November 2003

THE UNIVERSITY of MANCHESTER

Computer Science
University of Manchester

Interaction between Knowledge, Action and Commitment within Agent Dynamic Logic

Renate A. Schmidt
Dmitry Tishkovsky
Ullrich Hustadt

Department of Computer Science
University of Manchester
Preprint Series
CSPP-27

Interaction between Knowledge, Action and Commitment within Agent Dynamic Logic

Renate A. Schmidt

Department of Computer Science, University of Manchester, United Kingdom
schmidt@cs.man.ac.uk

Dmitry Tishkovsky

Department of Computer Science, University of Manchester, United Kingdom
dmitry@cs.man.ac.uk

Ullrich Hustadt

Department of Computer Science, University of Liverpool, United Kingdom
U.Hustadt@csc.liv.ac.uk

November 2003

Abstract

This paper considers a new class of agent dynamic logics which provide a formal means of specifying and reasoning about the agents' activities and informational, motivational and practical aspects of the behaviour of the agents. We present a Hilbert-style deductive system for a basic agent dynamic logic and consider a number of extensions of this logic with axiom schemata formalising interactions between knowledge and commitment (expressing an agent's awareness of her commitments), and interactions between knowledge and actions (expressing no learning and persistence of knowledge after actions). The deductive systems are proved sound and complete with respect to a Kripke-style semantics. Each of the considered logics is shown to have the small model property and therefore decidable.

1 Introduction

Logical methods are widely used for studying and formalising multi-agent systems. Modal logics, in particular, have been popular for this purpose and offer a number of advantages [23, 25]. Among the more well-known agent formalisms with a modal flavour are the BDI model [18, 19], the *KARO* framework [16, 25], and temporal logics of knowledge and belief [3, 7, 8]. Examples of more recent work includes the proposal of a modal logic framework of belief and change in [11], and an epistemic dynamic logic in [10].

Despite the existence of many different agent theories, there has been little progress in studying the logical properties of agent theories. Few sound and complete deductive systems for agent theories exist, and those that do exist are formulated for languages of modest expressiveness. Also, at the moment little is known about the decidability of the reasoning problem in agent theories. Generally, agent theories are very expressive and therefore decidability has not been a primary concern in the area of agent-based systems. Nevertheless, decidability is an important property, particularly, when agent theories are (to be) used in practice. Even though in many instances decidable agent theories may be too weak to adequately specify more complex properties of agents, identifying decidable fragments of agent theories still provides valuable information for safety-critical applications and the development of verification tools for agent-based systems.

This paper considers a new class of *agent dynamic logics* and presents results on soundness, completeness and decidability. Agent dynamic logics are intended to provide formal means of specifying and reasoning about the agents' activities and informational, motivational and practical aspects of the behaviour of the agents. The agent dynamic logics presented in this paper are most closely related to the *KARO* framework proposed by Meyer, van der Hoek and van Linder [16, 25]. One of the contributions of the *KARO* framework is the formalisation of the motivational attitudes of agents. An agent's motivational attitudes determine her motivation to perform an action: What are the agent's intentions? What are her wishes? What are her goals? When an agent knows that she has some goal she may be willing to commit herself to an action in order to achieve the goal, or she may be unwilling to commit herself to an action which is infeasible in connection with the goal. However, being committed to an action does not on its own imply that the agent performs this action. In addition, she needs the opportunity and ability to do so. The *KARO* framework presents an approach to modelling the actions of agents and the related motivational and intentional aspects based on propositional dynamic logic *PDL* enriched by operators for concepts such as knowledge, wishes, goals, abilities and commitments.

Although a proof theory has been developed for a part of *KARO* [25], at this moment it is not known whether the whole of *KARO* can be formalised as a logical system. This paper can be viewed as progress in this respect. Because *KARO* contains some sophisticated operators for managing the agents' goals and commitments, *KARO* is not easily axiomatisable by adding axiom schemata to those of deterministic *PDL*. In order to use the powerful machinery developed in the area of modal logic, our approach has been to omit some operators from *KARO* and give modal logic characterisations of the properties of the remaining operators. Although this necessarily reduces the expressivity, the logics we obtain have sound and complete axiomatisations, and some other nice properties, namely, the small model property, hence the finite model property, and decidability. Moreover, we believe the resulting logics remain expressive enough to be useful in applications. The considered logics are more expressive than the core fragment of *KARO* considered in recent work [12, 13] and the part of *KARO* for which a proof system is presented in [25].

The basic logic introduced in this paper is called *agent dynamic logic* (*ADL*). The logic includes for each agent i and each action α the dynamic logic operators $[\alpha]_i$ and $\langle\alpha\rangle_i$. The intended interpretation is that i represents an agent and $[\alpha]_i\phi$ says property ϕ holds always after agent i has performed the action α. $\langle\alpha\rangle_i\phi$ says there is a state in which property ϕ holds, after agent i has performed the action α. That an agent has the opportunity to perform an action can be expressed by $\langle\alpha\rangle_i\top$. *ADL* includes for each agent i the operator A_i which can be used to represent the ability to perform an action (by writing $\mathsf{A}_i\alpha$). In order to describe formally the commitments of an agent, *ADL* has a static commitment constructor Comm_i which states the commitment of an agent to an action (written $\mathsf{Comm}_i\alpha$). There are also two dynamic constructors c and u which allow the representation of the actions of committing or undoing a commitment to an action (written $\mathsf{c}\alpha$ and $\mathsf{u}\alpha$). Their intended use is in the scope of an action modality, for example, $[\mathsf{c}\alpha]_i\mathsf{Comm}_i\alpha$ states that always after agent i commits to the action α she is committed to performing that action. The operators K_i and W_i are used to express an agent's knowledge and wishes. $\mathsf{K}_i\phi$ is meant to be read as: agent i knows ϕ, while $\mathsf{W}_i\phi$ is meant to be read as: agent i wishes ϕ to hold.

In modelling intelligent agents a key concern is the connection between knowledge and action. In *ADL* there is no interdependence between an agent's knowledge and actions performed by the agent. However it may be natural to ascribe properties of no learning and perfect recall to intelligent agents (see e.g. [10, 23]). No learning is commonly formulated by the axiom schema

(no learning) $$[\alpha]_i\mathsf{K}_i\phi \rightarrow \mathsf{K}_i[\alpha]_i\phi$$

which says that agent i knows the result of her action in advance, in other words, there is no learning. Perfect recall, also called accordance, is commonly formulated by the axiom schema

(perfect recall) $$\mathsf{K}_i[\alpha]_i\phi \rightarrow [\alpha]_i\mathsf{K}_i\phi$$

which expresses persistence of the agent's knowledge after the execution of an action. This schema is useful in situations where agents can devise plans to achieve certain goals: if the agent knows that an action is in accordance with a goal then after performing the action, the agent knows that the goal was brought about [25]. It may also be natural to ascribe the property of awareness of commitments to agents. This can be formulated by the axiom schema

(known commitments) $$\mathsf{Comm}_i\alpha \rightarrow \mathsf{K}_i\mathsf{Comm}_i\alpha$$

which expresses that agent i knows about her commitments [16]. In this paper we consider *ADL* and extensions with the above axiom schemata.

To illustrate the capabilities of the language of agent dynamic logic consider the scenario of an examination situation for two agents, a teacher t and a student s.

- $\mathsf{A}_t\mathtt{give\_task}$ represents that the teacher is able to give a task to the student.
- $\langle\mathtt{give\_task}\rangle_t\top$ represents that the teacher has an opportunity to give a task.
- $\langle\mathtt{perform\_task}\rangle_s\top$ represents that the student has an opportunity to perform the task.
- $\mathsf{K}_s[\mathtt{give\_task}]_t\mathsf{Comm}_s\mathtt{perform\_task}$ represents that the student knows he is committed to performing the task whenever it is given by the teacher (though he may not have the ability to perform the task).
- $\mathsf{A}_s\alpha \to \mathsf{K}_s\mathsf{A}_s\alpha$ represents that the student is aware of his abilities.

Although this example is small, we believe that it is possible to formally describe contract agreements in the language of agent dynamic logics, and then to use the appropriate logic for analysis and verification purposes. For example, the specification of teamwork might be a use for agent dynamic logics. In the development of a complex project, the duties of the individual developers are usually fixed at the outset. So, if we can formally represent the duties and abilities of the developers in the team, as well as the goals of the project then we can reason about the stages of the project and their interdependencies. On the other hand, as soon as an optimal sequence of stages has been determined one can assign tasks to developers and fix conditions of teamwork such that the optimum is realised.

From a logical perspective, *ADL* and its extensions with the above axiom schemata, particularly, perfect recall and no learning, are products of logics with complex modalities. Products [5] have found applications in knowledge representation where they are used to add an epistemic dimension to program or description languages. It is known that it is extremely difficult to obtain decidability results for products of logics and the small model property usually fails in products [6]. Generally, when combining modal logics, the commutativity axioms above are notorious for making the reasoning problem much more complex. For example, allowing this kind of interaction between propositional linear time temporal logic and *S5* turns the satisfiability problem from a PSPACE-complete problem into a 2EXPTIME-complete problem [8]. Even worse, letting three *S5* modalities commute pairwise turns a PSPACE-complete logic (the independent join of several copies of *S5*) into an undecidable logic [14]. In the literature, investigations of products of logics use a subtle quasi-model technique [28], filtrations of a special kind or other advanced techniques (see [6, 27, 5]). For us, another difficulty arises because the dynamic logic modalities are allowed to be complex. This makes it impossible to apply the techniques developed for products of standard modal logics directly. Going from standard modal logic to program logics and to program logics with additional axiom schemata the constructions in the proofs of completeness and the filtration property become increasingly complex and require specialised 'filters'. Whereas filtration over subformulae is sufficient for modal logics, for program logics suitable sets of formulae need to be identified over which to filtrate. The next step when non-standard operators are added is to extend the proof techniques to accommodate the new operators, in our case the operators of ability and commitment. Next when considering extra axioms, even in the case where these axioms are Sahlqvist, there are difficulties in proving completeness, because although the canonical model has the expected correspondence property, the canonical model is not a standard model. To construct a standard model from the canonical model filtration is used, but we must take care to ensure the preservation of the correspondence property. Because of the presence of the commutativity axioms our proofs use the filtration technique of [6] developed for products of modal logics. If we only consider instances of commutativity axioms where α is an atomic action, this technique preserves the correspondence properties for these axioms in the filtrated model. This proves completeness for logics with commutativity axioms under *atomic* substitutivity. The preservation of the correspondence property becomes more intricate when we move from atomic actions to complex actions. Sometimes it is not possible to prove the preservation of correspondence for all actions. For example, as is shown in this paper, commutativity for knowledge and atomic actions gives a logic different from a logic in which knowledge and arbitrary actions commute.

The structure of the paper is as follows. The axiomatics and semantics of the class of agent dynamic logics considered in this paper is defined in Section 2. Soundness results and Sahlqvist correspondence results are proved. In Section 3, we first establish completeness with respect to non-standard models for the considered logics. A modification of the filtration procedure allows us then to prove completeness with respect to standard models, the small model property and decidability. *ADL* is proved to have these properties in Section 4, while Section 5 considers three different extensions of *ADL* and *test-free ADL* with the axioms of perfect recall, no learning and known commitments. Depending on whether these axioms are assumed to be closed under full substitutivity or are assumed to hold for atomic actions only, and also whether the logic includes the test operator, logics with different properties are obtained. In turns out, the knowledge operator is vacuous in some logics. In Section 6 we establish that the satisfiability problem of *ADL* is EXPTIME-complete by reduction of the satisfiability problem of *ADL* to that of *PDL* with converse. Section 7 discusses the relationship between agent dynamic logic and the *KARO* framework. We also make some observations about test and knowledge, and the awareness of abilities, actions and goals. Section 8 summarises the contributions of this paper and the final section lists the references.

This paper assumes the reader has a basic familiarity with modal logic and propositional dynamic logic. Good textbooks of modal logic and dynamic logic include [1, 2, 9, 15].

2 Axiomatisation and semantics

The language $\mathcal{L}$ of the logic *ADL* is defined over the following primitive types: a countable set $\mathrm{Var} = \{p, q, r, \ldots\}$ of propositional variables, a countable set $\mathrm{AtAc} = \{a_0, a_1, a_3, \ldots\}$ of atomic action variables, and a countable set Ag of agents. $\mathcal{L}$ is constructed over the Boolean connectives, $\rightarrow$ and $\perp$, the dynamic logic connectives, $\cup$ (non-deterministic choice), ; (sequential execution), $*$ (repetition), ? (test or confirm)[1] and $[\_]_i$ (box), where i denotes any agent, and modal connectives K_i (knowledge) and W_i (wishes). In addition, there are four unary operations on actions, A_i (read 'agent i has the ability to perform'), Comm_i (read 'agent i is committed to'), c (read 'commit to') and u (read 'uncommit').

The set For of formulae and the set Ac of actions in $\mathcal{L}$ are the smallest sets that satisfy the following conditions ($i \in \mathrm{Ag}$).

- $\mathrm{AtAc} \subseteq \mathrm{Ac}$, $\mathrm{Var} \cup \{\perp\} \subseteq \mathrm{For}$.
- If ϕ and ψ are formulae in For and α and β are action terms in Ac then: $\phi?$, α^*, $\mathsf{c}\alpha$, $\mathsf{u}\alpha$, $\alpha \cup \beta$, $\alpha;\beta$ are action terms in Ac, and $\phi \rightarrow \psi$, $\mathsf{K}_i\phi$, $\mathsf{W}_i\phi$, $[\alpha]_i\phi$, $\mathsf{A}_i\alpha$, $\mathsf{Comm}_i\alpha$ are formulae in For.

By definition, an *atomic action* is an action variable, and a *semi-atomic action* is an atomic action or a test action $\phi?$.

As usual we use the following abbreviations: $\neg\phi$ for $\phi \rightarrow \perp$, $\phi \vee \psi$ for $\neg\phi \rightarrow \psi$, $\phi \wedge \psi$ for $\neg(\neg\phi \vee \neg\psi)$, $\langle\alpha\rangle_i\phi$ for $\neg[\alpha]_i\neg\phi$, and $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. In accordance with [16], a few additional operators useful for formulating aspects of practical reasoning relating to the agents' motivational attitudes can be defined.

$$\begin{aligned}
\mathsf{PracPoss}_i(\alpha, \phi) &=^{\mathrm{def}} \langle\alpha\rangle_i\phi \wedge \mathsf{A}_i\alpha \\
\mathsf{Can}_i(\alpha, \phi) &=^{\mathrm{def}} \mathsf{K}_i\mathsf{PracPoss}_i(\alpha, \phi) \\
\mathsf{Cannot}_i(\alpha, \phi) &=^{\mathrm{def}} \mathsf{K}_i\neg\mathsf{PracPoss}_i(\alpha, \phi) \\
\mathsf{Goal}_i\phi &=^{\mathrm{def}} \mathsf{W}_i\phi \wedge \neg\phi \\
\mathsf{Intend}_i(\alpha, \phi) &=^{\mathrm{def}} \mathsf{PracPoss}_i(\alpha, \phi) \wedge \mathsf{K}_i\mathsf{Goal}_i\phi
\end{aligned}$$

The $\mathsf{PracPoss}_i$ operator expresses the practical possibility for the agent to bring about a result ϕ by executing the action α and is defined as agent i being able to perform α and ϕ possibly holding after α is executed by agent i. The Can_i and Cannot_i operators express the agent's awareness about the practical possibilities and impossibilities to execute an action α and obtain a result ϕ. The Goal_i operator defines goals to be those wishes that are unfulfilled.[2] The $\mathsf{Intend}_i(\alpha, \phi)$ operator is meant to formalise that agent i intends to

[1] In the agent applications the test operator of dynamic logic is more appropriately referred to as the confirm operator, because intuitively the test of ϕ by a conscious agent causes an expansion of the knowledge of the agent [25]. Here, $\phi?$ is meant to represent the confirmation of ϕ. Thus, all agents can confirm a property if it is true in the current state.

[2] A more fine-grained selection condition consisting of unfulfillment and implementability is used in [16]. In order to obtain a modal logic formalisation we make the simplifying assumption that goals need not be implementable.

perform α to achieve ϕ. Hence, this operator is based on practical possibility with respect to the agent's goals.

A Hilbert-style axiomatisation of *ADL* is presented in Figure 1. Thus, *ADL* is a combination of infinitely many copies of *PDL*, *S5*, and *K* with extra axioms (Axioms A5–A7) formulating properties of the ability and commitment operators. Axiom A5.1 expresses the fact that being able to perform a non-deterministic choice of two actions is exactly the same as being able to perform the first action or being able to perform the second one. Axioms A5.2–A5.4 represent an optimistic (or bold) interpretation of what it means to be able to perform sequences of actions. In particular, Axiom A5.2 declares that agent i is omnipotent in dead-end states, more precisely, when an agent does have the ability but not the opportunity to perform an action α then the agent has the ability to perform the sequence $\alpha;\beta$ for arbitrary actions β.[3] Axioms A5.3 and A5.4 formalise the same property, i.e. the persistence and sequential behaviour of the agent's abilities with respect to the operator of unbounded repetition. We have chosen to formalise the static commitment operator by a similar set of axioms. Axiom A6.1 lets the agent choose to what action she is committed if she is committed to the non-deterministic choice of two actions. The three axioms A6.2, A6.3 and A6.4 postulate the sequential behaviour of commitments, in particular, the persistence of agent's commitments with respect to sequential execution of actions. The group of axioms A7 determine the interplay of the dynamic commitment operators with the ability and static commitment operators. Axioms A7.1 and A7.2 specify that an agent cannot be uncommitted after committing to an action and, vice versa, cannot be committed just after uncommitting. Axioms A7.3 and A7.4 ensure that all agents are always able to perform actions of committing and uncommitting. Since being committed to committing to an action or being committed to uncommitting to an action is not constructive, Axioms A7.5 and A7.6 reduce such expressions to simpler forms. The axioms say to be committed to committing to an action is exactly the same as to be committed to this action, and similarly, to be committed to uncommitting to an action is exactly the same as not being committed to the concerned action.

Observe that *ADL* is not just an independent join of logics because it contains non-standard operators to express the agents' abilities and commitments. Moreover, the modal operators of *ADL* interact via the axioms for ability and commitment. This means that the logical properties of *ADL* cannot be obtained straightforwardly from the properties of the constituent logics.

Let L be a logic in the language of *ADL*. An *L-proof* (from a set Γ of formulae) is a finite sequence $\phi_0, \ldots, \phi_n$ of *ADL* formulae, where each ϕ_i ($0 \leq i \leq n$) is either a substitution instance of an axiom of L (or ϕ belongs to Γ) or it follows by one of the inference rules from preceding formulae in the sequence. A formula ψ is *L-derivable* (from a set Γ of formulae) iff there is an L-proof $\phi_0, \ldots, \phi_n$ (from Γ) such that $\psi = \phi_n$. An L-derivable formula is also called an *L-theorem* or a *theorem of L*. Otherwise it is not

[3] A more pessimistic (or cautious) agent, by contrast, would need certainty that there is an opportunity to perform action α before she can conclude anything about her ability following the execution of α. This meaning is described by $\mathsf{A}_i\alpha;\beta \leftrightarrow \mathsf{A}_i\alpha \wedge \langle\alpha\rangle_i\mathsf{A}_i\beta$ [25].

A1. All classical tautologies over $\mathcal{L}$

A2. The *PDL* axioms (see [17, 9]) for each $[\alpha]_i$ ($\alpha \in \mathrm{Ac}$, $i \in \mathrm{Ag}$):

1. $[\alpha]_i(\phi \to \psi) \to ([\alpha]_i\phi \to [\alpha]_i\psi)$
2. $[\alpha \cup \beta]_i\phi \leftrightarrow [\alpha]_i\phi \wedge [\beta]_i\phi$
3. $[\alpha;\beta]_i\phi \leftrightarrow [\alpha]_i[\beta]_i\phi$
4. $[\alpha^*]_i\phi \to \phi \wedge [\alpha]_i\phi$
5. $[\alpha^*]_i\phi \to [\alpha]_i[\alpha^*]_i\phi$
6. $\phi \wedge [\alpha^*]_i(\phi \to [\alpha]_i\phi) \to [\alpha^*]_i\phi$
7. $[\phi?]_i\psi \leftrightarrow (\phi \to \psi)$

A3. The *S5* axioms for each K_i ($i \in \mathrm{Ag}$):

1. $\mathsf{K}_i(\phi \to \psi) \to (\mathsf{K}_i\phi \to \mathsf{K}_i\psi)$
2. $\mathsf{K}_i\phi \to \mathsf{K}_i\mathsf{K}_i\phi$
3. $\mathsf{K}_i\phi \to \phi$
4. $\neg\mathsf{K}_i\phi \to \mathsf{K}_i\neg\mathsf{K}_i\phi$

A4. The K axiom for each W_i ($i \in \mathrm{Ag}$):

1. $\mathsf{W}_i(\phi \to \psi) \to (\mathsf{W}_i\phi \to \mathsf{W}_i\psi)$

A5. The following axioms for the ability operators A_i ($i \in \mathrm{Ag}$):

1. $\mathsf{A}_i\alpha \cup \beta \leftrightarrow \mathsf{A}_i\alpha \vee \mathsf{A}_i\beta$
2. $\mathsf{A}_i\alpha;\beta \leftrightarrow \mathsf{A}_i\alpha \wedge [\alpha]_i\mathsf{A}_i\beta$
3. $\mathsf{A}_i\alpha^* \to \mathsf{A}_i\alpha \wedge [\alpha]_i\mathsf{A}_i\alpha^*$
4. $\mathsf{A}_i\alpha \wedge [\alpha^*]_i(\mathsf{A}_i\alpha \to [\alpha]_i\mathsf{A}_i\alpha) \to \mathsf{A}_i\alpha^*$

A6. A similar group of axioms for the static commitment operators Comm_i ($i \in \mathrm{Ag}$):

1. $\mathsf{Comm}_i\alpha \cup \beta \leftrightarrow \mathsf{Comm}_i\alpha \vee \mathsf{Comm}_i\beta$
2. $\mathsf{Comm}_i\alpha;\beta \leftrightarrow \mathsf{Comm}_i\alpha \wedge [\alpha]_i\mathsf{Comm}_i\beta$
3. $\mathsf{Comm}_i\alpha^* \to \mathsf{Comm}_i\alpha \wedge [\alpha]_i\mathsf{Comm}_i\alpha^*$
4. $\mathsf{Comm}_i\alpha \wedge [\alpha^*]_i(\mathsf{Comm}_i\alpha \to [\alpha]_i\mathsf{Comm}_i\alpha) \to \mathsf{Comm}_i\alpha^*$

A7. Axioms for the dynamic commit and uncommit operators, c and u:

1. $[\mathsf{c}\alpha]_i\mathsf{Comm}_i\alpha$
2. $[\mathsf{u}\alpha]_i\neg\mathsf{Comm}_i\alpha$
3. $\mathsf{A}_i\mathsf{c}\alpha$
4. $\mathsf{A}_i\mathsf{u}\alpha$
5. $\mathsf{Comm}_i\mathsf{c}\alpha \leftrightarrow \mathsf{Comm}_i\alpha$
6. $\mathsf{Comm}_i\mathsf{u}\alpha \leftrightarrow \neg\mathsf{Comm}_i\alpha$

R. The inference rules:

1. modus ponens: $\phi, \phi \to \psi \vdash \psi$
2. the necessitation rules for the modal connectives ($\alpha \in \mathrm{Ac}$, $i \in \mathrm{Ag}$):

$$\phi \vdash [\alpha]_i\phi \qquad\qquad \phi \vdash \mathsf{K}_i\phi \qquad\qquad \phi \vdash \mathsf{W}_i\phi$$

Figure 1: Hilbert-style deductive system for *ADL*

L-derivable (from Γ). A set of formulae Γ is *L-consistent* iff $\bot$ is not derivable from Γ, and *L-inconsistent* otherwise. A formula ψ is *L-consistent* iff the set $\{\psi\}$ is consistent, and *L-inconsistent* otherwise.

The semantics of *ADL* is a Kripke-style semantics, that is, models are defined over relational structures, called frames. An *ADL-frame* is a tuple

$$F =^{\mathrm{def}} \langle S, Q_i, R_i^{\mathsf{K}}, R_i^{\mathsf{W}}, A_i, Comm_i, Q_i^{\mathsf{c}}, Q_i^{\mathsf{u}} \rangle_{i \in \mathrm{Ag}},$$

of a non-empty set S of states, families of functions Q_i from the set AtAc into S^2, families of binary relations R_i^{W} on S, families of equivalence relations R_i^{K} on S, families of functions A_i and $Comm_i$ from the set of semi-atomic action into the set of subsets of S, and families of functions Q_i^{c} and Q_i^{u} from the set of all actions into the set of binary relations on S.

For any subset X of S, let Id_X denote the identity relation on X, i.e. the relation $\{(s,s) \mid s \in X\}$. For any relation R, let $\mathrm{Dom}(R)$ be the domain of R, i.e. $\mathrm{Dom}(R) =^{\mathrm{def}} \{s \mid \exists t\, (s,t) \in R\}$. Let $R \circ R'$ denote the relational composition of two relations R and R', and let R^* denote the reflexive transitive closure of R, i.e. $R \circ R' =^{\mathrm{def}} \{(x,y) \mid \exists z\, (x,z) \in R \;\&\; (z,y) \in R'\}$ and $R^* =^{\mathrm{def}} \bigcup_{n \geq 0} R^n$, where $R^0 =^{\mathrm{def}} \mathrm{Id}_{\mathrm{Dom}(R)}$ and $R^{n+1} =^{\mathrm{def}} R \circ R^n$ for $n > 0$.

For action terms α and β, let the relations $Q_i(\alpha)$, $Q_i(\beta)$ and sets $A_i(\alpha)$, $A_i(\beta)$, $Comm_i(\alpha)$, $Comm_i(\beta)$ be given. Define Q_i, A_i, and $Comm_i$ on action terms built from α and β as in Figure 2.

Now, an *ADL-model* is a tuple $M =^{\mathrm{def}} \langle F, \models \rangle_{i \in \mathrm{Ag}}$ where F is an *ADL*-frame and $\models$ is the truth relation which is closed under the conditions of Figures 2 and 3. Taking into account that $Comm_i(\alpha) = \{s \mid s \models \mathsf{Comm}_i\alpha\}$, $A_i(\alpha) = \{s \mid s \models \mathsf{A}_i\alpha\}$, and similar observations for Q_i^{c} and Q_i^{u}, any *ADL*-model can also be viewed as a tuple $\langle S, Q_i, R_i^{\mathsf{K}}, R_i^{\mathsf{W}}, \models \rangle_{i \in \mathrm{Ag}}$ which looks like a normal Kripke model.

As usual we write $M \models \phi$ iff $\forall s \in M\; M, s \models \phi$. For any class K of models and any set $\Gamma \cup \{\phi\}$ of formulae we use an abbreviation $\Gamma \models_K \phi$ for

$$\forall M \in K\; ((\forall \gamma \in \Gamma\; M \models \gamma) \Rightarrow M \models \phi).$$

We say that a logic L is *sound* with respect to the class K of models iff all the theorems of the logic L are valid in every model of K. L is *complete* with respect to the class K iff all the formulae valid in all models from K are the theorems of L. L is *strongly complete* with respect to K iff for any set $\Gamma \cup \{\phi\}$ of formulae $\Gamma \models_K \phi$ implies that ϕ is L-derivable from Γ.

The fragment of *ADL* with all the operators and connectives of *ADL* except the test operator is called the *test-free* fragment. In the frames and models of *test-free ADL* the set of $Q_i(\alpha)$ relations is closed only under union, sequential composition, reflexive transitive closure and dynamic commitment operators.

The extensions of *(test-free) ADL* we consider in this paper are defined by variants of the following axiom schemata.

(known commitments) $\qquad\qquad \mathsf{Comm}_i\alpha \to \mathsf{K}_i\mathsf{Comm}_i\alpha$

$$
\begin{aligned}
Q_i(\alpha \cup \beta) &= Q_i(\alpha) \cup Q_i(\beta)\\
Q_i(\alpha;\beta) &= Q_i(\alpha) \circ Q_i(\beta)\\
Q_i(\alpha^*) &= (Q_i(\alpha))^*\\
Q_i(\mathsf{u}\alpha) &= Q_i^{\mathsf{u}}(\alpha) \circ \mathrm{Id}_{S \setminus Comm_i(\alpha)}\\
Q_i(\mathsf{c}\alpha) &= Q_i^{\mathsf{c}}(\alpha) \circ \mathrm{Id}_{Comm_i(\alpha)}\\
A_i(\alpha \cup \beta) &= A_i(\alpha) \cup A_i(\beta)\\
A_i(\alpha;\beta) &= A_i(\alpha) \cap \{s \mid \forall t \in S\,(sQ_i(\alpha)t \Rightarrow t \in A_i(\beta))\}\\
A_i(\alpha^*) &= A_i(\alpha) \cap \{s \mid \forall t, u \in S\,(\, sQ_i(\alpha^*)t \;\&\; t \in A_i(\alpha) \Rightarrow\\
&\qquad\qquad (tQ_i(\alpha)u \Rightarrow u \in A_i(\alpha))\,)\}\\
A_i(\mathsf{c}\alpha) &= S\\
A_i(\mathsf{u}\alpha) &= S\\
Comm_i(\alpha \cup \beta) &= Comm_i(\alpha) \cup Comm_i(\beta)\\
Comm_i(\alpha;\beta) &= Comm_i(\alpha) \cap \{s \mid \forall t \in S\,(sQ_i(\alpha)t \Rightarrow t \in Comm_i(\beta))\}\\
Comm_i(\alpha^*) &= Comm_i(\alpha) \cap \{s \mid \forall t, u \in S\,(\, sQ_i(\alpha^*)t \;\&\; t \in Comm_i(\alpha) \Rightarrow\\
&\qquad\qquad (tQ_i(\alpha)u \Rightarrow u \in Comm_i(\alpha))\,)\}\\
Comm_i(\mathsf{c}\alpha) &= Comm_i(\alpha)\\
Comm_i(\mathsf{u}\alpha) &= S \setminus Comm_i(\alpha)
\end{aligned}
$$

Figure 2: Definition of Q_i, A_i and $Comm_i$

$$
\begin{array}{rll}
M, s \not\models \bot & & \\
sQ_i(\phi?)t & \text{iff} & s = t \text{ and } M, t \models \phi\\
M, s \models \phi \to \psi & \text{iff} & M, s \models \phi \Rightarrow M, s \models \psi\\
M, s \models \mathsf{K}_i\phi & \text{iff} & \forall t\,(sR_i^{\mathsf{K}}t \Rightarrow M, t \models \phi)\\
M, s \models \mathsf{W}_i\phi & \text{iff} & \forall t\,(sR_i^{\mathsf{W}}t \Rightarrow M, t \models \phi)\\
M, s \models [\alpha]_i\phi & \text{iff} & \forall t\,(sQ_i(\alpha)t \Rightarrow M, t \models \phi)\\
M, s \models \mathsf{A}_i\alpha & \text{iff} & s \in A_i(\alpha)\\
M, s \models \mathsf{Comm}_i\alpha & \text{iff} & s \in Comm_i(\alpha)
\end{array}
$$

Figure 3: Definition of $\models$

(no learning)	$[\alpha]_i \mathsf{K}_i \phi \to \mathsf{K}_i [\alpha]_i \phi$
(perfect recall)	$\mathsf{K}_i [\alpha]_i \phi \to [\alpha]_i \mathsf{K}_i \phi$

When α ranges over the set of all actions we refer to these schemata as $KC(\alpha)$, $NL(\alpha)$ and $PR(\alpha)$, respectively. If α ranges only over the set of atomic actions then these schemata are referred to as $KC(a)$, $NL(a)$ and $PR(a)$, respectively. If $KC(\alpha)$ holds then we say the axiom of known commitments holds under *full substitution* (of the actions), whereas if $KC(a)$ holds then we say the axiom of known commitments holds under *atomic substitution* (of the actions). Similarly, for the other axiom schemata.

In addition to ADL, the following agent dynamic logics are considered.

(ADL natural)	$ADL^\natural =^{\text{def}} ADL + KC(\alpha) + NL(\alpha) + PR(\alpha)$
(ADL sharp)	$ADL^\sharp =^{\text{def}} ADL + KC(a) + NL(a) + PR(a)$
(ADL flat)	$ADL^\flat =^{\text{def}} \textit{test-free } ADL + KC(\alpha) + NL(\alpha) + PR(\alpha)$

Axiom schemata $KC(\alpha)$, $NL(\alpha)$, $PR(\alpha)$, $KC(a)$, $NL(a)$ and $PR(a)$ are just sets of Sahlqvist formulae. By the well-known Sahlqvist correspondence theorem [20] these schemata are satisfied in a modal logic model iff the model satisfies the following (first-order) frame properties.

$$Comm_i(\alpha) = S \setminus \mathrm{Dom}(R_i^{\mathsf{K}} \circ \mathrm{Id}_{S \setminus Comm_i(\alpha)})$$

(right commutativity)	$R_i^{\mathsf{K}} \circ Q_i(\alpha) \subseteq Q_i(\alpha) \circ R_i^{\mathsf{K}}$
(left commutativity)	$Q_i(\alpha) \circ R_i^{\mathsf{K}} \subseteq R_i^{\mathsf{K}} \circ Q_i(\alpha)$

We represent these properties by $clc(\alpha)$,[4] $com^r(\alpha)$ and $com^l(\alpha)$ when α is allowed to range over all the actions, and $clc(a)$, $com^r(a)$ and $com^l(a)$ when α ranges over the set of atomic actions only.

Theorem 1 (Sahlqvist Theorem) The axiom schema Ax is satisfied in an ADL-model M iff M satisfies $f(Ax)$, where f is defined by the following:

$KC(\alpha) \mapsto clc(\alpha)$	$KC(a) \mapsto clc(a)$
$NL(\alpha) \mapsto com^r(\alpha)$	$NL(a) \mapsto com^r(a)$
$PR(\alpha) \mapsto com^l(\alpha)$	$PR(a) \mapsto com^l(a)$

By definition, an $ADL^\natural$*-model* ($ADL^\flat$*-model*) is an ADL-model (a *test-free* ADL-model) which satisfies properties $clc(\alpha)$, $com^r(\alpha)$ and $com^l(\alpha)$. An $ADL^\sharp$*-model* is an ADL-model which satisfies properties $clc(a)$, $com^r(a)$ and $com^l(a)$.

Soundness for these logics can be proved as usual.

Theorem 2 (Soundness) Let L be one of the logics *test-free* ADL, ADL, $ADL^\natural$, $ADL^\sharp$, or $ADL^\flat$. Then, L is sound with respect to the class of all L-models.

In the next sections we will prove completeness, the small model theorem and decidability.

[4] The abbreviation *clc* stands for 'clustering of commitments'.

3 Completeness with respect to non-standard models

To begin with, we will prove completeness with respect to non-standard models. Here we use a standard technique of constructing a canonical model of ADL. Recall that canonical models are built from maximally consistent sets of formulae. A *maximal consistent set (MCS)* is a consistent set Γ of formulae such that Γ includes all theorems of ADL, and any superset of Γ is inconsistent. Any consistent set of ADL-formulae can be extended to a MCS in a standard way.

Let S be the set of all MCSs. We define the appropriate relations on S by the following equivalences:

$$sR_i^{\mathsf{K}}t \ \text{ iff } \ \forall \mathsf{K}_i\phi \in s,\ \phi \in t \qquad sR_i^{\mathsf{W}}t \ \text{ iff } \ \forall \mathsf{W}_i\phi \in s,\ \phi \in t$$
$$sQ_i(\alpha)t \ \text{ iff } \ \forall [\alpha]_i\phi \in s,\ \phi \in t \qquad s \models \phi \ \text{ iff } \ \phi \in s$$

The model $M = \langle S, Q_i, R_i^{\mathsf{K}}, R_i^{\mathsf{W}}, \models\rangle$ thus obtained is the *canonical model* of ADL.

Lemma 3 (Existence Lemma) For any state $s \in S$ in the canonical model of ADL the following are true.

1. If $\neg\mathsf{K}_i\neg\phi \in s$ then $\exists t \in S\ (sR_i^{\mathsf{K}}t\ \&\ \phi \in t)$.
2. If $\neg\mathsf{W}_i\neg\phi \in s$ then $\exists t \in S\ (sR_i^{\mathsf{W}}t\ \&\ \phi \in t)$.
3. If $\neg[\alpha]_i\neg\phi \in s$ then $\exists t \in S\ (sQ_i(\alpha)t\ \&\ \phi \in t)$.

Lemma 4 For any state $s \in S$ in the canonical model of ADL the following are true.

1. R_i^{K} is an equivalence relation.
2. $s \models \mathsf{Comm}_i\alpha$ whenever $tQ_i(\mathsf{c}\alpha)s$ for some $t \in S$.
3. $s \not\models \mathsf{Comm}_i\alpha$ whenever $tQ_i(\mathsf{u}\alpha)s$ for some $t \in S$.
4. $Q_i(\alpha \cup \beta) = Q_i(\alpha) \cup Q_i(\beta)$.
5. $Q_i(\alpha;\beta) = Q_i(\alpha) \circ Q_i(\beta)$.
6. $s \models [\alpha^*]_i\phi$ iff $s \models \phi \wedge [\alpha]_i[\alpha^*]_i\phi$.
7. $s \models [\alpha^*]_i\phi$ iff $s \models \phi \wedge [\alpha^*]_i(\phi \rightarrow [\alpha]_i\phi)$.
8. $s \models \mathsf{Comm}_i\alpha \cup \beta$ iff $s \models \mathsf{Comm}_i\alpha \vee \mathsf{Comm}_i\beta$.
9. $s \models \mathsf{Comm}_i\alpha;\beta$ iff $s \models \mathsf{Comm}_i\alpha \wedge [\alpha]_i\mathsf{Comm}_i\beta$.
10. $s \models \mathsf{Comm}_i\alpha^*$ iff $s \models \mathsf{Comm}_i\alpha \wedge [\alpha^*]_i(\mathsf{Comm}_i\alpha \rightarrow [\alpha]_i\mathsf{Comm}_i\alpha)$.

11. $s \models \mathsf{Comm}_i\mathsf{c}\alpha$ iff $s \models \mathsf{Comm}_i\alpha$.

12. $s \models \mathsf{Comm}_i\mathsf{u}\alpha$ iff $s \not\models \mathsf{Comm}_i\alpha$.

13. $s \models \mathsf{A}_i\alpha \cup \beta$ iff $s \models \mathsf{A}_i\alpha \vee \mathsf{A}_i\beta$.

14. $s \models \mathsf{A}_i\alpha;\beta$ iff $s \models \mathsf{A}_i\alpha \wedge [\alpha]_i\mathsf{A}_i\beta$.

15. $s \models \mathsf{A}_i\alpha^*$ iff $s \models \mathsf{A}_i\alpha \wedge [\alpha^*]_i(\mathsf{A}_i\alpha \to [\alpha]_i\mathsf{A}_i\alpha)$.

16. $s \models \mathsf{A}_i\mathsf{c}\alpha$ for all $s \in S$.

17. $s \models \mathsf{A}_i\mathsf{u}\alpha$ for all $s \in S$.

Proof. All items are direct consequences of the corresponding *ADL* axioms. □

Using Lemmata 3 and 4 we can prove that the canonical model of *ADL* is almost an *ADL*-model. The only property which is not valid in the canonical model is $Q_i(\alpha^*) = (Q_i(\alpha))^*$, so it is not a standard model for *ADL*. Any model which satisfies all the properties of Lemma 4 will be called a *non-standard ADL-model*. Similarly for *test-free ADL*.

The following completeness theorems hold.

Theorem 5 *(Test-free) ADL* is complete with respect to its canonical model.

Theorem 6 *(Test-free) ADL* is strongly complete with respect to the class of all non-standard *(test-free) ADL*-models.

By Sahlqvist correspondence the canonical model for extensions of *(test-free) ADL* by the considered axioms will possess the corresponding properties. Therefore:

Theorem 7 Any extension of *(test-free) ADL* by subsets of $\{KC(\alpha), NL(\alpha), PR(\alpha), KC(a), NL(a), PR(a)\}$ is strongly complete with respect to its canonical model. Moreover, such extensions are complete with respect to all non-standard *(test-free) ADL*-models which satisfy the properties corresponding to the additional axioms.

4 Completeness with respect to standard models and decidability of *ADL*

To obtain completeness with respect to standard models we use the filtration technique commonly used in modal logic (see e.g. [1, 2, 9, 15]). More specifically, we combine the Fischer-Ladner technique for *PDL* with the special filtration method of Gabbay and Shehtman [6] which is suitable to prove the small model property for axiomatic products of *K* and *S5*. In the next section we extend this method to prove the Small Model Theorem for extensions of *ADL*.

Definition 1 Let $\prec$ be the smallest transitive relation over $\{0,1\} \times$ For satisfying the following.

$$\begin{aligned}
(0,\phi),(0,\psi) &\prec (0,\phi\to\psi)\\
(0,\phi),(1,[\alpha]_i\phi) &\prec (0,[\alpha]_i\phi)\\
(0,\phi) &\prec (1,[\phi?]_i\psi)\\
(1,[\alpha]_i\phi),(1,[\beta]_i\phi) &\prec (1,[\alpha\cup\beta]_i\phi)\\
(1,[\alpha]_i[\beta]_i\phi),(1,[\beta]_i\phi) &\prec (1,[\alpha;\beta]_i\phi)\\
(1,[\alpha]_i[\alpha^*]_i\phi) &\prec (1,[\alpha^*]_i\phi)\\
(0,\mathsf{Comm}_i\alpha) &\prec (1,[\mathsf{c}\alpha]_i\phi),(1,[\mathsf{u}\alpha]_i\phi)\\
(0,\phi) &\prec (0,\mathsf{Comm}_i\phi?)\\
(0,\mathsf{Comm}_i\alpha),(0,\mathsf{Comm}_i\beta) &\prec (0,\mathsf{Comm}_i\alpha\cup\beta)\\
(0,\mathsf{Comm}_i\alpha),(0,[\alpha]_i\mathsf{Comm}_i\beta) &\prec (0,\mathsf{Comm}_i\alpha;\beta)\\
(0,[\alpha^*]_i\mathsf{Comm}_i\alpha) &\prec (0,\mathsf{Comm}_i\alpha^*)\\
(0,\mathsf{Comm}_i\alpha) &\prec (0,\mathsf{Comm}_i\mathsf{c}\alpha),(0,\mathsf{Comm}_i\mathsf{u}\alpha)\\
(0,\phi) &\prec (0,\mathsf{A}_i\phi?)\\
(0,\mathsf{A}_i\alpha),(0,\mathsf{A}_i\beta) &\prec (0,\mathsf{A}_i\alpha\cup\beta)\\
(0,\mathsf{A}_i\alpha),(0,[\alpha]_i\mathsf{A}_i\beta) &\prec (0,\mathsf{A}_i\alpha;\beta)\\
(0,[\alpha^*]_i\mathsf{A}_i\alpha) &\prec (0,\mathsf{A}_i\alpha^*)\\
(0,\phi) &\prec (0,\mathsf{K}_i\phi),(0,\mathsf{W}_i\phi)
\end{aligned}$$

Let $\preceq$ be the reflexive closure of $\prec$. For a set Σ of formulae, the *Fischer-Ladner closure* of Σ is defined as

$$FL(\Sigma) =^{\text{def}} \{\psi \mid \exists j \in \{0,1\}\, \exists\phi\in\Sigma\ (j,\psi)\preceq(0,\phi)\}.$$

We write $FL(\phi)$ instead $FL(\{\phi\})$.

It is easy to see that $\prec$ is well-founded. Consequently, $FL(\Sigma)$ is finite whenever Σ is finite. The next lemma allows us to find an upper bound for the cardinality of $FL(\Sigma)$ in terms of the lengths of formulae in Σ.

We denote by $|\phi|$ and $|\alpha|$ the length of ϕ and α, respectively (as the number of the symbols excluding parentheses, brackets and agent variables). By $\#\phi$ we denote the maximal number of occurrences of the symbols ; and $*$ below the operators Comm_i, A_i, c and u in ϕ.

Lemma 8

1. $\mathrm{Card}(\{(j,\psi) \mid (j,\psi)\preceq(0,\phi)\}) \leq |\phi|2^{\#\phi}$.
2. $\mathrm{Card}(\{(j,\psi) \mid (j,\psi)\preceq(1,[\alpha]_i\phi)\}) \leq |\alpha|2^{\#[\alpha]_i\phi}$.
3. $\mathrm{Card}(\{(j,\psi) \mid (j,\psi)\preceq(0,\mathsf{Comm}_i\alpha)\}) \leq |\alpha|2^{\#\mathsf{Comm}_i\alpha}$.

4. $\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0,\mathsf{A}_i\alpha)\}) \leq |\alpha| 2^{\#\mathsf{A}_i\alpha}$.

Proof. All properties can be proved simultaneously by induction on the well-founded relation $\prec$. The cases $\phi = \bot$ and $\phi = p$ are clear. In the induction step, we distinguish whether ϕ is of the form $\xi \to \eta$, $[\alpha]_i\eta$, $\mathsf{Comm}_i\alpha$, $\mathsf{A}_i\alpha$, $\mathsf{K}_i\eta$, or $\mathsf{W}_i\eta$. Concerning property 1 we have:

$$\begin{aligned}
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0,\xi \to \eta)\}) \\
&\quad = \mathrm{Card}(\{(0,\xi \to \eta)\} \cup \{(j,\psi) \mid (j,\psi) \preceq (0,\xi)\} \cup \{(j,\psi) \mid (j,\psi) \preceq (0,\eta)\}) \\
&\quad \leq 1 + |\xi| 2^{\#\xi} + |\eta| 2^{\#\eta} \leq (1 + |\xi| + |\eta|) 2^{\#\xi\to\eta} = |\xi \to \eta| 2^{\#\xi\to\eta} \\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0,[\alpha]_i\eta)\}) \\
&\quad = \mathrm{Card}(\{(0,[\alpha]_i\eta)\} \cup \{(j,\psi) \mid (j,\psi) \preceq (1,[\alpha]_i\eta)\} \cup \{(j,\psi) \mid (j,\psi) \preceq (0,\eta)\}) \\
&\quad \leq 1 + |\alpha| 2^{\#[\alpha]_i\eta} + |\eta| 2^{\#\eta} \leq (1 + |\alpha| + |\eta|) 2^{\#[\alpha]_i\eta} = |[\alpha]_i\eta| 2^{\#[\alpha]_i\eta} \\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0,\mathsf{Comm}_i\alpha)\}) \leq |\alpha| 2^{\#\mathsf{Comm}_i\alpha} \leq |\mathsf{Comm}_i\alpha| 2^{\#\mathsf{Comm}_i\alpha} \\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0,\mathsf{A}_i\alpha)\}) \leq |\alpha| 2^{\#\mathsf{A}_i\alpha} \leq |\mathsf{A}_i\alpha| 2^{\#\mathsf{A}_i\alpha} \\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0,\mathsf{K}_i\eta)\}) = \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0,\mathsf{W}_i\eta)\}) \\
&\quad \leq 1 + |\eta| 2^{\#\eta} \leq |\mathsf{K}_i\eta| 2^{\#\mathsf{K}_i\eta} = |\mathsf{W}_i\eta| 2^{\#\mathsf{W}_i\eta}
\end{aligned}$$

With respect to properties 2, 3, and 4 we distinguish whether α is of the form a, where a is an atomic action, $\xi?$, $\beta \cup \gamma$, $\beta;\gamma$, β^*, or $\mathsf{c}\beta$ (the case for α having the form $\mathsf{u}\beta$ is identical to the case $\mathsf{c}\beta$). For property 2 we have:

$$\begin{aligned}
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (1,[a]_i\eta\}) = 1 \leq |a| 2^{\#[a]_i\eta} \\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (1,[\xi?]_i\eta\}) \\
&\quad = 1 + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0,\xi)\}) = 1 + |\xi| 2^{\#\xi} \leq |\xi?| 2^{\#[\xi?]_i\eta} \\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (1,[\beta \cup \gamma]_i\eta)\}) \\
&\quad = 1 + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (1,[\beta]_i\eta)\} \cup \{(j,\psi) \mid (j,\psi) \preceq (1,[\gamma]_i\eta)\}) \\
&\quad \leq 1 + |\beta| 2^{\#[\beta]_i\eta} + |\gamma| 2^{\#[\gamma]_i\eta} \leq (1 + |\beta| + |\gamma|) 2^{\#[\beta\cup\gamma]_i\eta} = |\beta \cup \gamma| 2^{\#[\beta\cup\gamma]_i\eta} \\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (1,[\beta;\gamma]_i\eta)\}) \\
&\quad = 1 + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (1,[\beta]_i[\gamma]_i\eta)\} \cup \{(j,\psi) \mid (j,\psi) \preceq (1,[\gamma]_i\eta)\}) \\
&\quad \leq 1 + |\beta| 2^{\#[\beta]_i[\gamma]_i\eta} + |\gamma| 2^{\#[\gamma]_i\eta} \leq (1 + |\beta| + |\gamma|) 2^{\#[\beta;\gamma]_i\eta} = |\beta;\gamma| 2^{\#[\beta;\gamma]_i\eta} \\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (1,[\beta^*]_i\eta)\}) \\
&\quad = 1 + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (1,[\beta]_i[\beta^*]_i\eta)\}) \\
&\quad \leq 1 + |\beta| 2^{\#[\beta]_i[\beta^*]_i\eta} + \leq (1 + |\beta|) 2^{\#[\beta]_i[\beta^*]_i\eta} = |\beta^*| 2^{\#[\beta^*]_i\eta} \\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (1,[\mathsf{c}\beta]_i\eta)\}) = \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (1,[\mathsf{u}\beta]_i\eta)\}) \\
&\quad = 1 + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0,\mathsf{Comm}_i\beta)\}) \leq 1 + |\beta| 2^{\#\mathsf{Comm}_i\beta} \\
&\quad \leq (1 + |\beta|) 2^{\#[\mathsf{c}\beta]_i\eta} = (1 + |\beta|) 2^{\#[\mathsf{u}\beta]_i\eta} = |\mathsf{c}\beta| 2^{\#[\mathsf{c}\beta]_i\eta} = |\mathsf{u}\beta| 2^{\#[\mathsf{u}\beta]_i\eta}
\end{aligned}$$

For property 3 we have:

$$
\begin{aligned}
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i a)\}) = 1 \leq |a| 2^{\#\mathsf{Comm}_i a}\\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i \xi?)\})\\
&\quad = 1 + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0,\xi)\}) \leq 1 + |\xi| 2^{\#\xi} \leq |\xi?| 2^{\#\mathsf{Comm}_i \xi?}\\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i \beta \cup \gamma)\})\\
&\quad = 1 + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i \beta)\} \cup \{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i \gamma)\})\\
&\quad \leq 1 + |\beta| 2^{\#\mathsf{Comm}_i \beta} + |\gamma| 2^{\#\mathsf{Comm}_i \gamma} \leq |\beta \cup \gamma| 2^{\#\mathsf{Comm}_i \beta \cup \gamma}\\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i \beta;\gamma)\})\\
&\quad \leq\ 1 + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i \beta)\})\\
&\qquad\quad + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, [\beta]_i \mathsf{Comm}_i \gamma)\})\\
&\quad \leq\ 1 + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i \beta)\})\\
&\qquad\quad + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (1, [\beta]_i \mathsf{Comm}_i \gamma)\})\\
&\qquad\quad + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i \gamma)\})\\
&\quad \leq\ 1 + |\beta| 2^{\#\mathsf{Comm}_i \beta} + |\beta| 2^{\#[\beta]_i \mathsf{Comm}_i \gamma} + |\gamma| 2^{\#\mathsf{Comm}_i \gamma}\\
&\quad \leq (1 + |\beta| + |\gamma|) 2^{\#\mathsf{Comm}_i \beta;\gamma} = |\beta;\gamma| 2^{\#\mathsf{Comm}_i \beta;\gamma}\\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i \beta^*)\})\\
&\quad \leq 1 + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, [\beta^*]_i \mathsf{Comm}_i \beta)\})\\
&\quad \leq 1 + |\beta^*| 2^{\#[\beta^*]_i \mathsf{Comm}_i \beta} = 1 + |\beta^*| 2^{\#\mathsf{Comm}_i \beta^* - 1} \leq |\beta^*| 2^{\#\mathsf{Comm}_i \beta^*}\\
&\mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i \mathsf{c}\beta)\})\\
&\quad = \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i \mathsf{u}\beta)\})\\
&\quad \leq 1 + \mathrm{Card}(\{(j,\psi) \mid (j,\psi) \preceq (0, \mathsf{Comm}_i \beta)\}) \leq 1 + |\beta| 2^{\#\mathsf{Comm}_i \beta}\\
&\quad \leq (1 + |\beta|) 2^{\#\mathsf{Comm}_i \mathsf{c}\beta} = (1 + |\beta|) 2^{\#\mathsf{Comm}_i \mathsf{u}\beta}\\
&\qquad\qquad\qquad\qquad\quad = |\mathsf{c}\beta| 2^{\#\mathsf{Comm}_i \mathsf{c}\beta} = |\mathsf{u}\beta| 2^{\#\mathsf{Comm}_i \mathsf{u}\beta}
\end{aligned}
$$

We omit the calculations for $\mathsf{A}_i\alpha$ (case 4) as these are similar to those for $\mathsf{Comm}_i\alpha$. $\Box$

Corollary 9 $\mathrm{Card}(FL(\phi)) \leq |\phi| 2^{\#\phi}$.

Definition 2 Let $M = \langle S, Q_i, R_i^{\mathsf{K}}, R_i^{\mathsf{W}}, \models\rangle$ be a (non-standard) model for *ADL*, let Σ be a finite set of formulae and let $\mathsf{K}_{i_1}, \ldots, \mathsf{K}_{i_m}$ be all knowledge modalities occurring in formulae in $FL(\Sigma)$. For any s in M we denote by $\Sigma(s)$ the set $\{\phi \in FL(\Sigma) \mid s \models \phi\}$.

Define a relation $\sim_\Sigma$ on S by: $s \sim_\Sigma t$ iff

$$\Sigma(s) = \Sigma(t) \quad \text{and} \quad \forall j \in \{1, \ldots, m\}\ \{\Sigma(u) \mid s R_{i_j}^{\mathsf{K}} u\} = \{\Sigma(u) \mid t R_{i_j}^{\mathsf{K}} u\}.$$

Let $\|s\|_\Sigma$ (or, simply, $\|s\|$) denote the equivalence class for s, i.e. $\{t \mid s \sim_\Sigma t\}$. Let S^Σ be the set of equivalence classes over S, i.e. $S^\Sigma = \{\|s\| \mid s \in S\}$.

We define the *least filtration* $M^\Sigma = \langle S^\Sigma, Q_i^\Sigma, (R_i^{\mathsf{K}})^\Sigma, (R_i^{\mathsf{W}})^\Sigma, \models\rangle$ of M as follows. For any atomic action a,

$$Q_i^\Sigma(a) \ =^{\mathrm{def}}\ \{(\|s\|, \|t\|) \mid \exists s' \in \|s\|\ \exists t' \in \|t\|\ (s' Q_i t')\},$$

for any action α,

$$
\begin{aligned}
(Q_i^{\mathsf{c}}(\alpha))^\Sigma &=^{\mathrm{def}} \{(\|s\|,\|t\|) \mid \exists s' \in \|s\|\ \exists t' \in \|t\|\ (s' Q_i(\mathsf{c}\alpha) t')\} \\
(Q_i^{\mathsf{u}}(\alpha))^\Sigma &=^{\mathrm{def}} \{(\|s\|,\|t\|) \mid \exists s' \in \|s\|\ \exists t' \in \|t\|\ (s' Q_i(\mathsf{u}\alpha) t')\},
\end{aligned}
$$

and

$$
\begin{aligned}
(R_i^{\mathsf{K}})^\Sigma &=^{\mathrm{def}} \begin{cases} \{(\|s\|,\|t\|) \mid \exists s' \in \|s\|\ \exists t' \in \|t\|\ (s' R_i^{\mathsf{K}} t')\} \\ \qquad\qquad\qquad\quad \text{if } i \in \{i_1,\ldots,i_m\} \\ \mathrm{Id}_{S^\Sigma} \qquad\qquad \text{otherwise,} \end{cases} \\
(R_i^{\mathsf{W}})^\Sigma &=^{\mathrm{def}} \{(\|s\|,\|t\|) \mid \exists s' \in \|s\|\ \exists t' \in \|t\|\ (s' R_i^{\mathsf{W}} t')\}.
\end{aligned}
$$

Furthermore, for any propositional variable p,

$$\|s\| \models p \quad \text{iff} \quad s \models p \text{ and } p \in FL(\Sigma),$$

for any semi-atomic action α,

$$
\begin{aligned}
\|s\| \models \mathsf{Comm}_i\alpha \quad &\text{iff} \quad s \models \mathsf{Comm}_i\alpha \text{ and } \mathsf{Comm}_i\alpha \in FL(\Sigma) \\
\|s\| \models \mathsf{A}_i\alpha \quad &\text{iff} \quad s \models \mathsf{A}_i\alpha \text{ and } \mathsf{A}_i\alpha \in FL(\Sigma).
\end{aligned}
$$

The definition of the truth relation is extended to all formulae and all relations Q_i^Σ so that the conditions in Figures 2 and 3 are satisfied. This completes the definition of M^Σ.

At this point, in order to show that M^Σ is a standard *ADL*-model we need to prove that the $(R_i^{\mathsf{K}})^\Sigma$ are equivalence relations, and that the construction of M^Σ from M preserves the validity of formulae in Σ. This is done in the following.

Lemma 10 If Σ is finite then each equivalence class $\|s\|$ can be uniquely determined by some formula $\psi(\|s\|)$.

Proof. For any $i \in \{1,\ldots,m\}$, let $\Gamma_0 = \Sigma(s)$ and $\{\Gamma_1^i,\ldots,\Gamma_{k_i}^i\} = \{\Sigma(u) \mid s R_i^{\mathsf{K}} u\}$. The set $\mathcal{P}(FL(\Sigma))$ of all subsets of $FL(\Sigma)$ is finite, because Σ is finite. Let $\{\Delta_1^i,\ldots,\Delta_{l_i}^i\} = \mathcal{P}(FL(\Sigma)) \setminus \{\Gamma_1^i,\ldots,\Gamma_{k_i}^i\}$. Let γ_0 be the conjunction of formulae from Γ_0 together with negations of formulae from $FL(\Sigma) \setminus \Gamma_0$. Formulae γ_j^i and δ_j^i are introduced similarly for Γ_j^i and Δ_j^i, respectively. For $i \in \{1,\ldots,m\}$ define ψ_i as follows.

$$\psi_i =^{\mathrm{def}} \bigwedge_{1 \le j \le k_i} \neg\mathsf{K}_i\neg\gamma_j^i \wedge \bigwedge_{1 \le j \le l_i} \mathsf{K}_i\neg\delta_j^i$$

Finally, let

$$\psi(\|s\|) =^{\mathrm{def}} \gamma_0 \wedge \bigwedge_{1 \le i \le m} \psi_i.$$

Now it is routine to check that $\psi(\|s\|)$ is defined correctly and the only equivalence class in which it is true is $\|s\|$, i.e. $\{\|t\| \mid \|t\| \models \psi(\|s\|)\} = \{\|s\|\}$. $\square$

Lemma 11 $\sim_\Sigma \circ R_i^{\mathsf{K}} = R_i^{\mathsf{K}} \circ \sim_\Sigma$ for any $i \in \mathrm{Ag}$.

Proof. See the proof of Proposition 12.5 in [6]. $\square$

Lemma 12 All $(R_i^{\mathsf{K}})^\Sigma$ are equivalence relations on S^Σ.

Proof. It is easy to see from the definition that $(R_i^{\mathsf{K}})^\Sigma$ is reflexive and symmetric. To show transitivity assume that $\|s\|(R_i^{\mathsf{K}})^\Sigma\|t\|(R_i^{\mathsf{K}})^\Sigma\|u\|$, i.e. for some $s' \in \|s\|$, $t', t'' \in \|t\|$ and $u' \in \|u\|$ we have $s' R_i^{\mathsf{K}} t'$ and $t'' R_i^{\mathsf{K}} u'$. Then $t' \sim_\Sigma t''$ and, by Lemma 11, there exists u'' such that $t' R_i^{\mathsf{K}} u'' \sim_\Sigma u'$. The transitivity of R_i^{K} implies $s' R_i^{\mathsf{K}} u''$. By the definition of $(R_i^{\mathsf{K}})^\Sigma$, as $s' \in \|s\|$ and $u'' \in \|u\|$, $\|s\|(R_i^{\mathsf{K}})^\Sigma\|u\|$ follows. $\square$

Lemma 13 The following are theorems in *ADL*.

$$(\mathsf{A}_i\alpha \wedge [\alpha^*]_i(\mathsf{A}_i\alpha \to [\alpha]_i\mathsf{A}_i\alpha)) \leftrightarrow \mathsf{A}_i\alpha^*$$
$$(\mathsf{Comm}_i\alpha \wedge [\alpha^*]_i(\mathsf{Comm}_i\alpha \to [\alpha]_i\mathsf{Comm}_i\alpha)) \leftrightarrow \mathsf{Comm}_i\alpha^*$$

Proof. $\mathsf{A}_i\alpha \wedge [\alpha^*]_i(\mathsf{A}_i\alpha \to [\alpha]_i\mathsf{A}_i\alpha) \to \mathsf{A}_i\alpha^*$ is an axiom of *ADL*. The following proves the converse.

(1)	$\mathsf{A}_i\alpha^* \to \mathsf{A}_i\alpha$	by A5.3
(2)	$\mathsf{A}_i\alpha^* \to [\alpha]_i\mathsf{A}_i\alpha^*$	by A5.3
(3)	$[\alpha]_i\mathsf{A}_i\alpha^* \to [\alpha]_i\mathsf{A}_i\alpha$	by (1), A2.1
(4)	$\mathsf{A}_i\alpha^* \to [\alpha]_i\mathsf{A}_i\alpha$	by (2), (3)
(5)	$\mathsf{A}_i\alpha^* \to (\mathsf{A}_i\alpha \to [\alpha]_i\mathsf{A}_i\alpha)$	by (4)
(6)	$[\alpha^*]_i\mathsf{A}_i\alpha^* \to [\alpha^*]_i(\mathsf{A}_i\alpha \to [\alpha]_i\mathsf{A}_i\alpha)$	by (5), A2.1
(7)	$[\alpha^*]_i(\mathsf{A}_i\alpha^* \to [\alpha]_i\mathsf{A}_i\alpha^*)$	by (2)
(8)	$[\alpha^*]_i(\mathsf{A}_i\alpha^* \to [\alpha]_i\mathsf{A}_i\alpha^*) \to (\mathsf{A}_i\alpha^* \to [\alpha]_i\mathsf{A}_i\alpha^*)$	by A2.6
(9)	$\mathsf{A}_i\alpha^* \to [\alpha]_i\mathsf{A}_i\alpha^*$	by (7), (8)
(10)	$\mathsf{A}_i\alpha^* \to [\alpha^*]_i(\mathsf{A}_i\alpha \to [\alpha]_i\mathsf{A}_i\alpha)$	by (6), (9)
(11)	$\mathsf{A}_i\alpha^* \to \mathsf{A}_i\alpha \wedge [\alpha^*]_i(\mathsf{A}_i\alpha \to [\alpha]_i\mathsf{A}_i\alpha)$	by (1), (10)

The proof of the corresponding Comm_i formula is similar. $\square$

Lemma 14 The loop invariance rule $\phi \to [\alpha]_i\phi \vdash \phi \to [\alpha^*]_i\phi$ is derivable in *ADL*.

Proof. Assume $\phi \to [\alpha]_i\phi$. Then, $[\alpha^*]_i(\phi \to [\alpha]_i\phi)$ by the necessitation rule. Hence, $\phi \to \phi \wedge [\alpha^*]_i(\phi \to [\alpha]_i\phi)$. Using axiom A2.6 we obtain $\phi \to [\alpha^*]_i\phi$. $\square$

Lemma 15 Let M be a (non-standard) model. Then:

1. $M \models \mathsf{Comm}_i\alpha^* \leftrightarrow [\alpha^*]_i\mathsf{Comm}_i\alpha$
2. $M \models \mathsf{A}_i\alpha^* \leftrightarrow [\alpha^*]_i\mathsf{A}_i\alpha$

Proof. Using the *PDL*-like axioms of *ADL* it is easy to prove that

$$(\mathsf{Comm}_i\alpha \wedge [\alpha^*]_i(\mathsf{Comm}_i\alpha \to [\alpha]_i\mathsf{Comm}_i\alpha)) \leftrightarrow [\alpha^*]_i\mathsf{Comm}_i\alpha$$

is true in M. Now property 1 follows from Lemma 13 and the soundness of *ADL* with respect to all nonstandard models. Property 2 follows analogously. □

Lemma 16 Let Σ be a finite set of formulae.

1. For all $(0,\phi)$ such that $(0,\phi) \preceq (0,\sigma)$ for some $\sigma \in \Sigma$, $\|s\| \models \phi$ iff $s \models \phi$.
2. For all $(1,[\alpha]_i\eta)$ such that $(1,[\alpha]_i\eta) \prec (0,\sigma)$ for some $\sigma \in \Sigma$,
   (a) if $sQ_i(\alpha)t$ then $\|s\|Q_i^\Sigma(\alpha)\|t\|$;
   (b) if $\|s\|Q_i^\Sigma(\alpha)\|t\|$ and $s \models [\alpha]_i\eta$ then $t \models \eta$.

Proof. We prove 1 and 2 simultaneously using induction on the well-founded relation $\prec$. For the first property we distinguish whether ϕ is of the form $\bot$, p, where p is a propositional variable, $\xi \to \eta$, $[\alpha]_i\psi$, $\mathsf{Comm}_i\alpha$, $\mathsf{A}_i\alpha$, $\mathsf{K}_i\eta$, or $\mathsf{W}_i\eta$.

Case $\phi = \bot$. By the definition of models, $s \not\models \bot$ and $\|s\| \not\models \bot$.

Case $\phi = p$. $\|s\| \models p$ iff $s \models p$ holds by the construction of M^Σ.

Case $\phi = \xi \to \eta$. Because $(0,\xi),(0,\eta) \prec (0,\phi)$, property 1 holds for ξ and η by the induction hypothesis. Therefore, by the definition of M^Σ the following equivalences hold.

$$\begin{aligned}\|s\| \models \xi \to \eta \text{ iff } & (\|s\| \models \xi \Rightarrow \|s\| \models \eta)\\ \text{iff } & (s \models \xi \Rightarrow s \models \eta) \text{ iff } s \models \xi \to \eta\end{aligned}$$

Case $\phi = \mathsf{K}_i\eta$. Again, property 1 holds for η by the induction hypothesis because $(0,\eta) \prec (0,\mathsf{K}_i\eta)$. Let $\|s\| \models \mathsf{K}_i\eta$. Fix an arbitrary t such that $sR_i^{\mathsf{K}}t$. Then, by the definition of $(R_i^{\mathsf{K}})^\Sigma$, we have $\|s\|(R_i^{\mathsf{K}})^\Sigma\|t\|$. Thus, by the construction of M^Σ (see Figure 3), $\|t\| \models \eta$. By the induction hypothesis, it implies that $t \models \eta$. Consequently, $s \models \mathsf{K}_i\eta$. Conversely, suppose $s \models \mathsf{K}_i\eta$. Fix an arbitrary $\|t\|$ such that $\|s\|(R_i^{\mathsf{K}})^\Sigma\|t\|$. Then there exists $t' \in \|t\|$ such that $sR_i^{\mathsf{K}}t'$ and, hence, $t' \models \eta$. Thus, by the induction hypothesis, $\|t'\| \models \eta$. It remains to observe that $\|t\| = \|t'\|$.

Case $\phi = \mathsf{W}_i\eta$. The proof is as for the previous case.

Case $\phi = \mathsf{Comm}_i\alpha$. In this case we distinguish whether α is of the form a, where a is an atomic action, $\xi?$, $\beta \cup \gamma$, $\beta;\gamma$, β^*, $\mathsf{c}\beta$, or $\mathsf{u}\beta$.

Subcase $\alpha = a$. By the definition of M^Σ, $\|s\| \models \mathsf{Comm}_i a$ iff $s \models \mathsf{Comm}_i a$.

Subcase $\alpha = \xi?$. Again, the statement follows from the definition of M^Σ.

Subcase $\alpha = \beta;\gamma$. From $(0,[\beta]_i\mathsf{Comm}_i\gamma),(0,\mathsf{Comm}_i\beta) \prec (0,\mathsf{Comm}_i\beta;\gamma)$, by the induction hypothesis, property 1 holds for $\mathsf{Comm}_i\beta$ and property 2 holds for $[\beta]_i\mathsf{Comm}_i\gamma$. Let $\|s\| \models \mathsf{Comm}_i\alpha$. By definition $\|s\| \models \mathsf{Comm}_i\beta$ and $\|s\| \models [\beta]_i\mathsf{Comm}_i\gamma$. Hence, by 1, $s \models \mathsf{Comm}_i\beta$. By 2b,

$$s \models [\beta]_i\mathsf{Comm}_i\gamma \Rightarrow \forall t(\|s\|Q_i^\Sigma(\beta)\|t\| \Rightarrow t \models \mathsf{Comm}_i\gamma).$$

Let $sQ_i(\beta)t$. Then, by 2a $\|s\|Q_i^\Sigma(\beta)\|t\|$. Therefore, $\|t\| \models \mathsf{Comm}_i\gamma$ and, by induction, $t \models \mathsf{Comm}_i\gamma$. Hence, $s \models [\beta]_i\mathsf{Comm}_i\gamma$ and, therefore, $s \models \mathsf{Comm}_i\alpha$. Conversely, let $s \models \mathsf{Comm}_i\alpha$. Therefore, $s \models \mathsf{Comm}_i\beta$ and $s \models [\beta]_i\mathsf{Comm}_i\gamma$. By 1 we have $\|s\| \models \mathsf{Comm}_i\beta$. Let $\|s\|Q_i^\Sigma(\beta)\|t\|$. By 2b we get $t \models \mathsf{Comm}_i\gamma$ and by 1, $\|t\| \models \mathsf{Comm}_i\gamma$, which implies $\|s\| \models [\beta]_i\mathsf{Comm}_i\gamma$.

Subcase $\alpha = \beta \cup \gamma$. By the induction hypothesis, property 1 holds for $\mathsf{Comm}_i\beta$ and for $\mathsf{Comm}_i\gamma$ and, in this case, we get what we want quite easily:

$$\begin{aligned}\|s\| \models \mathsf{Comm}_i\alpha \text{ iff } & (\|s\| \models \mathsf{Comm}_i\beta \text{ or } \|s\| \models \mathsf{Comm}_i\gamma) \\ & \text{iff } (s \models \mathsf{Comm}_i\beta \text{ or } s \models \mathsf{Comm}_i\gamma) \text{ iff } s \models \mathsf{Comm}_i\alpha\end{aligned}$$

Subcase $\alpha = \beta^*$. $(0, \mathsf{Comm}_i\beta), (1, [\beta^*]_i\mathsf{Comm}_i\beta) \prec (0, \mathsf{Comm}_i\beta^*)$ holds by the properties of $\prec$. By the induction hypothesis, property 1 holds for $\mathsf{Comm}_i\beta$ and property 2 holds for $[\beta^*]_i\mathsf{Comm}_i\beta$. Let $\|s\| \models \mathsf{Comm}_i\alpha$. Then by Lemma 15 $\|s\| \models [\beta^*]_i\mathsf{Comm}_i\beta$. Assume $sQ_i(\beta^*)t$, where t is arbitrary. By property 2a we obtain $\|s\|Q_i^\Sigma(\beta^*)\|t\|$. So, $\|t\| \models \mathsf{Comm}_i\beta$. By property 1, $t \models \mathsf{Comm}_i\beta$. Thus, $s \models [\beta^*]_i\mathsf{Comm}_i\beta$ which is equivalent to $s \models \mathsf{Comm}_i\beta^*$ by Lemma 15. Conversely, suppose $s \models \mathsf{Comm}_i\alpha$. By Lemma 15 we have $s \models [\beta^*]\mathsf{Comm}_i\beta$. Let $\|s\|Q_i^\Sigma(\beta^*)\|t\|$. By property 2b, $t \models \mathsf{Comm}_i\beta$. Finally, by property 2a $\|t\| \models \mathsf{Comm}_i\beta$. Hence $\|s\| \models [\beta^*]_i\mathsf{Comm}_i\beta$ which is equivalent to $\|s\| \models \mathsf{Comm}_i\beta^*$ by Lemma 15.

Subcase $\alpha = \mathsf{c}\beta$. By the construction of M^Σ (see Figures 2 and 3), $\|s\| \models \mathsf{Comm}_i\mathsf{c}\beta$ iff $\|s\| \models \mathsf{Comm}_i\beta$ which is equivalent to $s \models \mathsf{Comm}_i\beta$ by the induction hypothesis. Finally, $s \models \mathsf{Comm}_i\mathsf{c}\beta$ iff $s \models \mathsf{Comm}_i\beta$ by the properties of non-standard models.

Subcase $\alpha = \mathsf{u}\beta$. Again, by the construction of M^Σ, $\|s\| \models \mathsf{Comm}_i\mathsf{u}\beta$ iff $\|s\| \not\models \mathsf{Comm}_i\beta$ which is equivalent to $s \not\models \mathsf{Comm}_i\beta$, by the induction hypothesis. The latter is equivalent to $s \models \mathsf{Comm}_i\mathsf{u}\beta$.

Case $\phi = \mathsf{A}_i\alpha$. In this case we again analyse the structure of the action α. Cases $\alpha = a$, $\alpha = \xi?$, $\alpha = \beta \cup \gamma$, $\alpha = \beta;\gamma$, $\alpha = \beta^*$ can be proved as for $\phi = \mathsf{Comm}_i\alpha$. The remaining cases, where $\alpha = \mathsf{c}\beta$ and $\alpha = \mathsf{u}\beta$, are obvious because $M \models \mathsf{A}_i\mathsf{c}\beta$ and $M \models \mathsf{A}_i\mathsf{u}\beta$ in any (non-standard) model M.

Case $\phi = [\alpha]_i\eta$. For the proof of 1 refer to e.g. [4, 9]. We use the induction hypothesis for $(1, [\alpha]_i\eta)$ and $(0, \eta)$. By the induction hypothesis, property 1 holds for η and property 2 holds for $[\alpha]_i\eta$. Thus, it follows that

$$s \models [\alpha]_i\eta \Rightarrow \forall t(\|s\|Q_i^\Sigma(\alpha)\|t\| \Rightarrow t \models \eta).$$

Conversely,

$$\forall t(\|s\|Q_i^\Sigma(\alpha)\|t\| \Rightarrow t \models \eta) \Rightarrow \forall t(sQ_i(\alpha)t \Rightarrow t \models \eta) \Rightarrow s \models [\alpha]_i\eta.$$

By the induction hypothesis $\|t\| \models \eta$ iff $t \models \eta$. Thus, we get

$$\begin{aligned}\|s\| \models [\alpha]_i\eta \text{ iff } & \forall t(\|s\|Q_i^\Sigma(\alpha)\|t\| \Rightarrow \|t\| \models \eta) \\ & \text{iff } \forall t(\|s\|Q_i^\Sigma(\alpha)\|t\| \Rightarrow t \models \eta) \text{ iff } s \models [\alpha]_i\eta.\end{aligned}$$

To prove property 2 we again distinguish different forms of α.

Subcase $\alpha = a$. Property 2a holds by the definition of $Q_i^\Sigma(a)$. To prove 2b we suppose that $\|s\|Q_i^\Sigma(a)\|t\|$. By the definition of Q_i^Σ, this means that there exist $s' \in \|s\|$ and $t' \in \|t\|$ such that $s'Q_i(a)t'$. It follows from $s \models [a]_i\eta$ and $s' \in \|s\|$ that $s' \models [a]_i\eta$. Thus, $t' \models \eta$ and, since $t' \in \|t\|$, $t \models \eta$.

Subcase $\alpha = \xi?$. As we have $(0, \xi) \prec (1, [\xi?]_i\eta)$, 1 holds for ξ. We obtain:

$$sQ_i(\xi?)s \text{ iff } s \models \xi \text{ iff } \|s\| \models \xi \text{ iff } \|s\|Q_i^\Sigma(\xi?)\|s\|.$$

Thus, 2 holds trivially for $[\xi?]_i\eta$.

Subcase $\alpha = \beta;\gamma$. We have $(1, [\beta]_i[\gamma]_i\eta), (1, [\gamma]_i\eta) \prec (1, [\beta;\gamma]_i\eta)$ by the definition of $\prec$. Therefore 2a holds for $(1, [\beta]_i[\gamma]_i\eta)$ and $(1, [\gamma]_i\eta)$ by the induction hypothesis, and, thus, 2a follows immediately:

$$\begin{aligned} sQ_i(\beta;\gamma)t &\text{ iff } \exists u\ sQ_i(\beta)uQ_i(\gamma)t \\ &\Rightarrow \exists\|u\|\ \|s\|Q_i^\Sigma(\beta)\|u\|Q_i^\Sigma(\gamma)\|t\| \text{ iff } \|s\|Q_i^\Sigma(\beta;\gamma)\|t\|. \end{aligned}$$

Further, by the induction hypothesis 2b holds for $(1, [\beta]_i[\gamma]_i\eta)$ and $(1, [\gamma]_i\eta)$. Suppose that $\|s\|Q_i^\Sigma(\beta;\gamma)\|t\|$ and $s \models [\beta;\gamma]_i\eta$. Then there exists $\|u\|$ from S^Σ such that $\|s\|Q_i^\Sigma(\beta)\|u\|Q_i^\Sigma(\gamma)\|t\|$ and $s \models [\beta]_i[\gamma]_i\eta$. Therefore, by 2b for β we have $\|u\|Q_i^\Sigma(\gamma)\|t\|$ and $u \models [\gamma]_i\eta$, and, finally, by 2b for γ, $t \models \eta$.

Subcase $\alpha = \beta \cup \gamma$. Because $(1, [\beta]_i\eta), (1, [\gamma]_i\eta) \prec (1, [\beta \cup \gamma]_i\eta)$, 2a holds for $(1, [\beta]_i\eta)$ and $(1, [\gamma]_i\eta)$. Then

$$\begin{aligned} sQ_i(\beta \cup \gamma)t &\text{ iff } sQ_i(\beta)t \text{ or } sQ_i(\gamma)t \\ &\Rightarrow \|s\|Q_i^\Sigma(\beta)\|t\| \text{ or } \|s\|Q_i^\Sigma(\gamma)\|t\| \text{ iff } \|s\|Q_i^\Sigma(\beta \cup \gamma)\|t\| \end{aligned}$$

By the induction hypothesis 2b holds for $(1, [\beta]_i\eta)$ and $(1, [\gamma]_i\eta)$. Suppose that $\|s\|Q_i^\Sigma(\beta \cup \gamma)\|t\|$ and $s \models [\beta \cup \gamma]_i\eta$. From the first assumption we have $\|s\|Q_i^\Sigma(\beta)\|t\|$ or $\|s\|Q_i^\Sigma(\gamma)\|t\|$, and from the second one $s \models [\beta]_i\eta$ and $s \models [\gamma]_i\eta$. Therefore, by 2b we obtain $t \models \eta$.

Subcase $\alpha = \beta^*$. Suppose $sQ(\beta^*)t$. Let

$$E =^{\text{def}} \{u \in S \mid \|s\|Q_i^\Sigma(\beta^*)\|u\|\}.$$

Then $\|s\|Q_i^\Sigma(\beta^*)\|t\|$ is equivalent to $t \in E$. By Lemma 10 each equivalence class $\|u\|$ can be completely defined by a formula $\psi(\|u\|)$. That is, $\psi(\|u\|)$ is true in all states from $\|u\|$ and false in all other states of M. Therefore, there is a formula $\psi(E)$ defining E in M because E is a union of equivalence classes. The formula $\psi(E)$ is a disjunction of formulae $\psi(\|u\|)$ for each $\|u\| \subseteq E$. By the definition of $Q_i(\beta^*)$, $\|s\|Q_i^\Sigma(\beta^*)\|s\|$, and, therefore, $s \in E$. Let $u \in E$ and $uQ_i(\beta)v$. Then, we have $\|s\|Q_i^\Sigma(\beta^*)\|u\|$ by the definition of E and $\|u\|Q_i^\Sigma(\beta)\|v\|$ by the induction hypothesis. Therefore, $\|s\|Q_i^\Sigma(\beta^*)\|v\|$, that is $v \in E$. Thus, $u \in E$ and $uQ_i(\beta)v$ implies $v \in E$. Because E is definable by a formula this is equivalent to

$$M \models \psi(E) \rightarrow [\beta]_i\psi(E).$$

By the loop invariance rule (Lemma 14):

$$M \models \psi(E) \to [\beta^*]_i \psi(E).$$

By assumption $sQ(\beta^*)t$, and $s \in E$. Therefore, $t \in E$, that is $\|s\|Q_i^\Sigma(\beta^*)\|t\|$. This completes the induction step for 2a. For 2b suppose $\|s\|Q_i^\Sigma(\beta^*)\|t\|$ and $s \models [\beta^*]_i\eta$. Then there exists a sequence of states $t_0, \ldots, t_n$ such that $s = t_0$, $t = t_n$ and $\|t_j\|Q_i^\Sigma(\beta)\|t_{j+1}\|$ for all $0 \leq j < n$. We have $(t_0 =)s \models [\beta^*]_i\eta$ by assumption. Then, $t_0 \models [\beta]_i[\beta^*]_i\eta$. We have $(1, [\beta]_i[\beta^*]_i\eta) \prec (1, [\beta^*]_i\eta)$, so applying the induction hypothesis we know $t_1 \models [\beta^*]_i\eta$. Continuing for n steps, gives $(t =)t_n \models [\beta^*]_i\eta$. Thus, $t \models \eta$.

Subcase $\alpha = \mathsf{c}\beta$. Taking into account that $M \models [\mathsf{c}\beta]_i \mathsf{Comm}_i\beta$ we have

$$\begin{aligned} sQ_i(\mathsf{c}\beta)t \;&\Rightarrow \|s\|(Q_i^{\mathsf{c}}(\beta))^\Sigma\|t\| \;\&\; t \models \mathsf{Comm}_i\beta \\ &\text{iff } \|s\|(Q_i^{\mathsf{c}}(\beta))^\Sigma\|t\| \;\&\; \|t\| \models \mathsf{Comm}_i\beta \text{ iff } \|s\|(Q_i^\Sigma(\mathsf{c}\beta))^\Sigma\|t\|. \end{aligned}$$

Now we prove 2b. Assume $s \models [\mathsf{c}\beta]_i\eta$ and $\|s\|Q_i^\Sigma(\mathsf{c}\beta)\|t\|$. By the definition of M^Σ, this implies that $\|s\|(Q_i^{\mathsf{c}}(\beta))^\Sigma\|t\|$ and $\|t\| \models \mathsf{Comm}_i\beta$. By the definition of $(Q_i^{\mathsf{c}}(\beta))^\Sigma$, this means that there are $s' \in \|s\|$ and $t' \in \|t\|$ such that $s'Q_i(\mathsf{c}\beta)t'$. Because $s' \in \|s\|$ we have $s' \models [\mathsf{c}\beta]_i\eta$. Consequently, $t' \models \eta$ and, because $t' \in \|t\|$, $t \models \eta$.

Subcase $\alpha = \mathsf{u}\beta$. The proof is similar to that of the previous subcase. □

Corollary 17 (Filtration Lemma) If the set Σ of formulae is finite then for all $\phi \in FL(\Sigma)$

$$\|s\| \models \phi \text{ iff } s \models \phi.$$

As is the case for standard modal logics and *PDL*, the Filtration Lemma has a number of useful consequences.

Theorem 18 (Small Model Theorem) Let ϕ be a formula, let $n = |\phi|2^{\#\phi}$, and let m be a number of different knowledge modalities in ϕ. If ϕ is satisfiable in some *ADL*-model then it is satisfiable in some finite *ADL*-model with no more than $2^n \cdot (2^{2^n})^m$ states.

Proof. Let $M = \langle S, Q_i, R_i^{\mathsf{K}}, R_i^{\mathsf{W}}, \models \rangle_{i \in \mathrm{Ag}}$ be an *ADL*-model of ϕ. Let Σ be $FL(\phi)$. Consider the least filtration M^Σ of M. It is easy to see that any equivalence class $\|s\|$ is uniquely determined by the tuple of sets $\langle \Sigma(s), \{\Sigma(u) \mid sR_{i_1}^{\mathsf{K}}u\}, \ldots, \{\Sigma(u) \mid sR_{i_m}^{\mathsf{K}}u\}\rangle$. Then there are 2^k possibilities for $\Sigma(s)$ and 2^{2^k} possibilities for $\{\Sigma(u) \mid sR_{i_j}^{\mathsf{K}}u\}$, if $k = \mathrm{Card}(FL(\Sigma))$. So, $\mathrm{Card}(S^\Sigma) \leq 2^k \cdot (2^{2^k})^m$. Thus, if $\Sigma = \{\phi\}$, $\mathrm{Card}(S^\Sigma) \leq 2^n \cdot (2^{2^n})^m$, by Lemma 8. □

Theorem 19 (Completeness) *ADL* is complete with respect to the class of all *ADL*-models.

Proof. Suppose ϕ is not derivable in *ADL*. Then the set $\{\neg\phi\}$ is *ADL*-consistent. Thus, by Theorem 6 it is possible to construct a suitable (non-standard) model for $\neg\phi$. By the Filtration Lemma, $\neg\phi$ is satisfiable in the model filtrated through $FL(\neg\phi)$. This filtrated model is standard by construction. □

Theorem 20 (Decidability) *ADL* is decidable.

Proof. Let ϕ be a formula, $n = |\phi|2^{\#\phi}$, and m the number of different knowledge modalities in ϕ. Applying the Small Model Theorem, we can test the satisfiability of a formula ϕ by enumerating all *ADL*-models with no more than $2^n \cdot (2^{2^n})^m$ states (there are only finitely many) and test whether ϕ is true in one of these. $\square$

5 Extensions of (test-free) *ADL*

This section is a study of the logical properties of different extensions of *ADL* and *test-free* *ADL* with the axioms *NL*, *PR* and awareness of commitments.

We say that a modal logic L *admits the elimination of* $\Box$, if $\phi \leftrightarrow \Box\phi$ is a theorem in L.

Lemma 21 Let L be a normal modal logic with reflexive modality $\Box$ satisfying one of the axiom schemata $\Box(\phi \to \psi) \to (\phi \to \Box\psi)$ or $(\phi \to \Box\psi) \to \Box(\phi \to \psi)$. Then, L admits the elimination of $\Box$.

Proof. Assume the first axiom schema hold for L. As L is closed under the substitution rule, the formula $\Box(p \to p) \to (p \to \Box p)$ belongs to L. The left side of the implication is always true. Thus, this formula is classically equivalent to $p \to \Box p$. Because $\Box$ is reflexive, the logic contains the formula $\Box p \to p$. Consequently, the equivalence $\Box p \leftrightarrow p$ is true which is what we aimed to prove. If the second axiom schema holds in L then substitute $\neg q$ for ϕ, and q for ψ. So, $(\neg q \to \Box q) \to \Box(\neg q \to q)$ is in L. This formula is equivalent to $(\neg q \to \Box q) \to \Box q$. The following equivalences complete the proof of the lemma: $((\neg q \to \Box q) \to \Box q) \leftrightarrow (\neg(q \vee \Box q) \vee \Box q) \leftrightarrow ((\neg q \wedge \neg\Box q) \vee \Box q) \leftrightarrow ((\neg q \vee \Box q) \wedge (\neg\Box q \vee \Box q)) \leftrightarrow (q \to \Box q)$. $\square$

As a consequence:

Lemma 22 If an extension L of *ADL* satisfies any of the axioms $NL(\alpha)$ or $PR(\alpha)$, then L admits the elimination of the knowledge operators.

Proof. Consider, for instance, the axiom $NL(\alpha)$, $[\alpha]_i \mathsf{K}_i\phi \to \mathsf{K}_i[\alpha]\phi$, and substitute a test expression $\psi?$ for α. This gives $[\psi?]_i \mathsf{K}_i\phi \to \mathsf{K}_i[\psi?]\phi$. Since $[\psi?]_i\phi \leftrightarrow (\psi \to \phi)$ is true in *ADL*, $(\psi \to \mathsf{K}_i\phi) \to \mathsf{K}_i(\psi \to \phi)$ is deducible in L. In the case of $PR(\alpha)$, $\mathsf{K}_i(\psi \to \phi) \to (\psi \to \mathsf{K}_i\phi)$ is deducible in L. Finally, apply Lemma 21. $\square$

5.1 $ADL^\natural$

The previous lemma implies that $ADL^\natural$ (i.e. *ADL* closed under $NL(\alpha)$, $PR(\alpha)$ and $KC(\alpha)$) admits the elimination of the knowledge operators. Hence:

Theorem 23 $ADL^\natural$ is deductively equivalent to *ADL* without the knowledge operators.

It is not difficult to prove that the sublogic of *ADL* without the knowledge operators is sound and complete, has the small model property and is decidable. So, we obtain the following theorem.

Theorem 24 (Completeness and decidability) $ADL^\natural$ is complete with respect to all $ADL^\natural$-models, has the small model property and is decidable.

It turns out that the size of the models decreases, thus, we state and prove the Small Model Theorem separately.

Theorem 25 (Small Model Theorem) Let ϕ be a formula and $n = |\phi| 2^{\#\phi}$. If ϕ is satisfiable in some $ADL^\natural$-model then it is satisfiable in a (finite) $ADL^\natural$-model with no more than 2^n states.

Proof. $ADL^\natural$ is equivalent to *ADL* without the knowledge operators. The filtration for *ADL* without these operators can be simplified in the following way: $s \sim_\Sigma t$ iff $\Sigma(s) = \Sigma(t)$. This decreases the size of the filtrated model to 2^m states, where m is the size of $FL(\Sigma)$. The size of $FL(\phi)$, in turn, is bounded from above by $|\phi| 2^{\#\phi}$. $\square$

The significance of Theorem 23 is discussed in Section 7.

5.2 $ADL^\sharp$

Recall that $ADL^\sharp$ is the extension of *ADL* by $KC(a)$, $NL(a)$, and $PR(a)$, that is, known commitments, no learning and perfect recall for atomic actions. For establishing decidability and the small model property for $ADL^\sharp$ it is enough to prove the following two lemmata.

Lemma 26 The filtration method of Definition 2 preserves the property $clc(a)$.

Proof. We show that for every $\|s\| \in S^\Sigma$ of an filtrated model M^Σ, $\|s\| \models \mathsf{Comm}_i a \to K_i \mathsf{Comm}_i a$.

Suppose that $\|s\| \models \mathsf{Comm}_i a$. This means by the definition of M^Σ that $s \models \mathsf{Comm}_i a$ and $\mathsf{Comm}_i a \in FL(\Sigma)$. Fix an arbitrary $\|t\|$ such that $\|s\| (R_i^{\mathsf{K}})^\Sigma \|t\|$. Our aim is to prove $\|t\| \models \mathsf{Comm}_i a$.

Assume that the modal operator K_i occurs in $FL(\Sigma)$. Then by the definition of $(R_i^{\mathsf{K}})^\Sigma$ there exist s' and t' such that $s' \in \|s\|$, $t' \in \|t\|$ and $s' R_i^{\mathsf{K}} t'$. From $s' \in \|s\|$ and $\mathsf{Comm}_i a \in FL(\Sigma)$ we conclude that $s' \models \mathsf{Comm}_i a$. Thus, $s' \models \mathsf{K}_i \mathsf{Comm}_i a$ because $M \models \mathsf{Comm}_i a \to \mathsf{K}_i \mathsf{Comm}_i a$ by the known commitments axiom schema. So, from $s' R_i^{\mathsf{K}} t'$ we obtain $t' \models \mathsf{Comm}_i a$. And, finally, because $t' \in \|t\|$ and $\mathsf{Comm}_i a \in FL(\Sigma)$ we get $\|t\| \models \mathsf{Comm}_i a$.

Now, assume K_i does not occur in Σ. Then by the definition of $(R_i^{\mathsf{K}})^\Sigma$ we have $s = t$ and, therefore, $s \sim_\Sigma t$. Thus, $t \models \mathsf{Comm}_i a$ and, consequently, $\|t\| \models \mathsf{Comm}_i a$. $\square$

Lemma 27 The filtration method of Definition 2 preserves the properties $com^r(a)$ and $com^l(a)$.

Proof. The proof is similar to the proof of Proposition 12.5 in [6]. We assume that $com^r(a)$ holds in M and prove that $com^r(a)$ holds in M^Σ. Suppose that $\|s\|(Q_i(a))^\Sigma\|t\|(R_i^{\mathsf{K}})^\Sigma\|u\|$. Then, by the definition of relations of the filtrated model, there exist $s' \in \|s\|$, $t', t'' \in \|t\|$ and $u' \in \|u\|$ such that $s'Q_i(a)t'$ and $t''R_i^{\mathsf{K}}u'$. Hence, $t' \sim_\Sigma t''R_i^{\mathsf{K}}u'$. By Lemma 11, there exists a u'' such that $t'R_i^{\mathsf{K}}u'' \sim_\Sigma u'$. From this, because $com^r(a)$ holds in M, it follows that there is a t_0 such that $s'R_i^{\mathsf{K}}t_0Q_i(a)u''$. Thus, $\|s\|(R_i^{\mathsf{K}})^\Sigma\|t_0\|(Q_i(a))^\Sigma\|u\|$.

Assume now that $com^l(a)$ holds in M. We prove that $com^l(a)$ holds in M^Σ. Suppose that $\|s\|(R_i^{\mathsf{K}})^\Sigma\|t\|(Q_i(a))^\Sigma\|u\|$. Then there exist $s' \in \|s\|$, $t', t'' \in \|t\|$ and $u' \in \|u\|$ such that $s'R_i^{\mathsf{K}}t'$ and $t''Q_i(a)u'$. This implies $s'R_i^{\mathsf{K}}t' \sim_\Sigma t''$, and by Lemma 11, there exists a s'' such that $s' \sim_\Sigma s''R_i^{\mathsf{K}}t''$. Now, because $s''R_i^{\mathsf{K}}t''Q_i(a)u'$ and $com^l(a)$ holds in M, there exists a t_0 such that $s''Q_i(a)t_0R_i^{\mathsf{K}}u'$. Thus, $\|s\|(Q_i(a))^\Sigma\|t_0\|(R_i^{\mathsf{K}})^\Sigma\|u\|$. $\square$

Then the proofs of the following results go through as for *ADL*.

Theorem 28 (Completeness) $ADL^\sharp$ is complete with respect to the class of all $ADL^\sharp$-models.

Theorem 29 (Small Model Theorem) Let ϕ be a formula, let $n = |\phi|2^{\#\phi}$, and let m be the number of different knowledge modalities in ϕ. If ϕ is satisfiable in some $ADL^\sharp$-model then it is satisfiable in a (finite) $ADL^\sharp$-model with no more than $2^n \cdot (2^{2^n})^m$ states.

Theorem 30 (Decidability) $ADL^\sharp$ is decidable.

5.3 $ADL^\flat$

Finally, we consider $ADL^\flat$, the extension of *test-free ADL* with the axioms known commitments, perfect recall and no learning under full substitution. Due to the absence of the test operator, $ADL^\flat$ does not admit the elimination of the knowledge operators.

Lemma 31 Let M be an *ADL*-model which satisfies $clc(a)$, $com^r(a)$ for all $a \in \mathrm{AtAc}$, and the following property:

$$R_i^{\mathsf{K}} \circ Q_i^{\mathcal{O}}(\alpha) \subseteq Q_i^{\mathcal{O}}(\alpha) \circ R_i^{\mathsf{K}} \qquad\qquad (\alpha \in \mathrm{Ac},\ \mathcal{O} \in \{\mathsf{c}, \mathsf{u}\})$$

Then, $clc(\alpha)$ and $com^r(\alpha)$ are true in M for all test-free actions α.

Proof. The proof uses induction on the length of α. The base case holds by the assumptions of the lemma. We skip the easy case where $\alpha = \beta \cup \gamma$ and first consider the case that $\alpha = \mathsf{c}\beta$. The case $\alpha = \mathsf{u}\beta$ can be proved in the same way. First we show that $R_i^{\mathsf{K}} \circ Q_i(\mathsf{c}\alpha) \subseteq Q_i(\mathsf{c}\alpha) \circ R_i^{\mathsf{K}}$. Fix arbitrary states s, t and u such that $sR_i^{\mathsf{K}}tQ_i(\mathsf{c}\beta)u$. This means that $sR_i^{\mathsf{K}}tQ_i^{\mathsf{c}}(\beta)u$ and $u \in Comm_i(\beta)$. By the induction hypothesis we have that $u \in S \setminus \mathrm{Dom}(R_i^{\mathsf{K}} \circ \mathrm{Id}_{S\setminus Comm_i(\beta)})$. By the assumptions of the lemma there exists a t' such that $sQ_i^{\mathsf{c}}(\beta)t'R_i^{\mathsf{K}}u$. Because R_i^{K} is symmetric, $uR_i^{\mathsf{K}}t'$ and, consequently, $t' \in Comm_i(\beta)$. (For, if $t' \notin Comm_i(\beta)$ then $(u, t') \in R_i^{\mathsf{K}} \circ \mathrm{Id}_{S\setminus Comm_i(\beta)}$ and $u \in$

$\mathrm{Dom}(R_i^{\mathsf{K}} \circ \mathrm{Id}_{S\setminus Comm_i(\beta)})$, which is a contradiction.) Hence, $sQ_i(\mathsf{c}\beta)t'R_i^{\mathsf{K}}u$. We also have: $Comm_i(\mathsf{c}\beta) = Comm_i(\beta) = S \setminus \mathrm{Dom}(R_i^{\mathsf{K}} \circ \mathrm{Id}_{S\setminus Comm_i(\beta)}) = S \setminus \mathrm{Dom}(R_i^{\mathsf{K}} \circ \mathrm{Id}_{S\setminus Comm_i(\mathsf{c}\beta)})$.

The case where $\alpha = \beta;\gamma$ highlights the reason for the assumption that $clc(\alpha)$ must be satisfied for all test-free action in Lemma 32 while in this lemma the property need only be true for all atomic actions. (The case $\alpha = \beta^*$ can be proved in a similar way.) It is easy to see, by taking into account the induction hypothesis, that $Q_i(\beta;\gamma) \circ R_i^{\mathsf{K}} = Q_i(\beta) \circ Q_i(\gamma) \circ R_i^{\mathsf{K}} \subseteq Q_i(\beta) \circ R_i^{\mathsf{K}} \circ Q_i(\gamma) \subseteq R_i^{\mathsf{K}} \circ Q_i(\beta) \circ Q_i(\gamma) = R_i^{\mathsf{K}} \circ Q_i(\beta;\gamma)$. A little more difficult to prove is that $Comm_i(\beta;\gamma) = S \setminus \mathrm{Dom}(R_i^{\mathsf{K}} \circ \mathrm{Id}_{S\setminus Comm_i(\beta;\gamma)})$. Here we essentially use the induction hypothesis for $com^r(\beta)$ and $com^r(\gamma)$. Let s be an arbitrary state with $s \models \mathsf{Comm}_i(\beta;\gamma)$. Then

$$\begin{aligned}
s \models \mathsf{Comm}_i(\beta;\gamma) \ &\text{iff} \ s \models \mathsf{Comm}_i\beta \wedge [\beta]_i\mathsf{Comm}_i\gamma \\
&\text{iff} \ s \models \mathsf{K}_i\mathsf{Comm}_i\beta \wedge [\beta]_i\mathsf{K}_i\mathsf{Comm}_i\gamma \\
&\Rightarrow s \models \mathsf{K}_i\mathsf{Comm}_i\beta \wedge \mathsf{K}_i[\beta]_i\mathsf{Comm}_i\gamma \\
&\text{iff} \ s \models \mathsf{K}_i(\mathsf{Comm}_i\beta \wedge [\beta]_i\mathsf{Comm}_i\gamma) \\
&\text{iff} \ s \models \mathsf{K}_i\mathsf{Comm}_i(\beta;\gamma).
\end{aligned}$$

□

The following lemma can be proved in a similar way.

Lemma 32 Let M be an *ADL*-model which satisfies $clc(\alpha)$ for all test-free actions α, $com^l(a)$ for all $a \in \mathrm{AtAc}$, and the following property:

$$Q_i^{\mathcal{O}}(\alpha) \circ R_i^{\mathsf{K}} \subseteq R_i^{\mathsf{K}} \circ Q_i^{\mathcal{O}}(\alpha) \qquad (\alpha \in \mathrm{Ac},\ \mathcal{O} \in \{\mathsf{c}, \mathsf{u}\})$$

Then, $com^l(\alpha)$ is true in M for all test-free actions α.

The canonical model for $ADL^\flat$ must satisfy all the properties $clc(\alpha)$, $com^r(\alpha)$ and $com^l(\alpha)$. As said previously this follows from the Sahlqvist Theorem for the axioms $KC(\alpha)$, $PR(\alpha)$ and $NL(\alpha)$. This would be enough for completeness if the canonical model was standard, but we know it is not. To obtain a standard model we have to filtrate the canonical model through appropriate FL-closed sets of formulae. This means, in the filtrated model these properties cannot be shown in a straightforward way. By Lemmata 26, 27 and by the definition of $(Q_i^{\mathsf{c}}(\alpha))^\Sigma$ and $(Q_i^{\mathsf{u}}(\alpha))^\Sigma$, it is only certain that the following properties are true in the filtrated model.

$$\begin{aligned}
Comm_i^\Sigma(a) &= S^\Sigma \setminus \mathrm{Dom}((R_i^{\mathsf{K}})^\Sigma \circ \mathrm{Id}_{S^\Sigma\setminus Comm_i^\Sigma(a)}) && (a \in \mathrm{AtAc}) \\
(R_i^{\mathsf{K}})^\Sigma \circ (Q_i(a))^\Sigma &= (Q_i(a))^\Sigma \circ (R_i^{\mathsf{K}})^\Sigma && (a \in \mathrm{AtAc}) \\
(R_i^{\mathsf{K}})^\Sigma \circ (Q_i^{\mathcal{O}}(\alpha))^\Sigma &= (Q_i^{\mathcal{O}}(\alpha))^\Sigma \circ (R_i^{\mathsf{K}})^\Sigma && (\alpha \in \mathrm{Ac},\ \mathcal{O} \in \{\mathsf{c}, \mathsf{u}\})
\end{aligned}$$

Nevertheless, Lemmata 31 and 32 allow us to extend these properties to all test-free actions.

Theorem 33 (Completeness) $ADL^\flat$ is complete with respect to the class of all $ADL^\flat$-models.

As for ADL, we obtain the following theorem from which decidability of the satisfiability problem of $ADL^\flat$ follows.

Theorem 34 (Small Model Theorem) Let ϕ be a formula of $ADL^\flat$, $n = |\phi|2^{\#\phi}$, and m be a number of different knowledge modalities in ϕ. If ϕ is satisfiable in some $ADL^\flat$-model then it is satisfiable in a finite $ADL^\flat$-model with no more than $2^n \cdot (2^{2^n})^m$ states.

Theorem 35 (Decidability) $ADL^\flat$ is decidable.

6 The computational complexity of *ADL*

The previous sections establish the decidability of ADL and possible extensions of ADL with interaction axioms for knowledge and actions. The Small Model Theorems for these extensions give us standard algorithms for satisfiability problems of the logics and, thus, provide upper bounds for the computational complexity of satisfiability. These bounds are one exponent higher than the bounds on the sizes of the models. There are three open problems which are connected with the computational complexity of the satisfiability problems in ADL, $ADL^\natural$, $ADL^\sharp$and $ADL^\flat$. The first open problem is whether there are better estimates for the sizes of models of the logics or to prove that they cannot be improved. Our conjecture is that for ADL this estimate can be decreased considerably while for the remainder of the logics it cannot. The second open problem is to improve the standard algorithms which are provided by the Small Model Theorems. For example, for PDL we have an upper bound on the size of a model of a given formula ϕ which is exponential in the size of ϕ. This implies the existence of a 2EXPTIME algorithm for testing satisfiability. However, the satisfiability problem of PDL is only EXPTIME-complete. We expect the same effect in the case of ADL-type logics. The third open problem which includes the above two is to give exact estimations for the complexity of the satisfiability problems in the considered logics.

In this section we establish the exact computational complexity of the logic ADL. The proof uses a reduction of the satisfiability problem of ADL to the satisfiability problem of PDL with converse ($CPDL$).[5] $CPDL$ is the extension of PDL with the *converse* operator $^{-1}$ on actions (programs) which, in addition to PDL axiom schemata, also satisfies the axiom schemata:

$$\phi \to [\alpha]\langle\alpha^{-1}\rangle\phi \qquad\qquad \phi \to [\alpha^{-1}]\langle\alpha\rangle\phi.$$

This axiomatisation of $CPDL$ is known to be complete and $CPDL$ has the Small Model Property with respect to the class of standard PDL-models where $^{-1}$ is interpreted as the converse operator on relations (see [9]). It is also known that the satisfiability problem of $CPDL$ is EXPTIME-complete [26].

[5]This reduction was suggested by one of the anonymous referees.

The key to reducing *ADL*-(un)satisfiability to *CPDL*-(un)satisfiability is to find a suitable renaming of certain subformula occurrences and action term occurrences and internalise the axioms for the ability operators and the commitment operators in the translation. The reduction introduces unique atomic actions b_i and d_i for the accommodating the modalities W_i and K_i ($i \in \mathrm{Ag}$). It introduces unique propositional symbols $p_{i,\alpha}$ and $q_{i,\alpha}$ for ability and commitment formulae, and unique atomic actions $c_{i,\alpha}$ and $u_{i,\alpha}$ are introduced for accommodating committing and uncommitting actions ($i \in \mathrm{Ag}$ and $\alpha \in \mathrm{Ac}$). The reduction, denoted by σ, of any *ADL* formula to a *CPDL* formula is defined by the following:

$$\begin{array}{llr} \sigma\bot = \bot \qquad \sigma p = p & & \sigma(\phi \to \psi) = \sigma\phi \to \sigma\psi \\ \sigma\mathsf{W}_i\phi = [b_i]\sigma\phi & & \sigma\mathsf{K}_i\phi = [(d_i \cup d_i^{-1})^*]\sigma\phi \\ \sigma\mathsf{A}_i\alpha = p_{i,\alpha} & & \sigma\mathsf{Comm}_i\alpha = q_{i,\alpha} \\ \sigma[\alpha]_i\phi = [\tau(i,\alpha)]\sigma\phi & & \end{array}$$

where τ is defined by:

$$\begin{array}{ll} \tau(i,a) = a_i \quad \text{for any atomic action } a & \\ \tau(i,\alpha \cup \beta) = \tau(i,\alpha) \cup \tau(i,\beta) & \tau(i,\alpha;\beta) = \tau(i,\alpha);\tau(i,\beta) \\ \tau(i,\alpha^*) = \tau(i,\alpha)^* & \tau(i,\phi?) = (\sigma\phi)? \\ \tau(i,\mathsf{c}\alpha) = c_{i,\alpha};q_{i,\alpha}? & \tau(i,\mathsf{u}\alpha) = u_{i,\alpha};(\neg q_{i,\alpha})? \end{array}$$

For any given *ADL*-formula ϕ, let $\mathrm{Ac}(\phi)$ denote the set of all actions occurring in ϕ. We write ϕ_{A} to denote the conjunction of formulae of the form below such that i occurs in ϕ and every action which occurs in ϕ_{A} belongs to $\mathrm{Ac}(\phi)$.

$$\begin{aligned} p_{i,\mathsf{c}\alpha} \wedge p_{i,\mathsf{u}\alpha} \wedge (p_{i,\alpha\cup\beta} &\leftrightarrow (p_{i,\alpha} \vee p_{i;\beta})) \\ \wedge\ (p_{i,\alpha;\beta} &\leftrightarrow (p_{i,\alpha} \wedge [\tau(i,\alpha)]p_{i,\beta})) \\ \wedge\ (p_{i,\alpha^*} &\leftrightarrow (p_{i,\alpha} \wedge [\tau(i,\alpha^*)](p_{i,\alpha} \to [\tau(i,\alpha)]p_{i,\beta}))) \end{aligned}$$

Similarly, we write ϕ_{Comm} to denote the conjunction of formulae of the form below such that i occurs in ϕ and every action which occurs in ϕ_{Comm} belongs to $\mathrm{Ac}(\phi)$.

$$\begin{aligned} (q_{i,\mathsf{c}\alpha} \leftrightarrow q_{i,\alpha}) \wedge (q_{i,\mathsf{u}\alpha} &\leftrightarrow \neg q_{i,\alpha}) \\ \wedge\ (q_{i,\alpha\cup\beta} &\leftrightarrow (q_{i,\alpha} \vee q_{i;\beta})) \\ \wedge\ (q_{i,\alpha;\beta} &\leftrightarrow (q_{i,\alpha} \wedge [\tau(i,\alpha)]q_{i,\beta})) \\ \wedge\ (q_{i,\alpha^*} &\leftrightarrow (q_{i,\alpha} \wedge [\tau(i,\alpha^*)](q_{i,\alpha} \to [\tau(i,\alpha)]q_{i,\beta}))) \end{aligned}$$

It is easy to see that all atomic actions occurring in ϕ_{A} and ϕ_{Comm} also occur in $\sigma(\phi)$. The following theorem can be proved semantically using the Completeness Theorem (Theorem 19) for *ADL* and known completeness results for *CPDL* [9].

Theorem 36 Let $a_0, \ldots, a_n$ be all atomic actions occurring in $\sigma(\phi)$. Then

$$ADL \vdash \phi \text{ iff } CPDL \vdash [(a_0 \cup \cdots \cup a_n \cup a_0^{-1} \cup \cdots \cup a_n^{-1})^*](\phi_{\mathsf{A}} \wedge \phi_{\mathsf{Comm}}) \to \sigma(\phi).$$

Proof. By the completeness theorems for *ADL* and *CPDL* it is enough to prove the equivalence of the following statements:

1. ϕ is valid in any *ADL*-model.
2. $[(a_0 \cup \cdots \cup a_n \cup a_0^{-1} \cup \cdots \cup a_n^{-1})^*](\phi_{\mathsf{A}} \wedge \phi_{\mathsf{Comm}}) \to \sigma(\phi)$ is valid in any *CPDL*-model.
3. $\sigma(\phi)$ is valid in any *CPDL*-model which validates $\phi_{\mathsf{A}} \wedge \phi_{\mathsf{Comm}}$.

The equivalence of 2 and 3 is a straightforward consequence of the deduction theorem for *CPDL*. The equivalence of 1 and 3 can easily be shown by transforming any *ADL*-countermodel for ϕ into a *CPDL*-countermodel for $\sigma(\phi)$ which validates $\phi_{\mathsf{A}} \wedge \phi_{\mathsf{Comm}}$, and vice versa. □

Since σ defines a reduction computable in polynomial time, it follows that the *ADL* satisfiability problem is in EXPTIME. Since *PDL* is a fragment of *ADL* it also follows that the satisfiability problem of *ADL* is EXPTIME-hard. This proves:

Theorem 37 The satisfiability problem of *ADL* is EXPTIME-complete.

It is not difficult to see that the axiom schema (known commitments) can be similarly encoded by conjoining $q_{i,\alpha} \to [(c_i \cup c_i^{-1})^*]q_{i,\alpha}$ for each $\alpha \in \mathrm{Ac}(\phi)$ to ϕ_{Comm}. The extension of *ADL* closed under this axiom is therefore also EXPTIME-complete.

Whether a suitable reduction to *CPDL* can be defined also for *ADL* logics closed under the knowledge-action interaction axioms is not clear. Some other techniques, possibly from [27, 5], might be necessary to establish better upper bounds.

7 Discussion

Observations about test and knowledge. A feature of agent dynamic logics is the implicit connection of the test and the knowledge operators under full substitutivity. As was shown above, the kind of interaction given by either of the axioms of perfect recall or no learning, results in the elimination the knowledge operators. This suggests assuming full substitutivity is inappropriate for the commutativity axioms. On the other hand, perhaps, the semantics of test is not appropriate in the presence of knowledge operators that satisfy the perfect recall or no learning axiom schemata. In [21, 22] the elimination of the knowledge operators is avoided with an alternative semantics for test such that in the resulting logic knowledge and test interact in a way so that the knowledge operators are independent from the other logical connectives. This modified test operator gives the agents the limited capability of confirming formulae in their knowledge states, rather than the strong capability of confirming all true properties in the current world [21].

Comparison between *ADL* and *KARO*. *ADL* and the *KARO* framework [16, 25] are similar in many respects. Both of them use dynamic logic to reason about action, and they contain identical operators to represent wishes, knowledge, abilities and commitments. Roughly speaking, *ADL* is a decidable non-deterministic counterpart of a subfragment of *KARO*, but there are some important differences between *ADL* and *KARO*.

First, in order to reason about actions, *KARO* extends the strictly deterministic version of *PDL* with deterministic 'while' and 'if' action constructors, while *ADL* uses non-deterministic *PDL*. This relaxation is motivated by both pragmatic and philosophical considerations. We did not want to constrain ourselves to a more specialised language if decidability and other good properties are not lost in the non-deterministic case. Moreover, non-determinism is a natural property in real world applications as it allows the exploration of alternative paths of executing actions. This means, (non-deterministic) *ADL* cannot be reduced to (deterministic) *KARO*, at least, not in a straightforward way. (On the side we note that one can consider a deterministic version of *ADL* and prove its soundness, completeness and decidability in the same manner as it has been done for deterministic *PDL* [9].)

Second, some of the *KARO* operators are simplified or not included in *ADL*. This concerns the operators for representing goals, making choices, and an implementability operator. The implementability operator can be formalised as existence of an action which brings about a desired property of a multi-agent system. In this paper we have taken the view that goals are unfulfilled wishes. In *KARO* goals are defined to be those wishes that are unfulfilled, implementable and chosen. However, it is not clear how best to axiomatise the planning element of the implementability operator which involves quantification over atomic actions, cf. the discussion in [12]. In *KARO* an additional mechanism of choosing wishes is meant to facilitate minimal change in order to avoid more than one agent's wishes being satisfied in the same state. The formalisation of these facilities in modal logic is problematic and left for future investigations. The absence of some of the *KARO* operators clearly means there is no straightforward translation of *KARO* into *ADL*.

Third, full *KARO* is essentially defined only by its semantics whereas *ADL* is a logic with both semantics and Hilbert-style axiomatisation. Because *ADL* has a purely relational semantics, the mathematical apparatus developed for standard modal logics can be applied and adapted for the purposes of *ADL*, as is shown in this paper.

Finally, one of the main features of *ADL* which distinguishes it from *KARO* is the treatment of the abilities with respect to non-determinism. Several options exist in defining the ability to perform actions [24, 25]. In a sense, our axiomatisation of ability is an attempt to merge the 'internal' (angelic) and 'external' (demonic) approaches considered by [24], which can be seen from comparing the defining axioms and the desired properties identified in [24, Theorems 4 and 9]. Our approach does not lead to inconsistency as the Soundness Theorem shows. The following question is open: Is it possible to define abilities so that they satisfy the properties of 'internal' or 'external' behaviour of actions [24, Theorems 4 and 9] and retain soundness, completeness and decidability of the logic?

Despite the many differences, *ADL* and *KARO* have some common properties. In constructing *ADL*, it was our intention to keep the most important properties of agents which are discussed and formalised in the *KARO* framework, possibly, slightly modifying them.

Awareness of abilities, actions and goals. In this paper we have considered the formalisation of awareness of commitments, but it is natural to also model other awareness conditions such as:

$$\begin{array}{ll} \mathsf{A}_i\alpha \to \mathsf{K}_i\mathsf{A}_i\alpha & \text{(agent } i \text{ is aware of her abilities),} \\ [\alpha]_i\phi \to \mathsf{K}_i[\alpha]_i\phi & \text{(agent } i \text{ is aware of her actions), or} \\ \mathsf{Goal}_i\phi \to \mathsf{K}_i\mathsf{Goal}_i\phi & \text{(agent } i \text{ is aware of her goals).} \end{array}$$

All the results of this paper hold if we assume awareness of abilities because this amounts to replacing the static commitment operator by the ability operator in the axiom of known commitments and the properties of ability parallel those of commitment. So, the axiom postulating the awareness of abilities together with the axiom of no learning can be added to *ADL* or any of its extensions considered in the paper. The small model property, completeness and decidability will be preserved.

The axiom expressing awareness of an agent's actions is more complicated. First of all, it can be seen that if substitutions of arbitrary actions is allowed for α then the knowledge operator trivially vanishes again. The key substitution to prove this is the substitution of $\top?$ for α in the axiom. Then $p \to \mathsf{K}_i p$ is valid and since K_i is reflexive $p \leftrightarrow \mathsf{K}_i p$ is a theorem. Moreover, it is not hard to see that the awareness of actions axiom implies the axiom of no learning is valid. It is also possible to prove the small model property, completeness and decidability for the extension of *(test-free) ADL* by the awareness of atomic actions. But these results cannot be generalised to the extensions of (test-free) *ADL* by the awareness of all actions in the same way as it has been done in Lemmata 31 and 32. A counterexample is a model M which consists only of a two-element knowledge cluster with empty relations for the atomic actions where elements of the cluster are distinguished by some propositional variable, i.e. $S = \{0, 1\}$, $R_1 = S^2$, $Q_1(a) = \emptyset$, $0 \models p$ and $1 \not\models p$. Then $M \models [a]_1 p \to \mathsf{K}_1[a]_1 p$ but $0 \not\models [a^*]_1 p \to \mathsf{K}_1[a^*]_1 p$.

The third axiom, the awareness of goals axiom, implies properties which are too ideal for real application. For example, it implies that in any state of reality either an agent knows everything about the current state or all her wishes are satisfied in this state. Hence, the definition of goals, here an approximation of a formalisation in [16], will require further investigation.

Beliefs. Adding beliefs, which are commonly specified by *KD45* modalities, poses no technical problems. It is not difficult to extend the results of this paper (soundness, completeness, the small model property and decidability) to hold also for independent joins of agent dynamic logics with *KD45*.

8 Conclusion

This paper has defined and analysed a set of agent dynamic logics as combinations of (infinitely many copies of) non-deterministic *PDL* with (infinitely many copies of) *S5* (for knowledge) and (infinitely many copies of) modal logic K (for wishes). There are additional operators, the primitives for formulating properties of ability and commitment, which do not normally arise in a logical context or in program semantics, but are needed in modelling and verifying the behaviour of intelligent agents. In agent dynamic logic there are various forms of interdependence between the operators which were considered. Applying suitable modifications of standard logical tools from modal logic and dynamic logic, results on soundness, completeness, the small model property and decidability were proved. Consequences of interest to pure (modal) logicians and anybody interested in the combination of dynamic logic and epistemic logic, are similar results for combinations of *PDL* and *S5* with commuting dynamic logic and *S5* modalities.

The results of this paper show that more expressive agent theories than have been considered previously can be successfully formalised and studied in the context of modal and dynamic logic. The advantage of the adopted modal logic approach is that the methods and results apply very generally, cf. Section 7.

Acknowledgements

We thank the anonymous referees for valuable comments. We are especially grateful to one referee who pointed out the existence of a polynomial reduction of *ADL* problems into *CPDL*. The research of the first two authors is supported by grants GR/M88761 and GR/R92035 from the UK Engineering and Physical Sciences Research Council.

References

[1] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*, vol. 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2001.

[2] A. Chagrov and M. Zakharyaschev. *Modal Logic*, vol. 35 of *Oxford Logic Guides*. Clarendon Press, 1997.

[3] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. *Reasoning about knowledge*. MIT Press, 1995.

[4] M. J. Fischer and R. E. Ladner. Propositional dynamic logic of regular programs. *Journal of Computer and System Sciences*, 18(2):194–211, 1979.

[5] D. M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. Many-dimensional modal logics: Theory and applications. Manuscript. `http://www.dcs.kcl.ac.uk/staff/mz/GKWZ/gkwz.html`, 2003.

[6] D. M. Gabbay and V. Shehtman. Products of modal logics, part 1. *Logic Journal of the IGPL*, 6(1):73–146, 1998.

[7] J. Y. Halpern and Y. Moses. A guide to completeness and complexity for modal logics of knowledge and belief. *Artificial Intelligence*, 54(2):319–379, 1992.

[8] J. Y. Halpern and M. Y. Vardi. The complexity of reasoning about knowledge and time. I. Lower bounds. *Journal of Computer and System Sciences*, 38(1):195–237, 1989.

[9] D. Harel, D. Kozen, and J. Tiuryn. *Dynamic Logic*. Foundations of Computing. MIT Press, 2000.

[10] A. Herzig, J. Lang, D. Longin, and T. Polacsek. A logic for planning under partial observability. In *Proceedings of the 17th National Conference on Artificial Intelligence (AAAI'00)*, pp. 768–773. AAAI Press/MIT Press, 2000.

[11] A. Herzig and D. Longin. Belief dynamics in cooperative dialogues. *Journal of Semantics*, 17(2):91–118, 2000.

[12] U. Hustadt, C. Dixon, R. A. Schmidt, M. Fisher, J.-J. C. Meyer, and W. van der Hoek. Reasoning about agents in the KARO framework. In C. Bettini and A. Montanari, eds, *Proceedings of the 8th International Symposium on Temporal Representation and Reasoning (TIME'01)*, pp. 206–213. IEEE Computer Society, 2001.

[13] U. Hustadt, C. Dixon, R. A. Schmidt, M. Fisher, J.-J. C. Meyer, and W. van der Hoek. Verification within the KARO agent theory. In J. L. Rash, C. A. Rouff, W. Truszkowski, D. F. Gordon, and M. G. Hinchey, eds, *Proceedings of the 1st International Workshop on Formal Approaches to Agent-Based Systems (FAABS'2000)*, vol. 1871 of *Lecture Notes in Artificial Intelligence*, pp. 33–47. Springer-Verlag, 2001.

[14] M. Kracht. Highway to the danger zone. *Journal of Logic and Computation*, 5(1):93–109, 1995.

[15] M. Kracht. *Tools and Techniques in Modal Logic*, vol. 142 of *Studies in Logic*. Elsevier Science Publ. B. V., 1999.

[16] J.-J. C. Meyer, W. van der Hoek, and B. van Linder. A logical approach to the dynamics of commitments. *Artificial Intelligence*, 113(1–2):1–40, 1999.

[17] R. Parikh. Propositional dynamic logics of programs: a survey. In E. Engeler, ed., *Proceedings of the Workshop on Logic of Programs*, vol. 125 of *Lecture Notes in Computer Science*, pp. 102–144. Springer-Verlag, 1979.

[18] A. S. Rao. Decision procedures for propositional linear-time Belief-Desire-Intention logics. In M. Wooldridge, J. P. Müller, and M. Tambe, eds, *Intelligent Agents II — Agent Theories, Architectures, and Languages, IJCAI'95 Workshop (ATAL-II),*

Proceedings, vol. 1037 of *Lecture Notes in Artificial Intelligence*, pp. 33–48. Springer-Verlag, 1996.

[19] A. S. Rao and M. P. Georgieff. Modeling rational agents within a BDI-architecture. In R. E. Fikes and E. Sandewall, eds, *Proceedings of the 2nd International Conference on the Principle of Knowledge Representation and Reasoning (KR'91)*, pp. 473–484. Morgan Kaufmann Publ., Inc., 1991.

[20] H. Sahlqvist. Completeness and correspondence in the first and second order semantics for modal logic. In S. Kanger, ed., *Proceedings of the 3d Scandinavian Logic Symposium*, pp. 110–143. North-Holland, 1975.

[21] R. A. Schmidt and D. Tishkovsky. Multi-Agent Logics of Dynamic Belief and Knowledge. In S. Greco, S. Flesca, N. Leone, and G. Ianni, eds, *Proceedings of the 8th European Conference on Logics in Artificial Intelligence (JELIA'02)*, vol. 2424 of *Lecture Notes in Artificial Intelligence*, pp. 38–49. Springer-Verlag, 2002.

[22] R. A. Schmidt and D. Tishkovsky. On Axiomatic Products of PDL and S5: Substitution, Tests and Knowledge. *Bulletin of the Section of Logic*, 31(1):27–36, 2002.

[23] W. van der Hoek. Logical foundations of agent-based computing. In M. Luck, V. Marík, O. Stepánková, and R. Trappl, eds, *Multi-Agent Systems and Applications*, vol. 2086 of *Lecture Notes in Artificial Intelligence*, pp. 50–73. Springer-Verlag, 2001.

[24] W. van der Hoek, B. van Linder, and J.-J. C. Meyer. On agents that have the ability to choose. *Studia Logica*, 66(1):79–119, 2000.

[25] B. van Linder, W. van der Hoek, and J.-J. C. Meyer. Formalizing abilities and opportunities of agents. *Fundamenta Informaticae*, 34(1–2):53–101, 1998.

[26] M. Y. Vardi. The taming of converse: Reasoning about two-way computations. In R. Parikh, ed., *Proceedings of the Workshop on Logic of Programs*, vol. 193 of *Lecture Notes in Computer Science*, pp. 413–424. Springer-Verlag, 1985.

[27] F. Wolter. The product of converse PDL and polymodal K. *Journal of Logic and Computation*, 10(2):223–251, 2000.

[28] M. Zakharyaschev, F. Wolter, and A. Chagrov. Advanced modal logic. In D. M. Gabbay and F. Guenthner, eds, *Handbook of Philosophical logic*, vol. 3, pp. 83–266. Kluwer Acad. Publ., 2nd ed., 2001.