\oplus1,j}$	⑤
	$C_{i\oplus1\oplus2,j\oplus1\oplus2} = C_{ij}$	①

Table 1: Types of the R- and $(R \circ R^-)$-successors of a C_{ij}-type grid center. In the last column, numbers on black ground mark different C_{ij}-types the first time they are met from the top of the table, whereas numbers on white ground refer to C_{ij}-types that have been met before.

Lemma 1 *Let c be an instance of $C_\boxplus$. Then it has at most one R-successor in each of the nine $A_{k\ell}$ concept extensions.*

Proof More precisely, c has exactly one R-successor in the extension of each of the four $A_{k\ell}$ concepts it is connected to by R (e.g. if w.l.o.g. $c \in (C_{ij})^\mathcal{I}$ then c has exactly one R-successor in the extension of A_{ij}, $A_{i\oplus1,j}$, $A_{i,j\oplus1}$, $A_{i\oplus1,j\oplus1}$ and no R-successor in any of the remaining five partitions of the extension of A). This follows from the fact that the nine $A_{k\ell}$ concepts are disjoint and c has a total number of at most four R-successors. □

Lemma 2 *Let c be an instance of $C_\boxplus$. Then it has exactly one $(R \circ R^-)$-successor in each of the nine $C_{k\ell}$ concept extensions.*

Proof Since c is an instance of C, it belongs to the extension of exactly one of the nine $C_{k\ell}$ concepts. W.l.o.g. let us assume $c \in (C_{ij})^\mathcal{I}$. Hence, owing to the $C_\boxplus$ and $A_\boxplus$ definitions and Lemma 1, c has surely R- and $(R \circ R^-)$-successors as shown in Tab. 1. In particular, c has $(R \circ R^-)$-successors in each of the nine $C_{k\ell}$ concept extensions. Since all the $C_{k\ell}$ extensions are disjoint and c has a total of at most nine $(R \circ R^-)$-successors, this means that c has exactly one $(R \circ R^-)$-successor in each of the nine $C_{k\ell}$ concept extensions (being c itself its unique $(R \circ R^-)$-successor in C_{ij}). □

Corollary 1 *Let a be an instance of $A_\boxplus$. Then (1) all its (R^-)-successors are instances of $C_\boxplus$ and (2) it has at most one (R^-)-successor in each of the nine $C_{k\ell}$ concept extensions.*

Proof It is an immediate consequence of Lemma 2. W.l.o.g. assume $a \in (A_{ij})^{\mathcal{I}}$. (1) If a had an (R^-)-successor $o \notin (C_{\boxplus})^{\mathcal{I}}$, then any of the four (R^-)-successors of a in $C_{\boxplus}$ would have at least ten $(R \circ R^-)$-successors (it has nine $(R \circ R^-)$-successors in $C_{\boxplus}$ by Lemma 2 plus o) and, thus, would violate the $C_{\boxplus}$ definition. (2) If a had, for instance, two distinct (R^-)-successors in $C_{k\ell}$ (i.e. $\exists c, c' \in (C_{k\ell})^{\mathcal{I}}$, $c \neq c'$, with $(c, a) \in R^{\mathcal{I}}$, $(c', a) \in R^{\mathcal{I}}$), then c would have two distinct $(R \circ R^-)$-successors in $C_{k\ell}$, c' and itself, contradicting Lemma 2.

More precisely, a has exactly one (R^-)-successor in the extension of each of the four $C_{k\ell}$ concepts it is connected to by R^-. $\qquad \square$

Corollary 2 *Let a be an instance of $A_{\boxplus}$. Then it has exactly one $(R^- \circ R)$-successor in each of the nine $A_{k\ell}$ concept extensions.*

Proof W.l.o.g. assume $a \in (A_{ij})^{\mathcal{I}}$. We show that if the thesis is false we come up with a contradiction. To this end, we must distinguish three cases. First of all, we can exclude a has another $(R^- \circ R)$-successor, say a', in A_{ij}: if this happened, each of the four (R^-)-successors of a (e.g. $c \in (C_{ij})^{\mathcal{I}}$) would have two distinct R-successors (a and a') in A_{ij}, thus violating Lemma 1. Second, we can also exclude a has two distinct $(R^- \circ R)$-successors in $A_{k\ell}$, say a' and a'', which can be reached through a common (R^-)-successor c (e.g. $c \in (C_{ij})^{\mathcal{I}}$): if this happened, c would have two distinct R-successors (a and a') in $A_{k\ell}$, against Lemma 1 again. In the third and last case, we must consider a having two distinct $(R^- \circ R)$-successors in $A_{k\ell}$, say a' and a'', which can be reached through distinct (R^-)-successors of a. W.l.o.g. we may assume such (R^-)-successors of a in C_{ij} and $C_{i,j\oplus2}$, and $a', a'' \in (A_{i\oplus1,j})^{\mathcal{I}}$. Hence we must have that $\exists c_0 \in (C_{ij})^{\mathcal{I}}$ with $\{(c_0, a), (c_0, a')\} \subseteq R^{\mathcal{I}}$ and $\exists c \in (C_{i,j\oplus2})^{\mathcal{I}}$ with $\{(c, a), (c, a'')\} \subseteq R^{\mathcal{I}}$. We then consider the application of Lemma 2 from $c_0 \in (C_{ij})^{\mathcal{I}}$. By construction, c_0 has c as $(R \circ R^-)$-successor through the path passing from $a \in (A_{ij})^{\mathcal{I}}$. Owing to Lemma 2, also the $(R \circ R^-)$ path passing from $a' \in (A_{i\oplus1,j})^{\mathcal{I}}$ (the path exists, as a' has an (R^-)-successor in $(C_{i,j\oplus2})^{\mathcal{I}}$) must lead to c and, thus, $(c, a') \in R^{\mathcal{I}}$. But this contradicts Lemma 1, as c would have two distinct R-successors (a' and a'') in $(A_{i\oplus1,j})^{\mathcal{I}}$. $\qquad \square$

Hence, we will interpret instances of $C_{\boxplus}$ as grid centers and instances of $A_{\boxplus}$ as grid points. In particular, nine different types of grid cells can be defined according to the type of their center: an (i, j)-type grid cell has a C_{ij}-type center, while its lower left, lower right, upper left and upper right vertices can be defined, respectively, as the instances of the A_{ij}, $A_{i\oplus1,j}$, $A_{i,j\oplus1}$ and $A_{i\oplus1,j\oplus1}$ concepts which are connected to the center via R (according to the $C_{\boxplus}$ definition). Therefore, the x- and y-successor relations on the grid can be defined by means of the $(R^- \circ R)$-paths connecting an A_{ij}-type grid point with an $A_{i\oplus1,j}$-type and an $A_{i,j\oplus1}$-type grid points, respectively. Such successors always exist and are uniquely defined, owing to Corollary 2.

In a similar way, Corollary 2 also allows us to uniquely define the x- and y-predecessors relations on the grid, by means of the $(R^- \circ R)$-paths connecting an A_{ij}-type grid point with an $A_{i\oplus2,j}$-type and an $A_{i,j\oplus2}$-type grid points, respectively (cf. $(a - 1) \mod 3 = (a + 2) \mod 3$).

Lemma 3 (Grid Closure) *For each grid point, the $(x \circ y)$- and $(y \circ x)$-successors are uniquely defined and coincide.*

Proof We can assume the grid point to represent the point $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ and call it $p_{(m,n)}$. W.l.o.g. we can further assume $p_{(m,n)}$ to be the bottom left vertex of an (i, j)-type

grid cell. Therefore, $p_{(m,n)}$ is an instance of A_{ij} and is an R-successor of the grid cell center, say $c_{(m,n)}$, which is an instance of C_{ij}. The x-successor of $p_{(m,n)}$, say $p_{(m+1,n)}$, is the R-successor of $c_{(m,n)}$ in $A_{i\oplus1,j}$ (by construction, it is an $(R^- \circ R)$-successor of $p_{(m,n)}$ and is unique by Corollary 2). Analogously, the y-successor of $p_{(m,n)}$, say $p_{(m,n+1)}$, is the R-successor of $c_{(m,n)}$ in $A_{i,j\oplus1}$. According to the $C_\boxplus$ definition, $c_{(m,n)}$ has also a fourth R-successor, say $\bar{p}$, in $A_{i\oplus1,j\oplus1}$. We consider now the $(x \circ y)$-successor of $p_{(m,n)}$, that is the y-successor of $p_{(m+1,n)}$, and call it $p'_{(m+1,n+1)}$. Owing to the y-successor definition, $p'_{(m+1,n+1)}$ must be an instance of $A_{i\oplus1,j\oplus1}$ connected to $p_{(m+1,n)}$ via $(R^- \circ R)$. However, both $\bar{p}$ and $p'_{(m+1,n+1)}$ are, by construction, $(R^- \circ R)$-successors of $p_{(m+1,n)}$ in $A_{i\oplus1,j\oplus1}$ and, thus, they must coincide thanks to Corollary 2. Analogously, the $(y \circ x)$-successor of $p_{(m,n)}$, that is the x-successor of $p_{(m,n+1)}$, say $p''_{(m+1,n+1)}$, must be an instance of $A_{i\oplus1,j\oplus1}$ connected to $p_{(m,n+1)}$ via $(R^- \circ R)$. Thence, Corollary 2 ensures that $\bar{p}$ and $p''_{(m+1,n+1)}$ coincide, as they are both $(R^- \circ R)$-successors of $p_{(m,n+1)}$ in $A_{i\oplus1,j\oplus1}$. Hence, $\bar{p}$ is the common $(x \circ y)$- and $(y \circ x)$-successor of $p_{(m,n)}$ on the grid, that can be called $p_{(m+1,n+1)}$ to represent the point $(m + 1, n + 1)$ of the plane. $\qquad\square$

Local compatibility is easily achieved by enforcing grid centers to be instances of a $C_{\mathcal{D}}$ concept defined as follows:

$$C_{\mathcal{D}} := \forall R.\left(\bigsqcup_{1 \le k \le m} \left(D_k \sqcap \left(\bigsqcap_{\substack{1 \le \ell \le m \\ k \ne \ell}} \neg D_\ell \right) \right) \right) \sqcap \bigsqcap_{0 \le i,j \le 2} \left(C_{ij} \Rightarrow \bigsqcap_{1 \le k \le m} \exists R.(A_{ij} \sqcap D_k) \right.$$

$$\left. \Rightarrow \left(\exists R.\left(A_{i\oplus1,j} \sqcap \left(\bigsqcup_{(D_k,D_\ell) \in H} D_\ell \right) \right) \sqcap \exists R.\left(A_{i,j\oplus1} \sqcap \left(\bigsqcup_{(D_k,D_\ell) \in V} D_\ell \right) \right) \right) \right)$$

Each domino type D_k is associated to an atomic concept with the same name. The value restriction in the first conjunct of $C_{\mathcal{D}}$ forces grid points to have a domino type. The second conjunct uses the definition of the x- and y-successors for the bottom left vertex of an (i,j)-type cell to enforce horizontal and vertical matching conditions via value restrictions.

Total Reachability will be achieved by constructing a "start" individual (s) and two "universal" roles: the former (U) which connects s to every grid center and the latter ($U \circ R$) which connects s to every grid point (see Fig. 2). The Lemmata and Corollaries which follow will justify the correctness of our construction.

Lemma 4 *Let s be an instance of*

$$D := \exists U \circ R \sqcap \exists^{\le1}(U \circ R) \circ (U \circ R)^- \sqcap \neg\exists R^- \sqcap \neg\exists U^- \sqcap \forall U.\neg\exists R^-$$

in a given interpretation $\mathcal{I}$. Then any $(U \circ R)$-successor x of s in $\mathcal{I}$ (D ensures that there is at least one) has s as its unique *$(U \circ R)$-predecessor.*

Proof Assume $s \in D^{\mathcal{I}}$ and x is a $(U \circ R)$-successor of s, that is $\exists o \in \Delta^{\mathcal{I}}$ such that $(s,o) \in U^{\mathcal{I}}$, $(o,x) \in R^{\mathcal{I}}$, with $o \ne s$ (as $s \notin (\exists U^-)^{\mathcal{I}}$), $x \ne o$ (as $s \in (\forall U.\neg\exists R^-)^{\mathcal{I}}$ and, thus, $o \notin (\exists R^-)^{\mathcal{I}}$) and $s \ne x$ (as $s \notin (\exists R^-)^{\mathcal{I}}$). If there were $s' \in \Delta^{\mathcal{I}}$, $s' \ne s$, such that s' is a $(U \circ R)$-predecessor of x (i.e. $\exists o' \in \Delta^{\mathcal{I}}$ such that $(s',o') \in U^{\mathcal{I}}$, $(o',x) \in R^{\mathcal{I}}$), then s and s' would be both $(R^- \circ U^-) = (U \circ R)^-$-successors of x in $\mathcal{I}$ and, thus, both $(U \circ R) \circ (U \circ R)^-$-successors of s in $\mathcal{I}$. Hence we should have $s \notin D^{\mathcal{I}}$, against the hypothesis. $\qquad\square$

Corollary 3 *Under the hypothesis of Lemma 4, any U-successor of s in $\mathcal{I}$ has s as its unique U-predecessor.*

Corollary 4 *Under the hypothesis of Lemma 4, let s be an instance of*

$$D' := D \sqcap \forall U.\forall R.\forall R^-.\exists U^- \sqcap \neg\exists R$$

in a given interpretation $\mathcal{I}$. Then any $(U \circ R \circ R^-)$-successor y of s in $\mathcal{I}$ (D ensures that there is at least one) is a U-successor of s in $\mathcal{I}$ and has s as its unique U-predecessor.

Proof Let $y \in \Delta^{\mathcal{I}}$ be a generic $(U \circ R \circ R^-)$-successor of s in $\mathcal{I}$, that is $\exists o, x \in \Delta^{\mathcal{I}}$ such that $(s, o) \in U^{\mathcal{I}}$, $(o, x) \in R^{\mathcal{I}}$, $(y, x) \in R^{\mathcal{I}}$ (we may assume $y \neq o$, as the Corollary is trivially true for o), with $s \neq y$ (as $s \notin (\exists R)^{\mathcal{I}}$). Since $s \in (\forall U.\forall R.\forall R^-.\exists U^-)^{\mathcal{I}}$, $y \in (\exists U^-)^{\mathcal{I}}$, that is $\exists s' \in \Delta^{\mathcal{I}}$ such that $(s', y) \in U^{\mathcal{I}}$. Notice that both s and s' have, by construction, x as $(U \circ R)$-successor. Since $s \in D^{\mathcal{I}}$, thanks to Lemma 4, s and s' must coincide. Hence y is a U-successor of s, which is also its unique U-predecessor by Corollary 3. $\qquad\square$

Lemma 5 (Plane Covering and Compatible Tiling) *Let s be an instance of*

$$
\begin{aligned}
E_{\mathcal{D}} := \quad & \exists U \circ R \sqcap \exists^{\leq 1}(U \circ R) \circ (U \circ R)^- \sqcap \neg\exists R \sqcap \neg\exists R^- \sqcap \neg\exists U^- \sqcap \\
& \forall U.\forall R.\forall R^-.\exists U^- \sqcap \forall U.(C_{\boxplus} \sqcap C_{\mathcal{D}} \sqcap \neg\exists R^-)
\end{aligned}
$$

in a given interpretation $\mathcal{I}$. Then, for the grid that tiles the plane $\mathbb{Z} \times \mathbb{Z}$, any grid center can be reached from s via U, any grid point can be reached from s via $U \circ R$ and local tiling conditions are imposed on all grid points (yielding a compatible tiling of the plane).

Proof Let us consider a grid center connected to $s \in (E_{\mathcal{D}})^{\mathcal{I}}$ via U ($E_{\mathcal{D}} \sqsubseteq \exists U \circ R \sqcap \forall U.C_{\boxplus}$ ensures that there is at least one). W.l.o.g. we can assume it to be the center of an (i, j)-type cell and call it $c_{(0,0)}$ ($c_{(0,0)} \in (C_{ij})^{\mathcal{I}}$). We can also call $p_{(0,0)}$ the bottom left vertex of this grid cell ($p_{(0,0)} \in (A_{ij})^{\mathcal{I}}$) and let it represent the origin $(0,0)$ of $\mathbb{Z} \times \mathbb{Z}$. We can now consider the x- and y-successors of $p_{(0,0)}$, say $p_{(1,0)}$ and $p_{(0,1)}$, respectively. By construction, we have $p_{(1,0)} \in (A_{i \oplus 1, j})^{\mathcal{I}}$, $p_{(0,1)} \in (A_{i, j \oplus 1})^{\mathcal{I}}$; moreover, either $p_{(0,0)}$, $p_{(1,0)}$ and $p_{(0,1)}$ are R-successors of $c_{(0,0)}$ and, thus, $(U \circ R)$-successors of s. In the (i, j)-type grid cell centered on $c_{(0,0)}$, $p_{(1,0)}$ and $p_{(0,1)}$ are the bottom right and top left vertices, but they are also the bottom left vertices of the two grid cells adjacent to the right and to the top, respectively. In particular, $p_{(1,0)}$ and $p_{(0,1)}$ are the bottom left vertices of an $(i \oplus 1, j)$- and an $(i, j \oplus 1)$-type grid cells, whose centers we can call $c_{(1,0)}$ and $c_{(0,1)}$, respectively (the existence and uniqueness of these cells and their centers is ensured by Lemma 2). Obviously, $c_{(1,0)}$ and $c_{(0,1)}$ are $((R \circ R^-)$-successors of $c_{(0,0)}$ and) (R^-)-successors of $p_{(1,0)}$ and $p_{(0,1)}$, respectively. Therefore, they are $(U \circ R \circ R^-)$-successors of the start individual s and, thanks to Corollary 4 (as $E_{\mathcal{D}} \sqsubseteq D'$), they are also U-successors of s.

Using the x- and y-predecessor definitions, we can easily see that the same holds for $c_{(-1,0)}$ and $c_{(0,-1)}$ grid centers. In any case, we can repeat the argument at will, starting with $c_{(1,0)}$, $c_{(0,1)}$, $c_{(-1,0)}$ and $c_{(0,-1)}$ in place of $c_{(0,0)}$, and show that the center of any grid cell on the plane can be reached from s via U. Hence, all grid points can be reached from s via $U \circ R$ and local tiling conditions are imposed on all of them by value restrictions (as $E_{\mathcal{D}} \sqsubseteq \forall U.C_{\mathcal{D}}$). $\qquad\square$

Thanks to Lemma 5, it is easy to see that a tiling system $\mathcal{D}$ has a compatible tiling iff concept $E_{\mathcal{D}}$ is satisfiable (i.e. there is an interpretation $\mathcal{I}$ such that $(E_{\mathcal{D}})^{\mathcal{I}} \neq \emptyset$).

Theorem 1 *Satisfiability (and, thus, subsumption) of concepts is undecidable for $\mathcal{ALCN}(\circ,^-)$ (and $\mathcal{ALCQ}(\circ,^-)$).*

3 Decidability of $\mathcal{ALCQ}(\circ)$

We will show in this Section how an effective decision procedure for $\mathcal{ALCQ}(\circ)$-concept satisfiability can be provided as a tableau-based algorithm. To this end, we consider $\mathcal{ALCQ}(\circ)$-concept descriptions in Negation Normal Form (NNF [21]), where the negation sign is allowed to appear before atomic concepts only. In fact, $\mathcal{ALCQ}(\circ)$-concept descriptions can be transformed into NNF in linear time via application of the same rules which can be used for $\mathcal{ALCQ}$ (pushing negations inwards):

$$\neg\exists^{\leq n}R.C \;=\; \exists^{\geq n+1}R.C \qquad \neg\exists^{\geq n}R.C \;=\; \exists^{\leq n-1}R.C \quad (\bot \text{ if } n = 0)$$
$$\neg\exists R.C \;=\; \forall R.\neg C \qquad\qquad\qquad \neg\forall R.C \;=\; \exists R.\neg C$$

in addition to the absorption rule for double negations and De Morgan's laws for $\sqcap$ and $\sqcup$. Obviously, unqualified number restrictions are treated as particular cases of qualified restrictions (with $C = \top$). We can further make use of the rules:

$$\exists R.C \;=\; \exists^{\geq 1}R.C \qquad \forall R.C \;=\; \exists^{\leq 0}R.\neg C$$

to get rid of (existential and) value restrictions. We define the concept descriptions obtained in this way as in NNF$^{\bowtie}$ and denote the NNF$^{\bowtie}$ of the $\mathcal{ALCQ}(\circ)$-concept description $\neg C$ as $\sim C$. We will use the symbol $\bowtie$ in number restrictions $\exists^{\bowtie n}R.C$ as a placeholder for either $\geq$ or $\leq$.

The Tableau algorithm we are going to introduce manipulates, as basic data structures, ABox assertions involving domain individuals. In fact, our algorithm is a simple extension of the tableau-based algorithm to decide $\mathcal{ALCN}(\circ)$-concept satisfiability presented by Baader and Sattler in [4]. The extension is based on the modification of the transformation rules for number restrictions ($\geq$- and $\leq$-rules) to take into account the "qualifying" conditions and on the introduction of a so-called *choose* rule (called ⊚-rule here), which makes sure that all "relevant" concepts that are implicitly satisfied by an individual are made explicit in the ABox. Basically, the proposed extension is similar to the one which extends the tableau-based $\mathcal{ALCN}$ sasfiability algorithm [13, 19] to an $\mathcal{ALCQ}$ satisfiability algorithm [18, 5].

Definition 3 Let NI be a set of individual names. An *ABox* $\mathcal{A}$ is a finite set of assertions of the form $C(a)$ –*concept assertion*– or $R(a, b)$ –*role assertion*– where C is a concept description, R a role name, and a, b are individual names. An interpretation $\mathcal{I}$, which additionally assigns elements $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ to individual names a, is a *model* of an ABox $\mathcal{A}$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$ (resp. $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$) for all assertions $C(a)$ (resp. $R(a, b)$) in $\mathcal{A}$. The ABox $\mathcal{A}$ is consistent iff it has a model. The individual a is an instance of the description C w.r.t. $\mathcal{A}$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$ holds for all models $\mathcal{I}$ of $\mathcal{A}$. We also consider in a ABox *inequality assertions* of the form $a \neq b$, with the obvious semantics that an interpretation $\mathcal{I}$ satisfies $a \neq b$, iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$. Inequality assertions are assumed to be symmetric, that is saying that $a \neq b \in \mathcal{A}$ is the same as saying $b \neq a \in \mathcal{A}$.

Sometimes in the DL field, a *unique name assumption* is made in works concerning reasoning with individuals, that is the mapping $\pi : \text{NI} \to \Delta^{\mathcal{I}}$ from individual names to domain elements is required to be injective. We dispense from this requirement as it has no effect for the $\mathcal{ALC}$ extensions studied here and the explicitly introduced inequality assertions can be used to enforce the uniqueness of names anyway.

Definition 4 The individual y is a $(R_1 \circ R_2 \circ \cdots \circ R_m)$-*successor* of x in $\mathcal{A}$ iff $\exists y_2 y_3 \ldots y_m$ variables in $\mathcal{A}$ such that $\{R_k(y_k, y_{k+1}) \mid 2 \leq k \leq m-1\} \cup \{R_1(x, y_2), R_m(y_m, y)\} \subseteq \mathcal{A}$.

Definition 5 An ABox $\mathcal{A}$ contains a *clash* iff, for an individual name $x \in \mathsf{NI}$, one of the two situations below occurs:

- $\{A(x), \neg A(x)\} \subseteq \mathcal{A}$, for a concept name $A \in \mathsf{NC}$;

- $(\exists^{\leq n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$ and x has p $(R_1 \circ \cdots \circ R_m)$-successors $y_1, \ldots, y_p$ with $p > n$ such that $\{C(y_i) \mid 1 \leq i \leq p\} \cup \{y_i \neq y_j \mid 1 \leq i < j \leq p\} \subseteq \mathcal{A}$, for role names $\{R_1, \ldots, R_m\} \subseteq \mathsf{NR}$, a concept description C and an integer $n \geq 0$.

To test the satisfiability of an $\mathcal{ALCQ}(\circ)$ concept C in NNF$^{\bowtie}$, the proposed $\mathcal{ALCQ}(\circ)$-algorithm works as follows. Starting from the initial ABox $\{C_0(x_0)\}$, it applies the *completion rules* in Fig. 3, which modify the ABox. It stops when no rule is applicable (when a clash is generated, the algorithm does not immediately stops but it always generate a complete ABox). An ABox $\mathcal{A}$ is called *complete* iff none of the completion rules is any longer applicable. The algorithm answers "C is satisfiable" iff a complete and clash-free ABox has been generated. The $\mathcal{ALCQ}(\circ)$-algorithm is non-deterministic, due to the $\sqcup$-, $\leq$- and $\copyright$-rules (for instance, the $\sqcup$-rule non-deterministically chooses which disjunct to add for a disjunctive concept).

Lemma 6 *Let C_0 be an $\mathcal{ALCQ}(\circ)$-concept in NNF$^{\bowtie}$, and let $\mathcal{A}$ be an ABox obtained by applying the completion rules to $\{C_0(x_0)\}$. Then*

1. *For each completion rule $\mathcal{R}$ that can be applied to $\mathcal{A}$ and for each interpretation $\mathcal{I}$, the following equivalence holds: $\mathcal{I}$ is a model of $\mathcal{A}$ iff $\mathcal{I}$ is a model of the ABox $\mathcal{A}'$ obtained by applying $\mathcal{R}$.*

2. *If $\mathcal{A}$ is a complete and clash-free ABox, then $\mathcal{A}$ has a model.*

3. *If $\mathcal{A}$ is complete but contains a clash, then $\mathcal{A}$ does not have a model.*

4. *The completion algorithm terminates when applied to $\{C_0(x_0)\}$.*

As a matter of fact, termination (4) yields that after finitely many steps we obtain a complete ABox. If C_0 is satisfiable, then $\{C_0(x_0)\}$ is also satisfiable and, thus, at least one of the complete ABoxes that the algorithm can generate is satisfiable by (1). Hence, such an ABox must be clash-free by (3). Conversely, if the application of the algorithm produces a complete and clash-free ABox $\mathcal{A}$, then it is satisfiable by (2) and, owing to (1), this implies that $\{C_0(x_0)\}$ is satisfiable. Consequently, the algorithm is a decision procedure for satisfiability of $\mathcal{ALCQ}(\circ)$-concepts.

Corollary 5 *Concept satifiability (and subsumption) for $\mathcal{ALCQ}(\circ)$ is decidable, and the Tableau algorithm based on the completion rules in Fig. 3 is an effective decision procedure.*

Proof of Part 1 of Lemma 6 We consider only the rules concerned with number restrictions and the $\copyright$-rule, as the proof for the first two rules is the same as for $\mathcal{ALC}$.

⊓-rule: **if** 1. $(C_1 \sqcap C_2)(x) \in \mathcal{A}$ and
            2. $\{C_1(x), C_2(x)\} \not\subseteq \mathcal{A}$
            then $\mathcal{A}' := \mathcal{A} \cup \{C_1(x), C_2(x)\}$

⊔-rule: **if** 1. $(C_1 \sqcup C_2)(x) \in \mathcal{A}$ and
            2. $\{C_1(x), C_2(x)\} \cap \mathcal{A} = \emptyset$
            then $\mathcal{A}' := \mathcal{A} \cup \{D(x)\}$ for some $D \in \{C_1, C_2\}$

≥ -rule: **if** 1. $(\exists^{\geq n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$ and
            2. x has exactly p $(R_1 \circ \cdots \circ R_m)$-successors $y_1, \ldots, y_p$ with $p < n$
                such that $\{C(y_i) \mid 1 \leq i \leq p\} \cup \{y_i \neq y_j \mid 1 \leq i < j \leq p\} \subseteq \mathcal{A}$
            then $\mathcal{A}' := \mathcal{A} \cup \{R_1(x, z_{i2}), R_2(z_{i2}, z_{i3}), \ldots, R_m(z_{im}, z_i), C(z_i) \mid 1 \leq i \leq n - p\}$
                    $\cup \{z_i \neq z_j \mid 1 \leq i < j \leq n - p\} \cup \{y_i \neq z_j \mid 1 \leq i \leq p, 1 \leq j \leq n - p\}$
                    where z_{ik}, z_i (for $1 \leq i \leq n - p, 2 \leq k \leq m$) are $m(n - p)$ fresh variables

≤ -rule: **if** 1. $(\exists^{\leq n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$ and
            2. x has more than n $(R_1 \circ \cdots \circ R_m)$-successors $y_1, \ldots, y_p$ such that
                $\{C(y_i) \mid 1 \leq i \leq p\} \subseteq \mathcal{A}$ and $\{y_i \neq y_j\} \cap \mathcal{A} = \emptyset$ for some i, j $(1 \leq i < j \leq p)$,
            then for some pair y_i, y_j $(1 \leq i < j \leq p)$ such that $\{y_i \neq y_j\} \cap \mathcal{A} = \emptyset$
                    $\mathcal{A}' := [y_i/y_j]\mathcal{A}$ (i.e. $\mathcal{A}'$ is obtained by replacing each occurrence of y_i by y_j)

©-rule: **if** 1. $(\exists^{\bowtie n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$ and
            2. y is an $(R_1 \circ \cdots \circ R_m)$-successor of x such that $\{C(y), \sim C(y)\} \cap \mathcal{A} = \emptyset$
            then $\mathcal{A}' := \mathcal{A} \cup \{D(y)\}$ for some $D \in \{C, \sim C\}$

Figure 3: The Completion Rules for $\mathcal{ALCQ}(\circ)$

3. **$\geq$-rule.** Assume that the rule is applied to the constraint $(\exists^{\geq n} R_1 \circ \cdots \circ R_m.C)(x)$ and that its application yields:

$$\begin{aligned} \mathcal{A}' \;=\; & \mathcal{A} \cup \{R_1(x, z_{i2}), R_2(z_{i2}, z_{i3}), \ldots, R_m(z_{im}, z_i), C(z_i) \mid 1 \leq i \leq n - p\} \\ & \cup \{z_i \neq z_j \mid 1 \leq i < j \leq n - p\} \cup \{y_i \neq z_j \mid 1 \leq i \leq p, 1 \leq j \leq n - p\} \end{aligned}$$

Since $\mathcal{A}$ is a subset of $\mathcal{A}'$, any model of $\mathcal{A}'$ is also a model of $\mathcal{A}$. Conversely, assume that $\mathcal{I}$ is a model of $\mathcal{A}$. On the one hand, since $\mathcal{I}$ satisfies $(\exists^{\geq n} R_1 \circ \cdots \circ R_m.C)(x)$, $x^{\mathcal{I}}$ has at least n $(R_1 \circ \cdots \circ R_m)$-successors in $\mathcal{I}$ which are instances of C. On the other hand, since the $\geq$-rule is applicable to $(\exists^{\geq n} R_1 \circ \cdots \circ R_m.C)(x)$, x has exactly p $(R_1 \circ \cdots \circ R_m)$-successors $y_1, \ldots, y_p$, with $p < n$, which are instances of C in $\mathcal{A}$. Thus, there exists $n - p$ $(R_1 \circ \cdots \circ R_m)$-successors $b_1, \ldots, b_{n-p}$ of $x^{\mathcal{I}}$ in $\mathcal{I}$ such that $b_i \in C^{\mathcal{I}}$ and $b_i \neq y_j$ for all i, j $(1 \leq i \leq n - p, 1 \leq j \leq p)$. For all i $(1 \leq i \leq n - p)$, let $\{b_{i2}, \ldots, b_{im}\} \subseteq \Delta^{\mathcal{I}}$ be such $(x^{\mathcal{I}}, b_{i2}) \in R_1^{\mathcal{I}}$, $(b_{i2}, b_{i3}) \in R_2^{\mathcal{I}}, \ldots$, $(b_{im}, b_i) \in R_m^{\mathcal{I}}$. We define the interpretation of the new variables added by the $\geq$-rule as $z_{i2}^{\mathcal{I}} = b_{i2}, \ldots, z_{im}^{\mathcal{I}} = b_{im}$, and $z_i^{\mathcal{I}} = b_i$ $(1 \leq i \leq n - p)$. Obviously, $\mathcal{I}$ satisfies $\mathcal{A}'$.

4. **$\leq$-rule.** Assume that the rule is applied to the constraint $(\exists^{\leq n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$ and let $\mathcal{I}$ be a model of $\mathcal{A}$. On the one hand, since the rule is applicable, x has more than n $(R_1 \circ \cdots \circ R_m)$-successors which are instances of C in $\mathcal{A}$. On the other hand, $\mathcal{I}$ satisfies $(\exists^{\leq n} R_1 \circ \cdots \circ R_m.C)(x)$ and, thus, there are two different $(R_1 \circ \cdots \circ R_m)$-successors y_i, y_j of x and instances of C in $\mathcal{A}$ such that $y_i^{\mathcal{I}} = y_j^{\mathcal{I}}$. Obviously, this implies that $y_i \neq y_j \notin \mathcal{A}$ and, thus, $\mathcal{A}' = \mathcal{A}[y_i/y_j]$ is the ABox obtained by applying the $\leq$-rule to $(\exists^{\leq n} R_1 \circ \cdots \circ R_m.C)(x)$. In addition, since $y_i^{\mathcal{I}} = y_j^{\mathcal{I}}$, $\mathcal{I}$ satisfies $\mathcal{A}'$. Conversely, assume that $\mathcal{A}' = \mathcal{A}[y_i/y_j]$ is obtained from $\mathcal{A}$ by applying the $\leq$-rule, and let $\mathcal{I}$ be a model of $\mathcal{A}'$. If we consider an interpretation $\mathcal{I}$ so that $y_j^{\mathcal{I}} = y_i^{\mathcal{I}}$ for the additional variable y_j that is present in $\mathcal{A}$ then obviously $\mathcal{I}$ satisfies $\mathcal{A}$.

5. **©-rule.** Assume that the rule is applied to the constraint $(\exists^{\bowtie n} R_1 \circ \cdots \circ R_m.C)(x)$ and that its application yields:

$$\mathcal{A}' \;=\; \mathcal{A} \cup \{D(y)\}$$

where $D(y) \notin \mathcal{A}$. Since $\mathcal{A}$ is a subset of $\mathcal{A}'$, any model of $\mathcal{A}'$ is also a model of $\mathcal{A}$. Conversely, assume that $\mathcal{I}$ is a model of $\mathcal{A}$. As far as y is concerned, either $y^{\mathcal{I}} \in C^{\mathcal{I}}$ or $y^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} = (\sim C)^{\mathcal{I}}$. If $y^{\mathcal{I}} \in C^{\mathcal{I}}$, for the ABox $\mathcal{A}'$ built with the choice $D = C$ we have that $\mathcal{I}$ satisfies $\mathcal{A}'$. Else, if $y^{\mathcal{I}} \in (\sim C)^{\mathcal{I}}$, $\mathcal{I}$ satisfies $\mathcal{A}'$ for the choice $D = \sim C$. In any case, $\mathcal{I}$ is a model of the ABox $\mathcal{A}'$ obtained by applying the ©-rule to $\mathcal{A}$.

$\square$

Proof of Part 2 of Lemma 6 Let $\mathcal{A}$ be a complete and clash-free ABox that is obtained by applying the completion rules to $\{C_0(x_0)\}$. We define the *canonical interpretation* $\mathcal{I}_{\mathcal{A}}$ of $\mathcal{A}$ as follows:

1. The domain $\Delta^{\mathcal{I}_{\mathcal{A}}}$ of $\mathcal{I}_{\mathcal{A}}$ consists of all the individual names $x \in \mathsf{NI}$ occurring in $\mathcal{A}$.

2. For all concept names $C \in \mathsf{NC}$ we define $C^{\mathcal{I}_{\mathcal{A}}} := \{x \mid C(x) \in \mathcal{A}\}$.

3. For all role names $R \in \mathsf{NR}$ we define $R^{\mathcal{I}_{\mathcal{A}}} := \{(x, y) \mid R(x, y) \in \mathcal{A}\}$.

4. For all individual names $x^{\mathcal{I}_{\mathcal{A}}} := x$ (i.e. the variable assignment π is the identity on NI).

We show that $\mathcal{I}_{\mathcal{A}}$ satisfies every constraint in $\mathcal{A}$.

By definition, $\mathcal{I}_{\mathcal{A}}$ satisfies all the role assertions of the form $R(x, y)$, iff $R(x, y) \in \mathcal{A}$. More generally, y is an $(R_1 \circ \cdots \circ R_m)$-successor of x in $\mathcal{A}$ iff y is an $(R_1 \circ \cdots \circ R_m)$-successor of x in $\mathcal{I}_{\mathcal{A}}$. Furthermore, $y \neq z$ implies $y^{\mathcal{I}_{\mathcal{A}}} \neq z^{\mathcal{I}_{\mathcal{A}}}$ by construction of $\mathcal{I}_{\mathcal{A}}$. By induction on the structure of concept descriptions, it can be easily shown that $\mathcal{I}_{\mathcal{A}}$ satisfies the concept assertions as well, provided that $\mathcal{A}$ is complete and clash-free. Again, we restrict our attention to number restrictions and the ⓞ-rule, since the induction base and the treatment of other constructors is the same as for $\mathcal{ALC}$.

- First, consider any assertion $(\exists^{\bowtie n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$ and all y's which are $(R_1 \circ \cdots \circ R_m)$-successors of x in $\mathcal{A}$. Then, for each of them, either $C(y) \in \mathcal{A}$ or $\sim C(y) \in \mathcal{A}$, otherwise the ⓞ-rule could be applied. Moreover, it can be easily proved (by induction on the structure of C) that $\{C(y), \sim C(y)\} \subseteq \mathcal{A}$ would lead to a clash.

- Consider $(\exists^{\geq n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$. Since $\mathcal{A}$ is complete, the $\geq$-rule cannot be applied to $(\exists^{\geq n} R_1 \circ \cdots \circ R_m.C)(x)$ and, thus, x has at least n $(R_1 \circ \cdots \circ R_m)$-successors which are instances of C in $\mathcal{A}$, which are also $(R_1 \circ \cdots \circ R_m)$-successors of x and instances of C in $\mathcal{I}_{\mathcal{A}}$ (by induction, $y \in C^{\mathcal{I}_{\mathcal{A}}}$ for each y with $C(y) \in \mathcal{A}$). Hence, $x \in (\exists^{\geq n} R_1 \circ \cdots \circ R_m.C)^{\mathcal{I}_{\mathcal{A}}}$

- Constraints with the form $(\exists^{\leq n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$ are satisfied since $\mathcal{A}$ is clash-free and complete. In fact, assume that x has more than n $(R_1 \circ \cdots \circ R_m)$-successors which are instances of C in $\mathcal{I}_{\mathcal{A}}$. Then x has more than n $(R_1 \circ \cdots \circ R_m)$-successors which are instances of C also in $\mathcal{A}$. If $\mathcal{A}$ contained inequality constraints $y_i \neq y_j$ for all these successors, then we would have a clash. Otherwise, the $\leq$-rule could be applied.

$\square$

Proof of Part 3 of Lemma 6 Assume that $\mathcal{A}$ contains a clash. If $\{A(x), (\neg A)(x)\} \subseteq \mathcal{A}$, then clearly no interpretation can satisfy both constraints. Thus assume that $(\exists^{\leq n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$ and x has $p > n$ $(R_1 \circ \cdots \circ R_m)$-successors $y_1, \ldots, y_p$ with $\{C(y_i) \mid 1 \leq i \leq p\} \cup \{y_i \neq y_j \mid 1 \leq i < j \leq p\} \subseteq \mathcal{A}$. Obviously, this implies that, in any model $\mathcal{I}$ of $\mathcal{A}$, $x^{\mathcal{I}}$ has $p > n$ $(R_1 \circ \cdots \circ R_m)$-successors which are instances of C in $\mathcal{I}$, which shows that $\mathcal{I}$ cannot satisfy $(\exists^{\leq n} R_1 \circ \cdots \circ R_m.C)(x)$. $\square$

Proof of Part 4 of Lemma 6 We must show that the Tableau algorithm that tests satisfiability of $\mathcal{ALCN}(\circ)$-concepts always terminates. In the following, we consider only ABoxes $\mathcal{A}$ that are obtained by applying the completion rules to $\{C_0(x_0)\}$. For a concept C, we define its and/or-size $|C|_{\sqcap, \sqcup}$ as the number of $\sqcap$ and $\sqcup$ constructors in C. The maximal role depth $depth(C)$ of C is defined as follows:

$$
\begin{aligned}
depth(A) = depth(\neg A) &:= 0 \quad \text{for } A \in \mathsf{NC} \\
depth(C_1 \sqcap C_2) = depth(C_1 \sqcup C_2) &:= \max\{depth(C_1), depth(C_2)\} \\
depth(\exists^{\bowtie n} R_1 \circ \cdots \circ R_m.C) &= m + depth(C)
\end{aligned}
$$

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Let C_0 be an $\mathcal{ALCN}(\circ)$-concept in NNF$^{\bowtie}$, and let $\mathcal{A}$ an ABox obtained by applying the completion rules to $\{C_0(x_0)\}$. As an easy consequence of the definition of the completion rules, we can observe the following facts:

1. Every variable $x \neq x_0$ that occurrs in $\mathcal{A}$ is an $(R_1 \circ \cdots \circ R_m)$-successor of x_0 for some role chain of length $m \geq 1$. In addition, every other role chain that connects x_0 with x has the same length.

2. If x can be reached in $\mathcal{A}$ by a role chain of length m from x_0, then for each constraint $C(x) \in \mathcal{A}$, the maximal role depth of C is bounded by the maximal role depth of C_0 minus m (i.e. $depth(C) \leq depth(C_0) - m$). Consequently, $m \leq depth(C_0)$.

Let m_0 be the maximal role depth of C_0. Because of the first fact, every individual x in a ABox $\mathcal{A}$ (reached from $\{C_0(x_0)\}$ by applying completion rules) has a unique role level $level(x)$, which is its distance from the root node x_0, i.e. the unique length of the role chains that connect x_0 with x. Owing to the second fact, the level of each individual is an integer between 0 and m_0.

In the following, we define a mapping K of ABoxes $\mathcal{A}$ to a $3(m_0 + 1)$-tuple of non-negative integers such that $\mathcal{A} \to \mathcal{A}'$ implies $K(\mathcal{A}) \succ K(\mathcal{A}')$, where $\succ$ denotes the lexicographic ordering on tuples. Since the lexicographic ordering is well-founded, this implies termination of the algorithm. In fact, if the algorithm did not terminate, then there would exist an infinite sequence $\mathcal{A}_0 \to \mathcal{A}_1 \to \cdots$, and this would yield an infinite descending $\succ$-chain of tuples.

Hence, let $\mathcal{A}$ be an ABox that can be reached from $\{C_0(x_0)\}$ by applying completion rules. We define:

$$K(\mathcal{A}) := (\overline{\kappa}_0, \overline{\kappa}_1, \ldots, \overline{\kappa}_{m_0-1}, \overline{\kappa}_{m_0}),$$

where (sub)tuple $\overline{\kappa}_\ell = (\kappa_\ell^1, \kappa_\ell^2, \kappa_\ell^3)$ and the components κ_ℓ^i are obtained as follows:

- κ_ℓ^1 is the number of individual variables x in $\mathcal{A}$ with $level(x) = \ell$.

- κ_ℓ^2 is the sum of the and/or sizes $|C|_{\sqcap,\sqcup}$ of all constraints $C(x) \in \mathcal{A}$ such that $level(x) = \ell$ and the $\sqcap$- or $\sqcup$-rule is applicable to $C(x)$.

- For a constraint $\alpha(x) = (\exists^{\geq n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$, let s be the cardinality of *maximal sets* $\{y_1, \ldots, y_s\}$, such that y_i is an $(R_1 \circ \cdots \circ R_m)$-successor of x, $C(y_i) \in \mathcal{A} \, (1 \leq i \leq s)$ and $y_i \neq y_j \in \mathcal{A} \, (1 \leq i < j \leq s)$. Then we associate with the constraint the number $r(\alpha(x)) = \max\{n - s, 0\}$, representing the number of individuals that (possibly) still have to be added to $\mathcal{A}$ to make the constraint $\alpha(x)$ satisfied, and define κ_ℓ^3 as follows:

$$\kappa_\ell^3 \;\; = \sum_{\alpha(x) \in \mathcal{A}, level(x) = \ell} r(\alpha(x))$$

In the following, we show that $\mathcal{A} \to \mathcal{A}'$ implies $K(\mathcal{A}) \succ K(\mathcal{A}')$ for each of the completion rules in Fig. 3.

1. **$\sqcap$-rule.** Assume the rule is applied to the constraint $(C_1 \sqcap C_2)(x)$, let $\mathcal{A}'$ be the ABox obtained by its application and let $\ell = level(x)$. First we compare $\overline{\kappa}_\ell$ and $\overline{\kappa}'_\ell$, i.e. the tuples associated with level ℓ in $\mathcal{A}$ and $\mathcal{A}'$, respectively. The first components

κ_ℓ^1 and $\kappa_\ell^{1\prime}$ agree since the number of individuals and their levels have not been changed. For the second component, we have a decrease (i.e. $\kappa_\ell^{2\prime} < \kappa_\ell^2$), since $|C_1 \sqcap C_2|_{\sqcap,\sqcup}$ is removed from the sum, and replaced by a number that is no larger than $|C_1|_{\sqcap,\sqcup} + |C_2|_{\sqcap,\sqcup}$ (depending on whether the top constructor of C_1 and C_2 is $\sqcup$ or $\sqcap$, or another constructor). Since tuples are compared with lexicographic ordering, a decrease in the second component makes sure that what happens in the third component is irrelevant. For the same reason, we need not consider tuples κ_q for $q > \ell$. Tuples at levels $q < \ell$ are either unchanged or have their third component decreased by the application of the rule, since the addition of the new constraints may add x to one of the maximal sets involved in the κ_q^3 definition (e.g. if $(\exists^{\geq 1} R.C_1)(x')$ with $R(x',x) \in \mathcal{A}$ but $C_1(x) \notin \mathcal{A}$, we might have a decrement in $\kappa_{\ell-1}^3$ when $C_1(x)$ is added to $\mathcal{A}'$).

2. $\sqcup$**-rule.** This rule can be treated like the $\sqcap$-rule.

3. $\geq$**-rule.** Assume the rule is applied to the constraint $(\exists^{\geq n} R_1 \circ \cdots \circ R_m.C)(x)$, let $\mathcal{A}'$ be the ABox obtained by its application and let $\ell = level(x)$. The first two components of $\overline{\kappa}_\ell$ remains unchanged. The third component decreases (i.e. $\kappa_\ell^{3\prime} < \kappa_\ell^3$), since the new individuals $z_1, \ldots, z_{n-q}$ can now be added to the maximal sets of explicitly distinct individuals which are instances of C and $(R_1 \circ \cdots \circ R_m)$-successors of x used in the computation of s. For this reason, the increase in the first component of tuples of levels larger than ℓ is irrelevant (z_{i2}'s are added at level $\ell + 1$, ..., z_{im}'s at level $\ell + m - 1$, and z_i's are added at level $\ell + m$). Tuples at levels smaller than ℓ are either unchanged or have their third component decreased by the application of the rule.

4. $\leq$**-rule.** Assume the rule is applied to the constraint $(\exists^{\leq n} R_1 \circ \cdots \circ R_m.C)(x)$, let $\mathcal{A}'$ be the ABox obtained by its application and let $\ell = level(x)$. On level $\ell + m$, the first component κ_ℓ^1 decreases, since variable y_i is removed. Thus, possible increases in other components of $\overline{\kappa}_\ell$ are irrelevant. Tuples associated with smaller levels $q < \ell$ remain unchanged or decrease. In fact, the third component of tuples of smaller level cannot increase since for the individuals y_i and y_j that have been identified there was no inequality $y_i \neq y_j \in \mathcal{A}$. Moreover, since no constraints are removed and, in particular, y_j in $\mathcal{A}'$ has all its old constraints plus the constraints of y_i in $\mathcal{A}$, y_i may contribute to one of the maximal sets involved in the κ_q^3 definition (e.g. if $\{(\exists^{\geq 1} R.C)(x'), R(x', y_i), C(y_j)\} \subseteq \mathcal{A}$ we might have a decrement in $\kappa_{\ell-1}^3$).

5. $\copyright$**-rule.** Assume the rule is applied to the constraint $(\exists^{\bowtie n} R_1 \circ \cdots \circ R_m.C)(x)$, let $\mathcal{A}'$ be the ABox obtained by its application and let $\ell = level(x)$. Obviously, the first two components remain unchanged at every level. Tuples at levels q smaller than $\ell + m$ have their third component unchanged or decreased, since the addition of the constraint $D(y)$ (with $level(y) = \ell + m$) may add some new individual to some of the maximal sets involved in the κ_q^3 definition.

$\square$

Complexity issues

The tableau-based satifiability algorithm proposed above for $\mathcal{ALCQ}(\circ)$ may require exponential time and space. The optimization strategies profitably employed for $\mathcal{ALCN}$

and $\mathcal{ALCQ}$ [24, 5] do not seem to be applicable to $\mathcal{ALCN}(\circ)$ and $\mathcal{ALCQ}(\circ)$. As a matter of fact, such strategies rely on the fact that the underying Logics have the *tree model* property, and, for the sake of satisfiability testing, the individuality of different role-successors of a given domain object is not relevant. Only the number of such successors counts (for $\geq$- and $\leq$-rule applicabilty and clash testing) and, thus, a single successor at a time can be used as "representative" also for its siblings, when continuing the algorithm for its further role-successors. In such a way, only one branch of the tree model at a time can be generated and investigated by the algorithm, giving rise to a non-deterministic procedure consuming only polynomial space and, thus, to PSPACE complexity (since NPSPACE =PSPACE, owing to Savitch's Theorem [23]). In our case, such an optimization does not seem to be possible, since $\mathcal{ALCN}(\circ)$ and $\mathcal{ALCQ}(\circ)$ do not have the tree model property, as number restrictions $\exists^{\geq p} R_1 \circ \cdots \circ R_{m-1} \sqcap \exists^{\leq q} R_1 \circ \cdots \circ R_{m-1} \circ R_m$ (with $p > q$) make some separate ($R_1 \circ \cdots \circ R_{m-1}$) role chains merge into confluent ($R_1 \circ \cdots \circ R_{m-1} \circ R_m$) chains to respect both kinds of number restrictions. In fact, the identifications of successors effected by the $\leq$-rule (say at level ℓ) may involve individuals generated by previous executions of the $\geq$-rule for different $\exists^{\geq n} R_1 \circ \cdots \circ R_m.C(x)$ constraints, with possibly different values of $level(x)$ and role chain lenghts (with the proviso that $level(x) + 1 \leq \ell \leq level(x) + m$). The enforcement of mutual constraints between possibly "intersecting" role chains strictly relies on the *individuation* of single successors, and cannot be surrogated, in general, via representatives. As a result, the algorithm in Fig. 3 is a non-deterministic procedure possibly producing complete ABoxes of *exponential size* in the length of the input concept description (even if binary coding of numbers is assumed).

Lemma 7 *Given a complete ABox $\mathcal{A}$ generated by the algorithm in Fig. 3, the size of $\mathcal{A}$ is exponential in the input size s, thanks to the following facts:*

1. *The number a of individuals in $\mathcal{A}$ is $O(2^{p(s)})$, where p is a polynomial funtion.*

2. *The number of constraints in $\mathcal{A}$ is a polynomial function of a.*

Let us define the *size c_0* of the concept description C_0 as the total number of symbols (operators, concept and role names) it contains, and let $N := \max\{n | \exists^{\geq n} R_1 \circ \cdots \circ R_m.C$ is a subconcept of $C_0\}$. Moreover, the number of subconcepts of C_0 is obviously bounded by c_0.

Proof of Fact 1 According to Fig. 3, new individuals (apart from x_0) are added to $\mathcal{A}$ by the application of the $\geq$-rule only. The algorithm execution generates a connected structure with the shape of a tree, rooted on x_0, where some node coincide (owing to $\leq$-rule applications). Each path in this tree-structure has a maximal length which is bounded by the maximal role depth m_0 of C_0. The out-degree is bounded by the maximal number of new successors that can be generated from an individual x. This number cannot exceed $c_0 N m_0$, since the number of of times the $\geq$-rule can be applied to a constraint on x is limited by the total number of $\exists^{\geq n}$ constructors in C_0 and, thus, by c_0 and, for each application of the $\geq$-rule, no more than $N m_0$ new individuals can be added. Hence, the total size of the tree-structure is bounded by $(c_0 N m_0)^{m_0} = 2^{m_0(\log c_0 + \log N + \log m_0)} \leq 2^{2c_0^2 + c_0 \log N}$, since $m_0 \leq c_0$. Obviously, the exponent is a polynomial function of the input size, even if binary coding of numbers is adopted. $\square$

Proof of Fact 2 For each individual x, $\mathcal{A}$ may at most contain a pair of constraints $\{C(x), \sim C(x)\}$ for each subconcept C of C_0. Hence, the total number of constraints

with the form $D(x)$ in $\mathcal{A}$ is bounded by $2c_0$. Moreover, for each pair of individuals x and y, the number of constraints with the form $R(x, y)$ (or $x \neq y$) in $\mathcal{A}$ is limited by the number of role names in C_0, which is strictly less than c_0, plus one (for inequality constraints). Hence, the size of $\mathcal{A}$ is surely bounded by $2c_0 a + c_0 a^2$ (we can derive a tighter bound if we take into account the role levels of individuals). $\qquad \square$

As it can be easily seen, the two facts together give a space consumption bounded by $2^{6s^2 + s}$.

Corollary 6 *By the given algorithm, deciding satisfiability (subsumption) of $\mathcal{ALCQ}(\circ)$ concepts is in the* NExpTime *(*co-NExpTime*) complexity class.*

3.1 An extension of the decidability result

We provide in this Section an extension of the algorithm given in Fig. 3 for $\mathcal{ALCQ}(\circ)$-concept satisfiability, such that it can also deal with union and/or intersection of role chains *of the same length*. The extension follows the same directions of the similar extension proposed for $\mathcal{ALCN}(\circ)$ in [4]. Analogously, also the soundness, completeness and termination proofs of our extended algorithm are very similar to the ones proposed for the basic algorithm in the previous Section and, thus, they will only be sketched.

The general form of a role expression $\mathcal{R}$ we consider here is the following:

$$\mathcal{R} \;=\; \bigsqcup_{i=1}^{M} \prod_{j=1}^{N_i} (R_1^{ij} \circ R_2^{ij} \circ \cdots R_m^{ij})$$

that is we assume, for the sake of simplicity, Boolean role chain combinations to be in Disjunctive Normal Form[1] (DNF). In the presence of role expressions of this kind, we modify the definition of role successor for a complex role chain $\mathcal{R}$ as follows.

Definition 6 *The individual y is a $\mathcal{R}$-successor of x in $\mathcal{A}$ (where $\mathcal{R}$ is defined as above) iff for some $\hat{\imath}$ ($1 \leq \hat{\imath} \leq M$), $\exists y_2^1 y_3^1 \ldots y_m^1 y_2^2 y_3^2 \ldots y_m^2 \, y_2^{N_{\hat{\imath}}} y_3^{N_{\hat{\imath}}} \ldots y_m^{N_{\hat{\imath}}}$ variables in $\mathcal{A}$ such that $\{R_k^{\hat{\imath}j}(y_k^j, y_{k+1}^j) \mid 2 \leq k \leq m-1, 1 \leq j \leq N_{\hat{\imath}}\} \cup \{R_1^{\hat{\imath}j}(x, y_2^j), R_m^{\hat{\imath}j}(y_m^j, y) \mid 1 \leq j \leq N_{\hat{\imath}}\} \subseteq \mathcal{A}$.*

Notice that, owing to this definition, role successors in $\mathcal{A}$ are also successors in every model $\mathcal{I}$ of $\mathcal{A}$: if $\mathcal{I}$ satisfies $\mathcal{A}$, and y is an $\mathcal{R}$-successor of x in $\mathcal{A}$, then $y^{\mathcal{I}}$ is an $\mathcal{R}$-successor of $x^{\mathcal{I}}$ in $\mathcal{I}$.

The Tableau algorithm is extended by replacing the completion rules dealing with number restrictions and the $\circledcirc$-rule with the rules shown in Fig. 4, so that the new complex role chains can be managed.

In order to prove that the new algorithm decides concept satisfiability for this $\mathcal{ALCQ}(\circ)$ extension, we must prove that all four parts of Lemma 6 still hold.

1. *Local correctness* of the $\geq'$-, $\leq'$- and $\circledcirc'$-rules can be shown as in the proof of Part 1 of the Lemma 6.

2. The *canonical model* induced by a complete and clash-free ABox is defined as in the proof of Part 2 of the Lemma 6. The proof that this canonical model satisfies the ABox is also similar to the one provided for Lemma 6. Note that the definition we used for $\mathcal{R}$-successors coincides with the notion of $\mathcal{R}$-successors in the canonical model $\mathcal{I}_{\mathcal{A}}$ induced by $\mathcal{A}$.

---

[1] General $\sqcap/\sqcup$ combinations of role chains can be put in DNF (which may require an exponential time) by rewriting concept C_0 before the execution of the satisfiability algorithm.

$\geq'$-rule: **if**   1. $(\exists^{\geq n}\mathcal{R}.C)(x) \in \mathcal{A}$ and

              2. x has exactly p $\mathcal{R}$-successors $y_1, \ldots, y_p$ with $p < n$

                 such that $\{C(y_i) \mid 1 \leq i \leq p\} \cup \{y_i \neq y_j \mid 1 \leq i < j \leq p\} \subseteq \mathcal{A}$

         then $\mathcal{A}' := \mathcal{A} \cup \{R_1^{i_i j}(x, z_{i2}^j), R_2^{i_i j}(z_{i2}^j, z_{i3}^j), \ldots, R_m^{i_i j}(z_{im}^j, z_i^j), C(z_i)$

$$\mid 1 \leq i \leq n - p, 1 \leq j \leq N_{i_i}\}$$

$$\cup \{z_i^\ell \neq z_j^\ell \mid 1 \leq i < j \leq n - p, 1 \leq \ell \leq N_{i_i}\}$$

$$\cup \{y_i \neq z_j^\ell \mid 1 \leq i \leq p, 1 \leq j \leq n - p, 1 \leq \ell \leq N_{i_i}\}$$

                 for some $\{i_1, i_2, \ldots, i_{n-p}\} \subseteq \{1, \ldots, M\}$,

                 where z_{ik}^j, z_i^j (for $1 \leq i \leq n - p, 2 \leq k \leq m, 1 \leq j \leq N_{i_i}$)

                 are $m \sum_{i=1}^{n-p} N_{i_i}$ fresh variables

$\leq'$-rule: **if**   1. $(\exists^{\leq n}\mathcal{R}.C)(x) \in \mathcal{A}$ and

              2. x has more than n $\mathcal{R}$-successors $y_1, \ldots, y_p$ such that

                 $\{C(y_i) \mid 1 \leq i \leq p\} \subseteq \mathcal{A}$ and $\{y_i \neq y_j\} \cap \mathcal{A} = \emptyset$ for some i, j $(1 \leq i < j \leq p)$,

         then   for some pair y_i, y_j $(1 \leq i < j \leq p)$ such that $\{y_i \neq y_j\} \cap \mathcal{A} = \emptyset$

                 $\mathcal{A}' := [y_i/y_j]\mathcal{A}$

$\copyright'$-rule: **if**   1. $(\exists^{\bowtie n}\mathcal{R}.C)(x) \in \mathcal{A}$ and

              2. y is an $\mathcal{R}$-successor of x such that $\{C(y), \sim C(y)\} \cap \mathcal{A} = \emptyset$

         then $\mathcal{A}' := \mathcal{A} \cup \{D(y)\}$ for some $D \in \{C, \sim C\}$

Figure 4: The Completion Rules for $\mathcal{ALCQ}(\circ)$ extended with complex role chains

3. The proof that an ABox containing a clash is unsatisfiable is the same as the one given above. This follows from the fact that role successors in an ABox $\mathcal{A}$ are also successors in every model $\mathcal{I}$ of $\mathcal{A}$.

4. The proof of *termination* is also very similar to the one considered before. The definition of the depth of a concept is extended in the obvious way to expressions involving complex roles:

$$depth(\exists^{\bowtie n}\mathcal{R}.C) = depth\left(\exists^{\bowtie n}\big(\overset{M}{\underset{i=1}{\sqcup}}\,\overset{N_i}{\underset{j=1}{\sqcap}}(R_1^{ij} \circ R_2^{ij} \circ \cdots R_m^{ij})\big).C\right) \;=\; m + depth(C)$$

Since role chains in complex roles are all of the same length, the two facts stated in the proof of Part 4 of Lemma 6 are still valid and, thus, we can define the same metric $K(\mathcal{A})$ as before also on all the ABoxes that are produced by the new completion rules. It can be seen that the proof that $\mathcal{A} \to \mathcal{A}'$ implies $\mathcal{K}(\mathcal{A}) \succ K(\mathcal{A}')$ carries over to the new rules. Actually, the proof given in Part 4 of Lemma 6 only relies on the fact that all role chains connecting any two individuals have the same length, which is still satisfied in the extended logics.

An immediate consequence of these observations is the Theorem that follows:

Theorem 2 *Concept satifiability (and subsumption) for the logic that extends $\mathcal{ALCQ}(\circ)$ with union/intersections of role chains of the same length is decidable, and the Tableau algorithm based on the completion rules in Fig. 4 is an effective decision procedure.*

As far as complexity of the algorithm is concerned, Lemma 7 holds also for the algorithm in Fig. 4. The only modification required is to the proof of Fact 2, in the tree-structure out-degree evaluation, since the application of each $\geq'$-rule may generate at most $Nm_0\hat{N}$ successors, where $\hat{N}$ is the maximal number of conjuncts occurring in a role chain combination. Since $\hat{N} \leq c_0$, the number of individuals in $\mathcal{A}$ is now bounded by $2^{3c_0^2 + c_0 \log N}$.

Corollary 7 *By the given algorithm, deciding concept satisfiability (subsumption) for the logic that extends $\mathcal{ALCQ}(\circ)$ with union/intersections of role chains of the same length is in the* NExpTime *(*co-NExpTime*) complexity class.*

4 Conclusions

In this paper we studied expressive Description Logics allowing for number restrictions on complex roles built with the composition operator ($\circ$), extended with other role constructors in $\{^-, \sqcup, \sqcap\}$ and qualified number restrictions.

In this framework, we improved the (un)decidability results by Baader and Sattler on logics of the $\mathcal{ALCN}$ family [4] by showing that $\mathcal{ALC\bar{N}}(\circ,^-)$ is undecidable via reduction of a domino problem, whereas the introduction of qualified number restrictions in $\mathcal{ALCQ}(\circ)$ (and in its extension with $\sqcap/\sqcup$ combinations of role chains with the same length) does not hinder decidability of reasoning. For $\mathcal{ALCQ}(\circ)$, a tableau-based satisfiability algorithm with a NExpTime upper bound has been proposed.

As we observed in the Introduction that known decidability results also lift up to $\mathcal{ALCQ}(^-, \sqcup, \sqcap)$, we shed some new light on the whole decidability scenario ranging from

$\mathcal{ALCN}$ to $\mathcal{ALCQ}(\circ,^-,\sqcup,\sqcap)$. In this picture, a big unanswered question concerns decidability of $\mathcal{ALCN}(\circ,\sqcup)$, whereas a small gap left open concerns decidability of "pure" $\mathcal{ALCN}(\circ,^-)$ (around the narrow borders of this gap, we proved in this work that the language with inverses in value restrictions and inverses and composition of roles under unqualified number restrictions is undecidable, whereas the language with inverses and role composition under value restrictions and inverses under qualified number restrictions is decidable, as it is a sublanguage of $\mathcal{CIQ}$ [16]). Another open question is the exact characterization of $\mathcal{ALCQ}(\circ)$ (and $\mathcal{ALCN}(\circ)$) complexity, as the NExpTime bound we derived may be far from being tight. Future work will also consider such issues.

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