Some Notes on (Double) Glueing along Hom-Functors

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1 Introduction

Glueing is a technique frequently applied in categorical logic and topos theory. We look at it from the point of view of finding new models for linear logic, but we only consider the multiplicatives and additives here. In these notes we introduce the simple notion of (double) glueing along hom-functors. The full generality (including a treatment of the exponentials) of the construction can be found in in [HS03], and short notes from a more abstract point of view are available from the author.

Definition 1 *A symmetric monoidal closed category* $\mathbf{C}$ *is* $*$**-autonomous** *if has an object* $\perp$ *such that the canonical arrows* $A \longrightarrow (A \multimap \perp) \multimap \perp$ *(the transposes of the evaluation maps) are isomorphisms.*

Definition 2 *A* categorical model of intuitionistic MALL *consists of a category which*

- *is symmetric monoidal closed;*
- *has finite products;*

A model of classical linear logic is a model of intuitionistic logic with a strong duality.

Definition 3 *A* model for classical MALL *consists of a category which*

- *is $*$-autonomous;*
- *has finite products and (so) finite coproducts;*

Exercise 1 *Recall that a $*$-autonomous category $\mathbf{C}$ has a self-duality $(-)^{\perp}\colon \mathbf{C}^{\mathsf{op}} \longrightarrow \mathbf{C}$ with the property that $(-)^{\perp\perp}$ is isomorphic to the identity on $\mathbf{C}$. Show that a category equipped with such a self-duality which has finite products also has finite coproducts, and that such a category also has a second tensor product. (Hint: DeMorgan!)*

To generate a $*$-autonomous category from a symmetric monoidal closed one $\mathbf{C}$ one can take $\mathbf{C} \times \mathbf{C}^{\mathsf{op}}$ (sketched in [HS03]). More sophisticated versions are the *Chu* (see [Chu79, Bar91]) and *dialectica* (see [dP91, dP89]) constructions.

2 Simple Glueing

Assume we have a model $\mathbf{C}$ for intuitionistic MALL. The aim here is to provide the objects with extra structure which has to be preserved by morphisms. When we use a more sophisticated version called *double glueing* it allows us to start with a compact closed category and extend it to a $*$-autonomous one where the two tensors given by $\otimes$ and $((-)^\perp \otimes (-)^\perp)^\perp$ are not the same.

We glue along the hom-functor $\mathbf{C}(\mathbf{I}, -)$ to obtain our first glued category.

Definition 4 *Given the category $\mathbf{C}$ define $\mathbf{GC}$ to be the category with*

- *objects: an object R of $\mathbf{C}$ together with a set $U \subseteq \mathbf{C}(\mathbf{I}, R)$.*
- *arrows: an arrow from (R, U) to (S, V) is an arrow $f\colon R \longrightarrow S$ in $\mathbf{C}$ such that for all $u \in U$, $fu \in V$.*

In other words, we have morphisms of the following kind:

$$\begin{tikzcd} R \arrow[d, "f"] \\ S \end{tikzcd} \qquad \text{such that} \qquad \begin{tikzcd} & \mathbf{I} \arrow[r, "u"] & R \in U \arrow[d, phantom, "\text{implies}"] \\ \mathbf{I} \arrow[r, "u"'] & R \arrow[r, "f"'] & S \in V \end{tikzcd}$$

Note that there is a forgetful functor $\mathbf{GC} \longrightarrow \mathbf{C}$, and this functor preserves all the categorical structure. To get something that goes in the opposite direction we can equip each object R of $\mathbf{C}$ with the empty subset of $\mathbf{C}(\mathbf{I}, R)$), or with the whole of $\mathbf{C}(\mathbf{I}, R)$.

In order to tie our investigation into glueing along hom-functors into the general theory we look more closely at the hom-functor under consideration.

Exercise 2 *Show that if $\mathbf{C}$ is symmetric monoidal then for every object R in $\mathbf{C}$ the hom-functor $\mathbf{C}(R, -)$ is symmetric monoidal, as is the contra-variant hom-functor $\mathbf{C}(-, R)$.*

If we start with with a symmetric monoidal (closed) category, then we get another such back. For the closed structure, recall that if $w\colon \mathbf{I} \longrightarrow R \multimap S$ is a morphism in $\mathbf{C}$ then because of the adjunction, it corresponds to an arrow $\hat{w}$ in

$$\mathbf{C}(\mathbf{I}, R \multimap S) \cong \mathbf{C}(\mathbf{I} \otimes R, S) \cong \mathbf{C}(R, S).$$

Proposition 2.1 *If $\mathbf{C}$ is symmetric monoidal then so is $\mathbf{GC}$. The structure is preserved by the forgetful functor $\mathbf{GC} \longrightarrow \mathbf{C}$ and it is given by*

- $(R, U) \otimes (S, V) = (R \otimes S, U \otimes V)$, *where*

$$U \otimes V = \{\mathbf{I} \xrightarrow{\cong} \mathbf{I} \otimes \mathbf{I} \xrightarrow{u \otimes v} \ \mid u \in U, v \in V\},$$

- $\mathbf{I} = (\mathbf{I}, \{\mathsf{id}_{\mathbf{I}}\})$.

Moreover, if $\mathbf{C}$ *is symmetric monoidal closed then so is* $\mathbf{GC}$*, and the additional structure is given by*

$$(R,U) \multimap (S,V) = (R \multimap S, W), \text{ where}$$

$$W = \{\mathbf{I} \xrightarrow{w} R \multimap S \mid \forall \mathbf{I} \xrightarrow{u} R \text{ in } U.\ \mathbf{I} \xrightarrow{u} R \xrightarrow{\hat{w}} S \in V\}$$

represents the hom-set $\mathbf{GC}(R,S)$ *and* $\hat{w}$ *is the obvious transpose of* w.

Exercise 3 *Convince yourself that we have defined a category, and that that category is indeed symmetric monoidal (closed), provided that the underlying category has the corresponding structure. What are the necessary natural isomorphisms?*

Exercise 4 *Describe the symmetric monoidal closed category* **GRel***. What's the best way to think about the objects? The morphisms? What does the multiplicative structure look like, redefined in those terms?*

(Recall that the symmetric monoidal closed structure on **Rel** *has a tensor given by cartesian product, with the obvious extension to morphisms, and that the linear function space of two sets is once again their cartesian product.)*

To interpret the additives we need $\mathbf{C}$ to have the corresponding structure.

Proposition 2.2 *(i) If* $\mathbf{C}$ *has finite products then so does* $\mathbf{GC}$*; the functor* $\mathbf{GC} \longrightarrow \mathbf{C}$ *preserves them. They are given by*

$$\begin{aligned}(R,U) \times (S,V) &= (R \times S, U \times V) \quad \text{where}\\ U \times V &= \{\langle u,v\rangle \colon \mathbf{I} \longrightarrow R \times S \mid u \in U, v \in V\}.\end{aligned}$$

The terminal object is $(\mathbf{1}, \mathbf{C}(\mathbf{I},\mathbf{1}))$.

(ii) If $\mathbf{C}$ *has finite coproducts then so has* $\mathbf{GC}$*, and* $\mathbf{GC} \longrightarrow \mathbf{C}$ *preserves them. They are given by*

$$\begin{aligned}(R,U) + (S,V) &= (R+S, U \oplus V) \quad \text{where}\\ U \oplus V &= \{\mathbf{I} \xrightarrow{u} R \xrightarrow{\mathsf{inl}} R+S \mid u \in U\} \cup \{\mathbf{I} \xrightarrow{v} S \xrightarrow{\mathsf{inr}} R+S \mid v \in V\}\end{aligned}$$

The unit for the coproduct is $(\mathbf{0}, \emptyset)$.

Exercise 5 *Convince yourself that Proposition 2.2 is true, and describe the additive structure of the category* **GRel***. (Recall that in* **Rel** *both sums and products of objects are given by disjoint unions, and that the projections/injections are the opposites of each other.)*

There is nothing special about the co-variant hom-functor here—we could just as well have used the contravariant one to glue along.

Exercise 6 *Assume now that an object of the glued category is an object* R *of* $\mathbf{C}$ *together with a subset* $X \subseteq \mathbf{C}(R,J)$*, where* J *is an arbitrary object of* $\mathbf{C}$*. Can you turn this into a category? (Hint: The implication that defines morphisms goes in the opposite direction now.) Can you make it symmetric monoidal closed? What about sums and products? (Hint: If you cannot do this exercise you may want to read ahead to double glueing.)*

Examples and applications

Logical relations. Glueing is the abstract mathematical counterpart of the technique of logical relations. However, this usually requires glueing along a functor other than a hom-functor and so giving details is beyond the scope of these notes.

Indecomposability. Glueing was first introduced by Freyd to give neat proofs of projectivity and indecomposability results for toposes. One can readily adapt this argument. Let $\mathbf{C}$ be the free symmetric monoidal closed category with coproducts on a collection of objects. As $\mathbf{C}$ is free we have a structure preserving functor $\mathbf{C} \longrightarrow \mathbf{G}$ given by taking generators A to $(A, \mathbf{C}(\mathbf{I}, A))$. Now the composite $\mathbf{C} \longrightarrow \mathbf{G} \longrightarrow \mathbf{C}$ is the identity. A map $\mathbf{I} \longrightarrow R + S$ in $\mathbf{C}$ thus maps to

$$(\mathbf{I}, \{\mathsf{id}_{\mathbf{I}}\}) \longrightarrow (R + S, U \oplus V)$$

in $\mathbf{G}$; $\mathsf{id}_{\mathbf{I}}$ maps to either U or V, and so $\mathbf{I}$ maps to one of R and S. Thus

$$\mathbf{C}(\mathbf{I}, R + S) \cong \mathbf{C}(\mathbf{I}, R) + \mathbf{C}(\mathbf{I}, S),$$

and $\mathbf{I}$ is indecomposable. This argument scales up to the free model for intuitionistic linear logic with coproducts.

Conservativity. Glueing along the composite of the hom-functor with another allows neat ways of proving that one category is a conservative extension of another.

3 Double Glueing

The problem with simple glueing is that even if we start with a $*$-autonomous category $\mathbf{C}$ there's in general no way of making $\mathbf{GC}$ $*$-autonomous. How would the duality work with the extra structure? We would somehow have to turn a subset of $\mathbf{C}(\mathbf{I}, A)$ into one of $\mathbf{C}(\mathbf{I}, A^{\perp})$, and there's no general way of doing that.

The answer is that we have to double the extra structure. Then we can use the existing self-duality on $\mathbf{C}$, and just swap the two parts of the extra structure. Here we only describe double glueing along hom-functors, but the construction works in general. See [HS03] for details.

Take $\mathbf{C}$ to be a category with an involution[0] $(-)^{\perp}$, so $(-)^{\perp\perp} = \mathsf{id}_{\mathbf{C}}$.

Definition 5 *The double glued category* $\mathbf{G}^{\mathsf{d}}\mathbf{C}$ *has*

- *objects:* (R, U, X), *where*
  - R *is an object of* $\mathbf{C}$ *and*
  - $U \subseteq \mathbf{C}(\mathbf{I}, R)$ *and*
  - $X \subseteq \mathbf{C}(R, \perp)$.
- *morphisms: A morphism from* $A = (R, U, X)$ *to* $B = (S, V, Y)$ *is an arrow* $f \colon R \longrightarrow S$ *in* $\mathbf{C}$ *such that*
  - *for all* $u \in U$ *we have* $fu \in V$ *and*

[0] In general, take a self-duality whose square is equivalent to the identity functor, but here we prefer identities over isomorphisms for a clearer exhibition. There's a coherence result [CHS06] that tells us this is not a problem.

– *for all* $y \in Y$ *we have* $yf \in X$.

Now morphisms are of the form:

$$
\begin{array}{c} R \\ \Big\downarrow f \\ S \end{array}
\quad \text{such that} \quad
\begin{array}{c} \mathbf{I} \xrightarrow{u} R \in U \\ \text{implies} \\ \mathbf{I} \xrightarrow[u]{} R \xrightarrow[f]{} S \in V \end{array}
\quad \text{and} \quad
\begin{array}{c} R \xrightarrow{f} S \xrightarrow{y} \perp \in X \\ \text{implies} \\ S \xrightarrow[y]{} \perp \in Y \end{array}
$$

So how can we now define an involution on $\mathbf{G^dC}$? The idea is that the duality acts on R as in $\mathbf{C}$, and that we *swap the two bits of extra structure* U *and* X. But why should that work?

Well, we have that U is a subset of $\mathbf{C}(\mathbf{I}, R)$, and we want to view this as a subset of

$$\mathbf{C}(R^\perp, \perp) = \mathbf{C}(R^\perp, \mathbf{I}^\perp) \cong \mathbf{C}(\mathbf{I}, R)$$

which is no problem. But by the same argument, we can view X, which is a subset of $\mathbf{C}(R, \perp)$ as a subset of $\mathbf{C}(\mathbf{I}, R)$.

We do abuse notation by suppressing the isomorphisms that appear above, and simply write

$$(R, U, X)^\perp = (R^\perp, X, U).$$

We make similar identifications without further comment below.

Note that there is a forgetful functor from $\mathbf{G^dC}$ to $\mathbf{GC}$ which forgets the second part of the extra structure, and that functor preserves all the categorical structure. There's another forgetful functor from $\mathbf{G^dC}$ to the glued category that is defined in Exercise 6.

We employ the following notation for a generalized notion of composition. If J is an arbitrary object in $\mathbf{C}$ then for $h\colon R \otimes S \longrightarrow J$ and $v\colon \mathbf{I} \longrightarrow S$, we define $\langle v|h\rangle_S\colon R \longrightarrow J$ to be

$$R \cong R \otimes \mathbf{I} \xrightarrow{\mathsf{id}_R \otimes v} R \otimes S \xrightarrow{h} J;$$

we can think of this as cutting on S. Provided with some $u\colon \mathbf{I} \longrightarrow R$ we can similarly define $\langle u|h\rangle_R\colon S \longrightarrow J$, this time[1] cutting on R.

Exercise 7 *In order to prove the following proposition, some properties for this generalized composition have to be established. It might be useful to keep track of those separately so that they can be reused as appropriate.*

Proposition 3.1 *If* $\mathbf{C}$ *is* $*$*-autonomous then so is* $\mathbf{G^dC}$*, and the forgetful functor to* $\mathbf{C}$ *preserves the* $*$*-autonomous structure, which is as follows.*

- *The involution* $(R, U, X)^\perp = (R^\perp, X, U)$.
- *The tensor unit* $\mathbf{I} = (\mathbf{I}, \{\mathsf{id}_\mathbf{I}\}, \mathbf{C}(\mathbf{I}, \perp))$.

[1] Clearly this notation is not good enough to say which copy of R we cut on if h happens to be a morphism $R \otimes R \longrightarrow J$, but this will not cause us any problems.

- *the tensor product of $A = (R, U, X)$ and $B = (S, V, Y)$ is*

$$
\begin{aligned}
A \otimes B &= (R \otimes S, U \otimes V, \mathbf{G}^{\mathsf{d}}\mathbf{C}(A, B^{\perp})) \quad \textit{where} \\
U \otimes V &= \{\mathbf{I} \cong \mathbf{I} \otimes \mathbf{I} \xrightarrow{u \otimes v} R \otimes S \mid u \in U, v \in V\} \quad \textit{and} \\
\mathbf{G}^{\mathsf{d}}\mathbf{C}(A, B^{\perp}) &= \{R \otimes S \xrightarrow{z} \perp \mid \forall \mathbf{I} \xrightarrow{u} R \in U.\ \langle u|z\rangle_R \colon S \longrightarrow \perp \in Y \\
&\qquad \forall \mathbf{I} \xrightarrow{v} S \in V.\ \langle v|z\rangle_S \colon R \longrightarrow \perp \in X\}.
\end{aligned}
$$

Up to natural identification the last component is the set of maps in $\mathbf{G}^{\mathsf{d}}\mathbf{C}$ from (R, U, X) to $(S, V, Y)^{\perp}$, hence the notation.

Exercise 8 *Convince yourself that the above proposition is true. Calculate the linear function space of two objects.*

Exercise 9 *Describe the category arising from applying double glueing to* **Rel**. *Do this by extending your answer to Exercise 4. In your version, is the self-duality anything more complicated than swapping the two pieces of extra structure on the objects? What does generalized composition amount to in this setting?*

Proposition 3.2 *If* $\mathbf{C}$ *has finite products then so has* $\mathbf{G}^{\mathsf{d}}\mathbf{C}$*, and the functor* $\mathbf{G}^{\mathsf{d}}\mathbf{C} \longrightarrow \mathbf{C}$ *preserves them. They are given by*

$$
\begin{aligned}
A \times B &= (R \times S, U \times V, X \oplus Y) \quad \textit{where} \\
U \times V &= \{\langle u, v\rangle \colon \mathbf{I} \longrightarrow R \times S \mid u \in U, v \in V\} \quad \textit{and} \\
X \oplus Y &= \{R \times S \xrightarrow{\pi_1} R \xrightarrow{x} \perp \mid x \in X\} \cup \{R \times S \xrightarrow{\pi_2} S \xrightarrow{y} \perp \mid y \in Y\}.
\end{aligned}
$$

The terminal object is $(\mathbf{1}, \mathbf{C}(\mathbf{I}, \mathbf{1}), \emptyset)$.

If $\mathbf{C}$ *has finite coproducts then so has* $\mathbf{G}^{\mathsf{d}}\mathbf{C}$*, and* $\mathbf{G}^{\mathsf{d}}\mathbf{C} \longrightarrow \mathbf{C}$ *preserves them. They are given by*

$$
\begin{aligned}
A + B &= (R + S, U \oplus V, X + Y) \quad \textit{where} \\
X + Y &= \{[x, y] \colon R + S \longrightarrow \perp \mid x \in X, y \in Y\} \quad \textit{and} \\
U \oplus V &= \{\mathbf{I} \xrightarrow{u} R \xrightarrow{\mathsf{inl}} R + S \mid u \in U\} \cup \{\mathbf{I} \xrightarrow{v} S \xrightarrow{\mathsf{inr}} R + S \mid v \in V\}
\end{aligned}
$$

The unit for the coproduct is $(\mathbf{0}, \emptyset, \mathbf{C}(\mathbf{0}, \perp))$.

Exercise 10 *Calculate sums and products in the category* $\mathbf{G}^{\mathsf{d}}\mathbf{Rel}$*, extending your work in Exercise 9.*

Examples and Applications. Double glueing allows us to define a notion of logical relation for $*$-autonomous categories, for example Loader's category of linear logical predicates [Loa94a, Loa94b] is a double glued category.

Double glueing allows the definition of fully complete models of fragments of linear logic ([Tan97] gives a number of examples for the multiplicative fragment, and [BHS] extends this technique to the additives). There is a formulation of 'process realizability' [Abr] which is based on double glueing, and there are plenty of other examples.

More sophistication. The double glueing construction allows us to cut down on morphisms by insisting that they preserve some extra structure that is put on objects. For some situations, the resulting category isn't quite as we'd like it to be.

For example, one might try to describe Girard's coherence spaces [Gir87, Gir95] as a double glued category. A *coherence space* is given by a set R and a subset U of the finite powerset of R, $\mathcal{P}R$, the 'coherent sets', or 'cliques'. The collection U has to satisfy the properties that

- every singleton is in U ('every singleton is coherent') and
- if $u' \subseteq u \in U$ then $u' \in U$ ('every subset of a coherent set is coherent') and
- if $u \subseteq R$ and for all $r, r' \in u$ we have $\{r, r'\} \in U$ then $u \in U$.

We can think of coherence spaces (R, U) as objects in the glued category[2] **GRel**: A morphism $\mathbf{I} \longrightarrow R$ can be thought of as a subset of R, namely the set of those elements of R who are related to the single element of $\mathbf{I}$ by the morphism in question. Hence a subset U of $\mathcal{P}R$ can be viewed as a subset of $\mathbf{Rel}(\mathbf{I}, R)$. However, to describe the correct morphisms for coherence spaces we have to think of this as a certain double glued category.

Given a set of coherent sets U we define another subset of $\mathcal{P}R$ by setting

$$U^\circ = {}^{[3]}\{x \subseteq_{\mathsf{f}} R \mid \forall u \in U, |u \cap x| \leq 1\}.$$

The sets in U° are the anti-cliques—they consist of elements which are pairwise incoherent. Recalling that $\perp$ also is a singleton set we note that we can view U° as a subset of $\mathbf{Rel}(R, \perp)$.

Exercise 11 *Prove that $U \subseteq \mathcal{P}R$ is the set of coherent subsets of some set R if and only if $U = U^{\circ\circ}$.*

We find it convenient to make this part of the structure of a coherence space, in other words, we use (R, U, U°), which is an object of $\mathbf{G^{d}Rel}$. At first sight, U° adds nothing to the available information, but that isn't quite true because we need it in order to specify the morphisms we wish to consider for coherence spaces.

A morphism $(R, U, U^\circ) \longrightarrow (V, V, V^\circ)$ is given by a relation

$$f\colon R \longrightarrow\!\!\!\!\!\!\!+\;\; V$$

such that

$$\begin{array}{ccc}
u \in U & \qquad\qquad & f[y] = \{r \in R \mid \exists s \in y.\, (r, s) \in f\} \in U^\circ \\
\big\Downarrow\ \text{implies} & & \big\Downarrow\ \text{implies} \\
[u]f = \{s \in V \mid \exists r \in u.\, (r, s) \in f\} \in V & & y \in V^\circ.
\end{array}$$

In other words, morphisms are precisely the morphisms of $\mathbf{G^{d}Rel}$. Hence we may think of coherence spaces as a certain subcategory[4] of $\mathbf{G^{d}Rel}$. However, the categorical structure

[2] This should be clear already to those who have done Exercise 4.

[3] Most people use $U^\perp$ for our U°, but the symbol $\perp$ is quite overloaded enough already!

[4] Or, for those who know the usual definition of coherence spaces, there is a full and faithful functor that embeds these into $\mathbf{G^{d}Rel}$.

of the double glued category is not that for coherence spaces. Given (R, U, U°) and (V, V, V°) their tensor product in $\mathbf{G^dRel}$, $(R \otimes V, U \otimes V, Z)$, has a second component

$$U \otimes V \subseteq \mathcal{P}(R \otimes V)$$

which is not closed with respect to forming subsets, so this does not describe a coherence space. In other words, the subcategory of coherence spaces is not closed under tensor (although it is closed under negation), and similar issues arise with the additive structure.

There is a way of defining a subcategory of the double glued category that does have the correct categorical structure: We take the definition in $\mathbf{G^dRel}$ and then apply the closure operation $(-)^{\circ\circ}$ to the second component, and apply $(-)^\circ$ to that to obtain the desired third component.[5] That does give rise to a category that is equivalent to the usual category of coherence spaces.

In general one can use the notion of an **orthogonality** (in our example that role is taken by $(-)^{\circ\circ}$) to create subcategories of a double glued category (when glueing along hom-functors) using the same idea. Details are again given in [HS03].

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[5] The interested reader may want to verify that sets of the form U° are already closed under that operation.

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