A Principle for Incorporating Axioms into the First-Order Translation of Modal Formulae*

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Abstract. In this paper we present a translation principle, called the *axiomatic translation*, for reducing propositional modal logics with background theories, including triangular properties such as transitivity, Euclideanness and functionality, to decidable logics. The goal of the axiomatic translation principle is to find simplified theories, which capture the inference problems in the original theory, but in a way that is more amenable to automation and easier to deal with by existing theorem provers. The principle of the axiomatic translation is conceptually very simple and can be largely automated. Soundness is automatic under reasonable assumptions, and termination of ordered resolution is easily achieved, but the non-trivial part of the approach is proving completeness.

1 Introduction

Modal logic is widely accepted to provide an appropriate formal framework for an ever increasing number of different application areas of computer science. In the field of knowledge representation, particularly in the subfield concerned with description logics, modal logics arise in the form of the basic multi-modal logic and propositional dynamic logic. Modal logics also form the basis of many agent logics. Various aspects of agents are formalised by modal operators, e.g. $S5$ modalities for knowledge, $KD45$ modalities for belief, and so on. Normally agent logics also include operators formalising notions such as ability and commitment which are given by non-standard modal operators. Agent-based systems are a particular application area, where there is a clear need for non-standard modal logics, that is, modal logics that do not just simply coincide with familiar modal logics. Thus, there is an increasing interest in non-standard combinations of interacting modal logics. These modal logics are commonly given through a set of modal axiom schemata in the form of a Hilbert axiomatisation. However, it is generally accepted that Hilbert axiomatisations are not amenable for automated reasoning. The question arises how reasoning systems can be constructed for a

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given combination of (possibly interacting) modal logics. The most common approach is to adopt a semantics-based approach. Tableau and sequent calculi can be seen as methods for constructing a Kripke model for a given formula. For this it is necessary to know what the Kripke models for a given Hilbert axiomatisation are. Another option is to use translation into first-order or second-order logic and then to use first-order or second-order theorem provers. The translation is again based on the semantics of the operators. In all these cases it is necessary to find a class of Kripke models with respect to which the given Hilbert axiomatisation is sound and complete. Again one would want to automate this process, or at least have some sort of automated support. A number of automated second-order quantifier elimination procedures were developed and studied that transform Hilbert axioms into corresponding properties of the accessibility relation in the Kripke semantics of modal logics. The present paper investigates an alternative idea that is based on an approach in which, starting from a Hilbert axiomatisation, by a process of partial translation and proof-theoretic transformation we can directly obtain proof methods for a wide variety of modal logics.

The paper is structured as follows. Section 2 gives an informal introduction and overview of the principle of the axiomatic translation. The axiomatic translation is then defined in two steps. First a semantics-based translation of modal formulae is defined in Section 3. Then in Section 4 axioms are incorporated into the translation and a general soundness result is proved. In Section 5 completeness results are established for a series of familiar modal logics and modal logics extended with generalised modal axioms. Since the axiomatic translation reduces modal logics to decidable first-order fragments and decidable clausal classes, general conditions can be identified under which modal logics are decidable and have the finite model property. These issues are the subject of Section 6. Section 7 discusses further consequences and related work, and Section 8 concludes with remarks on implementation and a summary.

It is assumed that the reader has a basic familiarity with modal logic, first-order logic and resolution. The notation used in this paper is the same as in [2]. Due to lack of space all proofs of theorems are omitted but can be found in [15].

2 The idea of the principle

The idea of the principle presented in this paper can be situated in the general framework of Ohlbach [14] for the combination of Hilbert-style deduction and semantic reasoning. In this framework, if a first-order semantics is known for some of the logical connectives, the semantic definitions are used as rewrite rules for eliminating these connectives. In addition, Hilbert axioms and rules are encoded directly with quantification over formula variables. The formulas are left as terms and are treated in a special way. Thus a problem specification has four components, the semantic rewrite rules determining the semantic encoding of some of the logical operators, the specification of a semantic structure, for example, in the form of frame correspondence properties, the encoded Hilbert axioms and rules, and finally the conjecture to be proved or refuted. The result is a first-order formulation of the problem and now ordinary first-order theorem proving methods can be applied.

To explain the translation principle, we need to give more details. Let 'holds' be a designated predicate capturing the semantic definition of the logical operators $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ and $\Box$ by the following first-order equivalences:

$$\forall p\, \forall x\, (\mathrm{holds}(\neg p, x) \;\leftrightarrow\; \neg\, \mathrm{holds}(p, x)) \tag{1}$$

$$\forall pq\, \forall x\, (\mathrm{holds}(p \star q, x) \;\leftrightarrow\; (\mathrm{holds}(p, x) \star \mathrm{holds}(q, x))) \tag{2}$$

$$\forall p\, \forall x\, (\mathrm{holds}(\Box p, x) \;\leftrightarrow\; \forall y\, (R(x, y) \rightarrow \mathrm{holds}(p, y))) \tag{3}$$

where $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Let S be the set of these equivalences, and let Δ be a set of modal axioms. A T*-encoded problem specification* of the satisfiability of φ in a (multi-)modal logic $K_{(m)}\Delta$ in the Ohlbach framework is given by

$$\mathrm{tr}_{K_{(m)}\Delta}(\varphi) = \exists \overline{p}\, \exists x\ \mathrm{holds}(\varphi, x) \wedge \bigwedge S \wedge \bigwedge H.$$

H is the set of T*-encodings of the axioms* in Δ. For any axiom $\mathcal{A}$ in Δ, its T-encoding is defined by $\forall \overline{p}\, \forall x\ \mathrm{holds}(\mathcal{A}, x)$, where $\overline{p}$ are the propositional variables occurring in $\mathcal{A}$. (Ohlbach also allows mixed semantics and T-encoded problem specifications.)

Theorem 1. *Let Δ be any set of modal logic formulae and let H be the set of T-encodings of Δ. Suppose $K_{(m)}\Delta$ is sound and complete. Then, for any modal formula φ, φ is satisfiable in $K_{(m)}\Delta$ iff $\mathrm{tr}_{K_{(m)}\Delta}(\varphi)$ is first-order satisfiable.*

Note that the modus ponens inference rule is not part of $\mathrm{tr}_{K_{(m)}\Delta}(\varphi)$. Thus, when showing the satisfiability/validity of $\mathrm{tr}_{K_{(m)}\Delta}(\varphi)$ we are not simply constructing Hilbert-style proofs in first-order logic.

The equivalences in S can be used as rewrite rules to transform the candidate formula to be proved and also the axioms in H. For example, if the candidate formula is $\varphi = \Box\neg\Box p$ then $\exists p\, \exists x\ \mathrm{holds}(\varphi, x)$ transforms to

$$\exists p\, \exists x \forall y\, (R(x, y) \rightarrow \neg(\forall z\, R(y, z) \rightarrow \mathrm{holds}(p, z))),$$

by exhaustively applying the equivalences (1)–(3) from left-to-right. Ohlbach calls this the S*-normalised* representation of φ. Hilbert axioms can be normalised in a similar fashion. For example, rewriting the T-encodings of the axioms $4 = \Box p \rightarrow \Box\Box p$ and $T = \Box p \rightarrow p$ one obtains:

$$\forall p\, \forall x\ \mathrm{holds}(\Box p \rightarrow \Box\Box p, x) \equiv \forall p\, \forall x\, (\mathrm{holds}(\Box p, x) \rightarrow \mathrm{holds}(\Box\Box p, x))$$

$$\equiv \forall p\, \forall x\, (\mathrm{holds}(\Box p, x) \rightarrow \forall y\, (R(x, y) \rightarrow \mathrm{holds}(\Box p, y))) \tag{4}$$

$$\begin{aligned} &\equiv \forall p\, \forall x\, (\forall y\, (R(x, y) \rightarrow \mathrm{holds}(p, y)) \\ &\qquad \rightarrow \forall y\, (R(x, y) \rightarrow (\forall z\, (R(y, z) \rightarrow \mathrm{holds}(p, z))))) \end{aligned} \tag{5}$$

$$\forall p\, \forall x\ \mathrm{holds}(\Box p \rightarrow p, x)$$

$$\equiv \forall p\, \forall x\, (\mathrm{holds}(\Box p, x) \rightarrow \mathrm{holds}(p, x)) \tag{6}$$

$$\equiv \forall p\, \forall x\, (\forall y\, (R(x, y) \rightarrow \mathrm{holds}(p, y)) \rightarrow \mathrm{holds}(p, x)) \tag{7}$$

If we replace the $\mathrm{holds}(p, x)$ literals in (5) and (7) by literals of the form $P(x)$ and the $\forall p$ quantifier by $\forall P$ then what we have is the standard translations of 4 and T. The standard translations of the axioms are thus second-order formulae, while the T-encodings of modal axioms are always first-order logic formulae.

Since $K4$, KT and $S4$ are decidable logics, an immediate question is: Which first-order methods decide T-encoded problem specifications in these logics? There are a number of solvable first-order fragments which can be decided with first-order methods. These include the two-variable fragment, the guarded fragment, Maslov's class $\overline{K}$, and fluted logic. But none of these classes allows formulae of the form (5) and (7). The usual techniques are not sufficient to reduce both formulae to these classes either. Analysis of the respective clausal forms reveals also that the normalised T-encoding of the axiom 4 does not belong to any known solvable clausal classes. However, the normalised T-encoding of T can be expressed, for example, in DL^* [2] or the clausal class of Maslov's class $\overline{K}$.

The idea of the approach described in this paper is the following. Take a modal axiom, form its T-encoding, but instead of normalising all modal connectives away, do a partial normalisation which stops with (4) and (6), in the case of 4 and T. The clausal forms are:

$$\neg\,\mathrm{holds}(\Box p, x) \vee \neg R(x, y) \vee \mathrm{holds}(\Box p, y) \qquad \text{for } 4$$
$$\neg\,\mathrm{holds}(\Box p, x) \vee \mathrm{holds}(p, x) \qquad \text{for } T$$

Now, if we regard $\Box p$ and p as *ground terms* then it is not difficult to see that both clauses belong to a large selection of solvable clausal classes and decidable first-order fragments. For example, DL^*, the class of guarded clauses, etc. They even belong to FO^2 and all other decidable fragments mentioned above. This can be more easily seen if the terms $\Box p$ and p are embedded in the predicate symbol, as in:

$$\neg Q_{\Box p}(x) \vee \neg R(x, y) \vee Q_{\Box p}(y) \quad \text{and} \quad \neg Q_{\Box p}(x) \vee Q_p(x).$$

The way to view literals of the form $Q_\psi(x)$, is to think of them as saying that the formula ψ holds at the world x. These clauses are examples of what we call *schema clauses*. The idea is that the properties of a transitive (resp. reflexive) $\Box$ operator are captured by including sufficiently many instances of the appropriate schema clause into the translation of a candidate formula. For the translation to work the schema clauses must be linked with the translation of the candidate formula. This is achieved by basing the translation on a particular form of structural transformation.

It is instructive to consider an example. Suppose we want to test the satisfiability of the formula $\varphi = \Box\neg\Box p$ in K. The following is a suitable structural form of the relational translation of φ. New symbols have been introduced for all $\Box$ subformulae of φ. (It is not exactly the translation defined formally below, but serves to illustrate the principle.)

$$Q_{\Box\neg\Box p}(a)$$
$$\forall x\, (Q_{\Box\neg\Box p}(x) \leftrightarrow \forall y\, (R(x, y) \to \neg Q_{\Box p}(y))) \qquad \text{definition for } \Box\neg\Box p$$

$$\forall x\,(Q_{\Box p}(x) \leftrightarrow \forall y\,(R(x,y) \rightarrow Q_p(y))) \qquad \text{definition for } \Box p$$

The symbol a denotes the Skolem constant representing the root world in a Kripke model. The translation used is satisfiability equivalence preserving. Thus, the formula φ is satisfiable in K iff the conjunction of the above three formulae is first-order satisfiable. The satisfiability of $\Box\neg\Box p$ in K can now be tested by applying a standard theorem prover to the above set.

What do we do to test satisfiability in $K4$? The standard approach is to add frame correspondence properties for R as a theory to the translation of the candidate formula. But instead, we adapt the translation of the candidate formula by incorporating the axiom 4 into the translation. This is achieved by adding sufficiently many instances of the schema clause of 4. It turns out that in the case of $K4$ it is enough to add one instance of the schema clause for every $\Box$ subformula in the candidate formula. In the case of our running example we need to add the following two instances, one for each $\Box$ subformula of $\varphi = \Box\neg\Box p$.

$$\neg Q_{\Box\neg\Box p}(x) \vee \neg R(x,y) \vee Q_{\Box\neg\Box p}(y) \qquad \text{schema 4 instance for } \Box p/\Box\neg\Box p$$
$$\neg Q_{\Box p}(x) \vee \neg R(x,y) \vee Q_{\Box p}(y) \qquad \text{schema 4 instance for } \Box p/\Box p$$

We refer to this form of encoding as the *axiomatic translation* of φ in $K4$. The general principle of the axiomatic approach for $K4$ is the following. *For every $\Box$ subformula, $\Box\psi$, of the candidate formula add the clause $\neg Q_{\Box\psi}(x) \vee \neg R(x,y) \vee Q_{\Box\psi}(y)$ to the translation, provided $\Box$ is a 4-modality, and similarly for other axioms.* For example, if $\Box$ is a T-modality then for every $\Box$ subformula, $\Box\psi$, of the candidate formula we need to add the clause $\neg Q_{\Box\psi}(x) \vee Q_{\psi}(x)$.

The main difficulty of the principle of the axiomatic encoding is to know how many instances of a schema clause need to be added to the translation. In the Hilbert axiomatisation the axioms are valid for any substitution instances. Since we do not have access to a substitution rule, we need to make sure from the outset that enough instances of the schema clauses are present in the translation of the candidate formula. Of course this does not preclude a lazy implementation which delays the translation of subformulae until absolutely necessary.

3 Translation to clause logic

Without loss of generality, attention is restricted to modal formulae formulated in terms of the connectives $\wedge$, $\neg$, and $\Box$ and the constant $\bot$. It is assumed that all occurrences of double negation have been eliminated. If ψ denotes a modal formula then, by definition, $\sim\psi$ denotes ϕ if $\psi = \neg\phi$, and $\neg\psi$ otherwise.

The standard translation into first-order logic is based on the standard Kripke semantics. It is combined with structural transformation, that is, new symbols are introduced for subformulae of the candidate formula. Structural transformation is a well-known technique with many advantages. It enables the preservation of the structure of the original formula in the first-order clausal form, thus improving the readability of first-order resolution proofs. When using hyperresolution, as is done in the completeness proofs (see below), it is often possible to

T	$\Box p \to p$	reflexivity	$\forall x\, R(x,x)$
4	$\Box p \to \Box\Box p$	transitivity	$\forall xyz\, (R(x,y) \land R(y,z) \to R(x,z))$
B	$\Diamond\Box p \to p$	symmetry	$\forall xy\, (R(x,y) \to R(y,x))$
D	$\Box p \to \Diamond p$	seriality	$\forall x \exists y\, R(x,y)$
alt_1	$\Diamond p \to \Box p$	functionality	$\forall xyz\, (R(x,y) \land R(x,z) \to y \approx z)$
5	$\Diamond\Box p \to \Box p$	Euclideanness	$\forall xyz\, (R(x,y) \land R(x,z) \to R(y,z))$

Fig. 1. Axiom schemata and relational background theories

read resolution proofs as modal tableau proofs. Structural transformation can also be used to keep the complexity of the reduction from one logic into another low.

The translation of a modal logic formula φ into first-order logic is specified as follows. For any subformula ψ of φ, let the notation $\mathrm{Def}_\psi(\varphi)$ represent the *definition* of Q_ψ, which is defined by:

$$\forall x\, (Q_\psi(x) \to \pi(\psi, x)) \land \forall x\, (Q_\psi(x) \to \neg Q_{\sim\psi}(x)) \land \forall x\, (Q_{\sim\psi}(x) \to \pi(\sim\psi, x)).$$

Q_ψ is a new predicate symbols uniquely associated with the modal formula ψ, and $\pi(\psi, x)$ is a first-order formula with one free variable x given by the following.

$$\pi(\bot, x) = \bot \qquad \pi(p, x) = \top \qquad \pi(\neg p, x) = \neg Q_p(x)$$
$$\pi(\psi \land \phi, x) = Q_\psi(x) \land Q_\phi(x) \qquad \pi(\neg(\psi \land \phi), x) = Q_{\sim\psi}(x) \lor Q_{\sim\phi}(x)$$
$$\pi(\Box\psi, x) = \forall y\, (R(x,y) \to Q_\psi(y)) \qquad \pi(\neg\Box\psi, x) = \exists y\, (R(x,y) \land Q_{\sim\psi}(y))$$

Thus, π is a function associating a first-order formula with any modal formula. Here the definition of π is based on the standard relational semantics.

Now let Π be the mapping defined by:

$$\Pi(\varphi) = \exists x\, Q_\varphi(x) \land \mathrm{Simpl}(\textstyle\bigwedge\{\mathrm{Def}_\psi(\varphi) \mid \psi \in \mathrm{Sf}(\varphi)\}),$$

where $\mathrm{Sf}(\varphi)$ denotes the set of all subformulae of φ. The purpose of Simpl is to eliminate obvious redundancies in the definitional forms which can be dealt with in linear time, for example, deletion of trivially tautologous formulae such as $\forall x\, (Q_p(x) \to \top)$.

Lemma 1. *For any modal formula φ, $\Pi(\varphi)$ can be computed in linear time.*

If Δ is a set of first-order definable modal axioms, then let $\mathrm{Th}(\Delta)$ denote the relational background theory corresponding to Δ, that is, $\psi \in \mathrm{Th}(\Delta)$ iff ψ is the frame correspondence property of some axiom $\mathcal{A} \in \Delta$. Some familiar axioms and their frame correspondence properties are listed in Figure 1. (When we use the symbol $\Diamond$, as is done in the figure, it is taken to be a shorthand for $\neg\Box\neg$.)

Theorem 2 (Soundness and completeness of Π). *Let L be a first-order definable propositional multi-modal logic $K_{(m)}\Delta$ which is sound and complete. For any L formula φ, φ is satisfiable in L iff $\mathrm{Th}(\Delta) \land \Pi(\varphi)$ is satisfiable in first-order logic.*

$\neg\psi$	$\neg Q_{\neg\psi}(x) \vee \neg Q_{\psi}(x)$	$\neg(\psi \wedge \phi)$	$\neg Q_{\neg(\psi\wedge\phi)}(x) \vee Q_{\sim\psi}(x) \vee Q_{\sim\phi}(x)$
$\bot$	$\neg Q_{\bot}(x)$	$\Box\psi$	$\neg Q_{\Box\psi}(x) \vee \neg R(x,y) \vee Q_{\psi}(y)$
$\psi \wedge \phi$	$\neg Q_{\psi\wedge\phi}(x) \vee Q_{\psi}(x)$ $\neg Q_{\psi\wedge\phi}(x) \vee Q_{\phi}(x)$	$\neg\Box\psi$	$\neg Q_{\neg\Box\psi}(x) \vee R(x, f_{\neg\Box\psi}(x))$ $\neg Q_{\neg\Box\psi}(x) \vee Q_{\sim\psi}(f_{\neg\Box\psi}(x))$

Fig. 2. Definitional clausal forms.

$\mathcal{A}$	Axiom	Schema clause $\mathrm{Ax}^{\mathcal{A}}(p)$
T	$\Box p \to p$	$\neg Q_{\Box p}(x) \vee Q_p(x)$
4	$\Box p \to \Box\Box p$	$\neg Q_{\Box p}(x) \vee \neg R(x,y) \vee Q_{\Box p}(y)$
B	$\neg\Box\neg\Box p \to p$	$\neg R(x,y) \vee \neg Q_{\Box p}(y) \vee Q_p(x)$
D	$\Box p \to \neg\Box\neg p$	$\neg Q_{\Box p}(x) \vee Q_{\neg\Box\neg p}(x)$
alt_1	$\neg\Box\neg p \to \Box p$	$\neg Q_{\neg\Box p}(x) \vee Q_{\Box\neg p}(x)$
5	$\neg\Box\neg\Box p \to \Box p$	$\neg R(x,y) \vee \neg Q_{\Box p}(y) \vee Q_{\Box p}(x)$
4^{κ}	$\Box p \to \Box^{\kappa}\Box p$	$\neg Q_{\Box p}(x) \vee \neg R^{\kappa}(x,y) \vee Q_{\Box p}(y)$
$\mathrm{alt}_1^{\kappa_1,\kappa_2}$	$\neg\Box^{\kappa_1}\Box p \to \Box^{\kappa_2}\Box\neg p$	$\neg R^{\kappa_1}(x,y) \vee \neg Q_{\neg\Box\psi}(y) \vee \neg R^{\kappa_2}(x,z) \vee Q_{\Box\neg\psi}(z)$
5^{κ}	$\neg\Box^{\kappa}\neg\Box p \to \Box p$	$\neg R^{\kappa}(x,y) \vee \neg Q_{\Box p}(y) \vee Q_{\Box p}(x)$

Fig. 3. Schema clauses ($\kappa \geq 1$ and $\kappa_1, \kappa_2 \geq 0$).

In this paper we assume the clausal form of a first-order formula φ, written $\mathrm{Cls}(\varphi)$, is computed by transformation into conjunctive normal form, inner Skolemisation, and clausifying the Skolemised formula. Figure 2 lists modal formulae and the clausal form of the corresponding definitional clausal forms.

Lemma 2. *Let φ be a modal formula. Each clause in $\mathrm{Cls}\,\Pi(\varphi)$ is either (i) the unit clause $Q_{\varphi}(a)$, for some Skolem constant a, or (ii) it is an instance of a definitional clause given in Figure 2.*

4 Incorporating axioms into the translation

Following the recipe described in Section 2 for the axioms 4 and T, schema clauses can be derived fully automatically from Hilbert axioms. Figure 3 lists the schema clauses of a selection of axiom schemata. The notation $\neg R^0(s,t) \vee C$ represents $\neg(s \approx t) \vee C$. When s is a variable that does not occur in t then $\neg R^0(s,t) \vee C$ is equivalent to $C\{s/t\}$. For $n \geq 0$, $\neg R^{n+1}(s,t) \vee C$ represents $\neg R(s,z) \vee \neg R^n(z,t) \vee C$ where z is a new variable that does not occur in the clause. The clauses are assumed to be closed under universal quantification of the free variables. The propositional variable p in the schema clauses is a *parameter*, which will be suitably instantiated in the axiomatic translation defined below.

In this paper we restrict our attention to modal axioms in one variable which are represented by one schema clause. However, in general, modal axioms may be in more than one variable and may reduce to a set of schema clauses. Even modal axioms in one variable may reduce to a set of schema clauses.

Now we formalise the principle of the axiomatic translation exemplified in Section 2. Let L be a normal propositional modal logic. That is, L is an extension $K_{(m)}\Delta$ of multi-modal logic $K_{(m)}$, where Δ denotes a set of axiom schemata. For each axiom $\mathcal{A} \in \Delta$, let $\mathfrak{X}_{\mathcal{A}}$ be an arbitrary set of L formulae. Let $\mathfrak{X} = \{\mathfrak{X}_{\mathcal{A}}\}_{\mathcal{A}\in\Delta}$. Each $\mathfrak{X}_{\mathcal{A}}$ is the instantiation set for an axiom $\mathcal{A}$ in Δ and $\mathfrak{X}$ is the collection of instantiation sets used in the axiomatic translation of a modal formula in $K_{(m)}\Delta$. (The set $\mathfrak{X}$ is always relative to the set of formulae Δ, but to avoid cluttering we do not make this explicit in the notation.)

Definition 1 (Axiomatic translation $\Pi^{\Delta}_{\mathfrak{X}}$). *Let $\Pi^{\Delta}_{\mathfrak{X}}$ be a function mapping modal formulae to first-order formulae defined as follows. If φ is an L formula then $\Pi^{\Delta}_{\mathfrak{X}}(\varphi)$ is (the simplification with* Simpl *of) the conjunction of the following:*

1. *The structural translation of φ.*

$$\Pi(\varphi) = \exists x\, Q_{\varphi}(x) \wedge \mathrm{Simpl}(\bigwedge\{\mathrm{Def}_{\psi}(\varphi) \mid \psi \in \mathrm{Sf}(\varphi)\})$$

2. *The structural translation of formulae in $X = \bigcup_{\mathcal{A}\in\Delta} \mathfrak{X}_{\mathcal{A}}$ except those that are not subformulae of φ and therefore already included in $\Pi(\varphi)$.*

$$\mathrm{Simpl}(\bigwedge\{\mathrm{Def}_{\psi}(X \setminus \mathrm{Sf}(\varphi)) \mid \psi \in X \setminus \mathrm{Sf}(\varphi)\})$$

3. *The reverse implications of the links between Q_{ψ} and $Q_{\sim\psi}$ in the definitions.*

$$\bigwedge\{\forall x\, (\neg Q_{\sim\psi}(x) \rightarrow Q_{\psi}(x)) \mid \psi \in X \cup \mathrm{Sf}(\varphi)\}$$

4. *For each axiom schema $\mathcal{A}$ in Δ and each schema clause C associated with $\mathcal{A}$, the ψ instances of C for each formula ψ in $\mathfrak{X}_{\mathcal{A}}$.*

$$\bigwedge\{\mathrm{Ax}^{\mathcal{A}}(\psi) \mid \mathcal{A} \in \Delta,\ \psi \in \mathfrak{X}_{\mathcal{A}}\}$$

   By definition, $\mathrm{Ax}^{\mathcal{A}}(\psi)$ is the conjunction of (the universal closure of) all clauses $C\{p/\psi\}$, where C is a clausal schema in the clausal schema set associated with the axiom schema $\mathcal{A}$.

The implicit limitation in the definition to axiom schemata with one free variable, or clauses with one parameter, is not crucial. The definition can be easily generalised for modal axioms in more than one variable.

Notice the clausal form of the formulae in 3 of Definition 1 are positive clauses. As a consequence hyperresolution generates also non-ground conclusions for the axiomatic encoding. They are omitted from the definition of $\mathrm{Def}_{\psi}(\varphi)$ so that inferences with non-ground positive premises using the standard translation approach need not be simulated in the completeness proofs. For some modal logics and modal axioms the formulae of Definition 1.3 are not strictly necessary. However they are included for reasons of uniformity and because experience shows theorem provers benefit from the presence of shortcut clauses because they then tend to terminate earlier and proofs are normally shorter.

Theorem 3 (Soundness of $\Pi_{\mathfrak{X}}^{\Delta}$). *Let L be a consistent propositional modal logic $K_{(m)}\Delta$ with Δ a finite set of modal formulae. Let φ be any L formula and assume $\bigcup_{\mathcal{A}\in\Delta}\mathfrak{X}_{\mathcal{A}}$ is a finite set of L formulae. If φ is L satisfiable then $\Pi_{\mathfrak{X}}^{\Delta}(\varphi)$ is first-order satisfiable.*

Basically the proof uses an argument that is standard for proving the soundness of renaming techniques. The only difference is that we start with a modal model and construct a first-order model of the formula $\Pi_{\mathfrak{X}}^{\Delta}(\varphi)$. This model is a conservative extension of the first-order model normally associated with the modal model. An instance of a clausal schema is satisfied in the first-order model, since the corresponding instance of the corresponding modal axiom is satisfied in every world of the modal model.

The soundness result is very general and very useful. For example, it is possible to use the axiomatic translation method for proving theorems in non-first-order definable modal logics. We can prove that atomicity, given by the formula $\Diamond(p \to \Box p)$, is a theorem in $KM4$ where M is McKinsey's axiom $\Box\Diamond p \to \Diamond\Box p$. The axiom M reduces to a set of schema clauses and a proof can be found with the $\Box$ subformula instances. However, at the moment we do not know for which set of instances (if any) the axiomatic translation would be complete for KM and $KM4$. This means that if no proof is found for a formula with respect to a particular set of formulae $\mathfrak{X}$ it is not possible to draw any conclusion.

5 Completeness

In order to prove completeness of the translation mapping $\Pi_{\mathfrak{X}}^{\Delta}$, a possible approach is to take a model-theoretic route and to show how to construct an L model for φ from a given first-order model for $\Pi_{\mathfrak{X}}^{\Delta}(\varphi)$. Alternatively, one can use a proof-theoretic argument; this is the approach taken in this paper. A proof-theoretic argument sufficient for obtaining completeness is to show how to map refutation proofs of $\mathrm{Th}(\Delta) \wedge \Pi(\varphi)$ to refutation proofs of $\Pi_{\mathfrak{X}}^{\Delta}(\varphi)$ in first-order logic. For various reasons we use hyperresolution style derivations combined with splitting. In particular, conclusions with hyperresolution are always positive clauses and, more importantly, the clausal form of $\mathrm{Th}(\Delta) \wedge \Pi(\varphi)$ for the logics we consider are sets of *near range-restricted* clauses. Recall, a clause C is said to be *range-restricted* iff the set of variables in the positive part of C is a subset of the set of variables of the negative part of C. For the logics we consider clause sets may also contain non-ground positive clauses $R(x, x)$, $R(x, f(x))$ and $R(f(x), x)$. Derivations with hyperresolution and splitting on such near range-restricted clauses have the following properties: At least one of the positive premises is always a ground unit clauses and non-redundant conclusions are then always positive ground clauses (with the exception of one case, namely for KDB using the classic approach, where non-redundant conclusions are ground or have the form $R(f(x), x)$). Hyperresolution and splitting not only detects unsatisfiability; in the case that φ is satisfiable, a (Herbrand) model can be immediately read off from any complete open branch in the derivation.

Moreover, the derivations can be mapped directly to derivations in first-order sentence tableaux.

In proving completeness, a key problem is to determine how many instances of the clausal schemata are needed for the axiomatic translation to work. That is, we need to specify the set $\mathfrak{X}$ of formulae for which instances of the schema clauses are to be formed. To do so in a systematic way, we need some more notation. If X is a set of modal formulae then $\Box X$ denotes the set $\{\Box\psi \mid \psi \in X\}$. Let φ be an arbitrary modal formula. We assume all axiom schemata are *unary*, i.e. any axiom schema contains only occurrences of one propositional variable. Suppose $\mathcal{A}$ denotes an arbitrary unary axiom schema with free variable p, and σ denotes a sequence of unary axiom schemata (without repetition). The empty sequence is denoted by ϵ. Now, define $\mathfrak{X}_\varphi^\sigma$ inductively by:

$$\begin{aligned}\mathfrak{X}_\varphi^\epsilon &= \{\psi \mid \Box\psi \in \mathrm{Sf}(\varphi)\} \quad \text{and} \\ \mathfrak{X}_\varphi^{\sigma.\mathcal{A}} &= \mathfrak{X}_\varphi^\sigma \cup \{\phi\{p/\psi\} \mid \Box\phi \in \Box\mathfrak{X}_\mathcal{A}^\epsilon,\ \Box\psi \in \mathrm{Sf}(\Box\mathfrak{X}_\varphi^\sigma)\}.\end{aligned}$$

$\mathfrak{X}_\varphi^\epsilon$ is thus the set of subformulae of φ that occur immediately below a $\Box$ operator. Note that the order of appearance of the axioms in the sequence is essential.

In the remainder of the section we present completeness theorems of the axiomatic translation for a selection of modal logics.

The axiomatic translation of any formula φ in $K4$, KT, KD, KB, $K\mathrm{alt}_1$ is determined by $\mathfrak{X}_4 = \mathfrak{X}_T = \mathfrak{X}_B = \mathfrak{X}_{\mathrm{alt}_1} = \mathfrak{X}_\varphi^\epsilon$. That is, $\Box\mathfrak{X}_\mathcal{A}$ is the set of $\Box$ subformulae of φ for $\mathcal{A} \in \{4, T, D, B, \mathrm{alt}_1\}$.

Theorem 4. *Suppose $\mathcal{A} \in \{T, B, D, 4, \mathrm{alt}_1\}$. Let φ be any modal formula and assume $\mathfrak{X} = \{\mathfrak{X}_\mathcal{A}\}$ where $\mathfrak{X}_\mathcal{A} = \mathfrak{X}_\varphi^\epsilon$. Then, $\Pi_\mathfrak{X}^\mathcal{A}(\varphi)$ is unsatisfiable in first-order logic, whenever φ is unsatisfiable in $K\mathcal{A}$.*

For each of the axioms the structure of the argument is always the same, but in detail the proofs can differ significantly depending on the nature of the modal axiom.

We can show completeness for the generalised axiom schemata 4^κ ($\kappa \geq 1$) and $\mathrm{alt}_1^{\kappa_1,\kappa_2}$ ($\kappa_1, \kappa_2 \geq 0$), cf. Figure 3, and $S4$ using $\mathfrak{X}_\varphi^\epsilon$.

Theorem 5. *Suppose $\mathcal{A} \in \{4^\kappa, \mathrm{alt}_1^{\kappa_1,\kappa_2}\}$. Let φ be any modal formula and assume $\mathfrak{X} = \{\mathfrak{X}_\mathcal{A}\}$ where $\mathfrak{X}_\mathcal{A} = \mathfrak{X}_\varphi^\epsilon$. If φ is unsatisfiable in $K\mathcal{A}$ then $\Pi_\mathfrak{X}^\mathcal{A}(\varphi)$ is unsatisfiable in first-order logic.*

Theorem 6. *Let φ be any modal formula and assume $\mathfrak{X} = \{\mathfrak{X}_T, \mathfrak{X}_4\}$, where $\mathfrak{X}_T = \mathfrak{X}_4 = \mathfrak{X}_\varphi^\epsilon$. Then, $\Pi_\mathfrak{X}^{T,4}(\varphi)$ is unsatisfiable in first-order logic, whenever φ is unsatisfiable in $S4$.*

Similarly, one can prove that the axiomatic translation using $\mathfrak{X}_\varphi^\epsilon$ for each axiom is complete for KTB, KDB, $KT\mathrm{alt}_1$, $KB\mathrm{alt}_1$, $KD\mathrm{alt}_1$ and $KD4$.

From these results, one might be tempted to think that the axiomatic translation provides a generic and modular approach for translating problem specifications into first-order logic. Unfortunately, in general the solution is not as smooth as above. In general, it is not enough to form one instance of the clausal schemata of an axiom for each $\Box$ subformula. It might not even be only $\Box$ subformulae instances that are required, though we do not consider any logics of this kind in this paper.

A logic of the former kind is $K5$. For $K5$, clausal schema instances need to be formed not only for $\Box$ subformulae, but also for $\Box\neg\Box\psi$ formulae, if $\Box\psi$ is a subformula of the candidate formula. A counter example that the axiomatic translation is not always complete for $\Box$ subformula instances alone is the formula $\varphi = \Diamond\Diamond\Diamond\neg q \wedge \Box\Box q$ which is unsatisfiable in $K5$, but $\Pi^5_{\mathfrak{X}}(\varphi)$, where $\mathfrak{X} = \{\mathfrak{X}^\epsilon_\varphi\}$, is satisfiable. However, $\Pi^5_{\mathfrak{X}}(\varphi)$ can be refuted if $\mathfrak{X} = \{\mathfrak{X}^5_\varphi\}$, where $\mathfrak{X}^5_\varphi = \mathfrak{X}^\epsilon_\varphi \cup \neg\Box\mathfrak{X}^\epsilon_\varphi$.

Theorem 7. *Let φ be any modal formula and let $\mathfrak{X} = \{\mathfrak{X}_5\}$, where $\mathfrak{X}_5 = \mathfrak{X}^5_\varphi$. Then, $\Pi^5_{\mathfrak{X}}(\varphi)$ is unsatisfiable in first-order logic, whenever φ is unsatisfiable in $K5$.*

The theorem is a consequence of a more general result.

Theorem 8. *Let φ be any modal formula and assume $\mathfrak{X} = \{\mathfrak{X}_{5^\kappa}\}$, where $\mathfrak{X}_{5^\kappa} = \mathfrak{X}^{5^\kappa}_\varphi$. If φ is unsatisfiable in $K5^\kappa$ then $\Pi^{5^\kappa}_{\mathfrak{X}}(\varphi)$ is unsatisfiable in first-order logic.*

Theorem 6 for $S4$ might lead one to speculate that the principle may be easily extended to $K_{(m)}\Delta$ where Δ contains multiple axiom schemata. But there is a counter example. While for KB and $K4$, it is enough to add $\Box$ subformula instances of the schema clauses, for $K4B$ it is not enough to instantiate the schema clauses for 4 and B with $\Box$ subformulae. For example, $5 = \neg\Box\neg\Box p \rightarrow \Box p$ is a theorem in $K4B$. But $\Pi^{4,B}_{\mathfrak{X}}(\neg 5)$ is not refutable with $\mathfrak{X} = \{\mathfrak{X}_4, \mathfrak{X}_B\}$ where $\mathfrak{X}_4 = \mathfrak{X}_B = \mathfrak{X}^\epsilon_\varphi$, nor with $\mathfrak{X}_4 = \mathfrak{X}^B_\varphi$ and $\mathfrak{X}_B = \mathfrak{X}^\epsilon_\varphi$. A refutation proof can only be found for $\Pi^{4,B}_{\mathfrak{X}}(\neg 5)$ if $\mathfrak{X}_4 = \mathfrak{X}^\epsilon_\varphi$ and $\mathfrak{X}_B = \mathfrak{X}^4_\varphi$.

Theorem 9. *Let φ be any modal formula and assume $\mathfrak{X} = \{\mathfrak{X}_B, \mathfrak{X}_4\}$, where $\mathfrak{X}_4 = \mathfrak{X}^\epsilon_\varphi$ and $\mathfrak{X}_B = \mathfrak{X}^4_\varphi$. Then, $\Pi^{4,B}_{\mathfrak{X}}(\varphi)$ is unsatisfiable in first-order logic, whenever φ is unsatisfiable in $K4B$.*

Theorem 10. *Let φ be any modal formula and assume $\mathfrak{X} = \{\mathfrak{X}_T, \mathfrak{X}_B, \mathfrak{X}_4\}$, where $\mathfrak{X}_T = \mathfrak{X}_4 = \mathfrak{X}^\epsilon_\varphi$ and $\mathfrak{X}_B = \mathfrak{X}^4_\varphi$. Then, $\Pi^{T,4,B}_{\mathfrak{X}}(\varphi)$ is unsatisfiable in first-order logic, whenever φ is unsatisfiable in $S5$.*

Thus we can summarise:

Theorem 11. *For each of the modal logics $K\Delta$ considered in this section ($K4$, KT, KD, KB, $K\mathrm{alt}_1$, $K4^\kappa$, $K\mathrm{alt}_1^{\kappa_1,\kappa_2}$, $KT4$, KTB, KDB, $KT\mathrm{alt}_1$, $KB\mathrm{alt}_1$, $KD\mathrm{alt}_1$, $KD4$, $K5$, $K5^\kappa$, $K4B$, $KT4B$) and any modal formula φ, there is an effectively computable set $\mathfrak{X}$ such that (i) φ is satisfiable in $K\Delta$ iff $\Pi^\Delta_{\mathfrak{X}}(\varphi)$ is first-order satisfiable. Moreover, (ii) $\Pi^\Delta_{\mathfrak{X}}(\varphi)$ can be computed in linear time.*

Finally, we note that it is possible to mix the axiomatic translation with the standard translation.

Theorem 12. *Let φ be any modal formula and assume $\mathfrak{X} = \{\mathfrak{X}_4\}$, where $\mathfrak{X}_4 = \mathfrak{X}^\epsilon_\varphi$. Then, $\mathrm{Th}(\{T, B\}) \wedge \Pi^4_{\mathfrak{X}}(\varphi)$ is unsatisfiable in first-order logic, whenever φ is unsatisfiable in $S5$.*

6 Decidability

Lemma 3. *(i) The axiomatic translation of any modal formula is equivalent to a GF^2 formula. (ii) The axiomatic translation of any modal formula can be embedded in the clausal class DL^*.*

Theorem 13. *Let L be a sound and complete propositional modal logic $K_{(m)}\Delta$. Then, L is decidable and has the finite model property, whenever the following conditions are satisfied. (i) Δ is finite. (ii) For any L formula φ, there are effectively computable sets $\mathfrak{X}_{\mathcal{A}}$ for each $\mathcal{A} \in \Delta$ such that if $\Pi^{\Delta}_{\mathfrak{X}}(\varphi)$ is satisfiable in first-order logic then φ is satisfiable in L, where $\mathfrak{X} = \{\mathfrak{X}_{\mathcal{A}}\}_{\mathcal{A}\in\Delta}$.*

Corollary 1. *The modal logics considered in the previous section, and their fusions, are decidable and have the finite model property.*

Decidability of extensions of K with the axioms T, B, D, 4, 5, alt_1 are well-known. In [6] the decidability of the logics $K4^\kappa$, $K5^\kappa$ and $K\mathrm{alt}_1^{\kappa_1,\kappa_2}$ was shown by using a reduction to SωS. It is also well-known that the operation of fusing normal propositional modal logics preserves decidability. Thus the decidability of the logics considered is known or follows from known results. As for the finite model property, we believe the finite model property of the logics $K5^\kappa$ and $K\mathrm{alt}_1^{\kappa_1,\kappa_2}$ has been open up to now.

What is interesting about the approach described in this paper, is the method of proving decidability and the potential of the approach for systematically developing and implementing decision procedures for modal logics. In most cases hyperresolution which is used in the completeness proofs (see [15]) is not a decision procedure for the axiomatic translation of modal problems, but ordered resolution is. In particular:

Theorem 14. *Resolution based on any refinement compatible with the ordering $>_d$ defined in [2] decides the axiomatic encoding of satisfiability problems in all modal logics satisfying the conditions of Theorem 13.*

To see this, use the axiomatic translation to embed the problem specification in DL^* or GF^2. Then note that any refinement of resolution based on $>_d$ decides DL^* and the guarded fragment [2]. The ordering $>_d$ is a standard ordering available in most modern theorem provers. This immediately gives practical decision procedures for testing satisfiability of the modal logics considered in the previous section and their fusions. With the blocking inference rule introduced in [9] hyperresolution can also be turned into a decision procedure for the axiomatic translation of the considered logics.

7 Other consequences and related work

The combination of the axiomatic translation and first-order inference methods promises to provide a powerful and versatile approach for studying and mechanising modal reasoning. However, there are also other consequences. We mention here just a few.

There is a close relationship between hyperresolution and tableau calculi. In [2] it is shown how this relationship can be exploited for systematically developing sound, complete and terminating tableau rule systems for PDL-like multi-modal logics extended with operators on the parameters of the modal operators. The same approach can be used for extracting tableau rules from the combination of the axiomatic translation and hyperresolution. The idea is to express a group of hyperresolution inference steps as a tableau rule. In this way tableau calculi may be obtained which coincide exactly with labelled semantic-based calculi found in the literature.

The axiomatic translation can also be interpreted on the modal level. On the modal level it corresponds to a reduction of the satisfiability of a modal formula with respect to an extension $K_{(m)}\varDelta$ of $K_{(m)}$, to global satisfiability in $K_{(m)}$, or satisfiability in $K_{(m)}$ enhanced with the universal modality (which can be further reduced to local satisfiability in $K_{(m)}$). On a modal level the axiomatic translation is closely related to the reduction functions introduced independently by Kracht in [10] and developed further in [11]. Kracht's reduction functions are defined differently, the most important difference being that they do not use new symbols for subformulae. Renaming would lead to an improvement of complexity of the reduction functions from polynomial to linear complexity. Kracht has proved a number of results which immediately carry over to the axiomatic translation. For example, some of the logics considered in the present paper satisfy Kracht's criteria for local interpolation.

An EXPTIME upper bound for the complexity of the considered logics or their fusions is immediate from the reduction to $K_{(m)}$ with the universal modality. This bound is not optimal and can be improved for certain logics. Inspection of the completeness proofs for KT, KB, KD, $K4$, $S4$, $K\mathrm{alt}_1$ reveals that short-cut clauses are not necessary for completeness. For the logics KT, KB, KD, $K\mathrm{alt}_1$ one can observe that the axiomatic reduction to K is a set of non-logical axioms which are acyclic. Thus a result of [13] for the corresponding description logic $\mathcal{ALC}$ with acyclic TBoxes implies:

Theorem 15. *The computational complexity of the satisfiability problem of each of KT, KB, KD, $K\mathrm{alt}_1$ and their fusions is PSPACE-complete.*

More and better space bounds can be given by using the axiomatic translation and the ideas of [12].

The axiomatic translation can be used to combine different styles of reasoning systems. For example, the axiomatic translation was used in a decision procedure for the join of discrete linear temporal logic with subsystems of $S5$ [8]. The system combines temporal resolution for the temporal formulae and first-order resolution on the modal formulae.

The idea of incorporating a theory into the translation of a formula, or eliminating a relation with special properties by internalisation, is of course not new. Often, reductions between logics can be seen to use this principle. For example, De Giacomo [1] embedded CPDL into PDL by using an axiomatic encoding. Demri and Goré [5] embed the provability logics G and Grz into GF^2. Both are second-order modal logics. This shows that with the axiomatic translation it is possible to get a first-order translation of second-order modal logics.

De Giacomo's encoding of converse in PDL uses instances of schemata defined over the Fisher-Ladner closure of the candidate formula. Although the size of the Fisher-Ladner closure is linear in the size of the candidate formula, in many instances the encodings used in this paper are much more efficient since in many cases we just formed instances with $\Box$ subformulae rather than all subformulae and their negations. This raises the question whether the encoding of CPDL into PDL can be optimised, and what is the smallest set of instances necessary. Such questions are of general significance to the efficiency of proof methods exploiting the axiomatic translation.

There is a (non-obvious) connection to [4], worth exploring further. Demri and De Nivelle show that modal regular grammar logics with converse are decidable by reduction to GF^2. For a given modal logic it is not immediate whether it is in fact a regular grammar modal logic (possibly with converse). The problem is that one has to find a regular language which describes the closure under the relational correspondence properties. We expect that one can extract a regular language description from the axiomatic encoding, and thus be able to prove membership in the class of regular grammar modal logic with converse for a larger class of modal logics. Such a connection is useful because regular grammar modal logics have been shown to have a number of nice properties by Demri [3].

8 Conclusions

The attractiveness of the translation principle described in this paper is its amenability for automation. The translation is not difficult to implement. As a prover any existing first-order logic prover can be used. Initial experiments with an implementation of the translation and two resolution provers show that the performance gain when using the axiomatic translation over the standard translation using explicit correspondence properties is considerable. When employing the axiomatic translation a significant speedup is noticeable for both provers especially when the logics include modalities defined by triangular properties. Alternatively, the axiomatic translation can be transferred to the level of modal logic or description logic and combined with a suitable theorem prover for that logic. What is required is either a description logic prover for $\mathcal{ALC}$ that can handle terminological axioms or a modal theorem prover that can handle multi-modal $K_{(m)}$ with non-logical axioms, i.e. global satisfiability/validity in $K_{(m)}$, or a prover for $K_{(m)}$ extended with the universal modality can be used.

To summarise, what we tried to achieve with the axiomatic translation is to find a simplified theory, which captures the semantic background theory, but in a way that is more amenable to automation. The principle of incorporating modal axioms into the first-order translation of modal formulae is conceptually very simple. The clausal schemata can be automatically deduced, and soundness is automatic under reasonable assumptions. The non-trivial part of the approach is proving completeness (determining the instantiation sets and simulate hyperresolution refutations based on the classical translation approach). The reward for succeeding in proving completeness (in this way or another) is automatic decidability, and other nice properties such as the finite model property, interpolation, sound and complete tableau systems. Besides proving completeness everything else can be easily automated and implemented. While in this paper we have focussed on the application of the axiomatic translation of well-known axiom schemata, we expect that the principle and the insights underlying the axiomatic translation is applicable to a wide range of non-standard axiom schemata, in particular, those found in agent-based systems, description logics, and related fragments of first-order logic such as the monadic guarded fragment introduced in [7].

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