CS3191 Section 1

Introduction to Games

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Overview of the Course

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- Games and evolution. Games used to model the evolution of traits.

About this course

Elements of teaching on this course:

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- a game tree (required for the formal definition of game);
- a strategy;
- a pay-off function;
- the normal form of a game.

What is a game?

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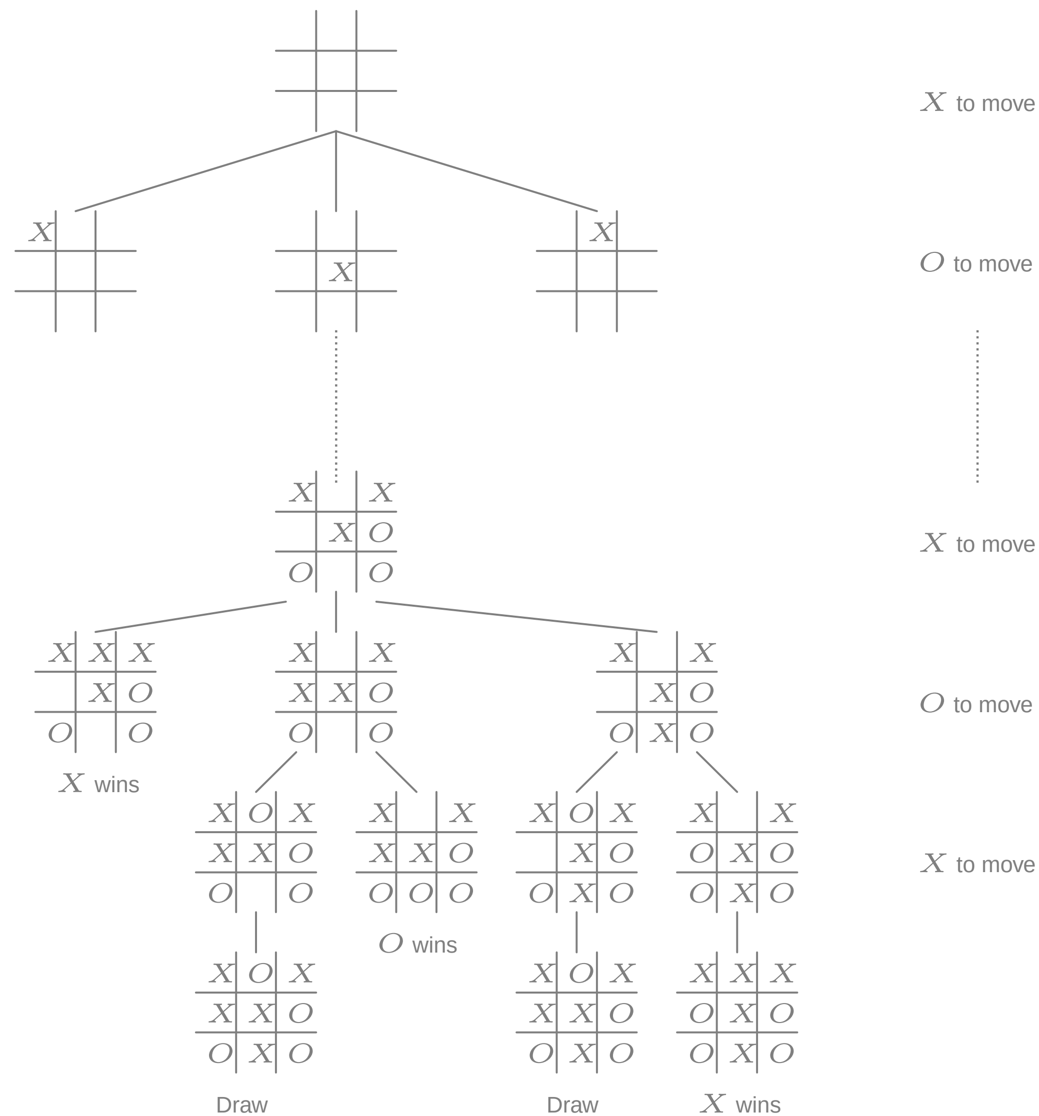
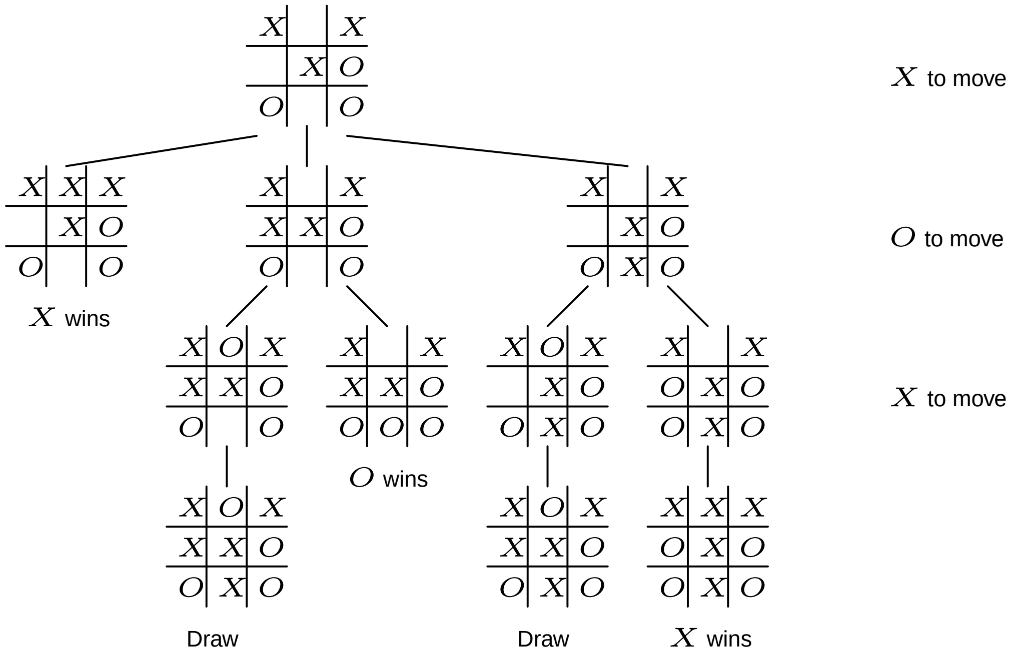
A game gives the rules for the interactions of the participating entities, the players.

Example



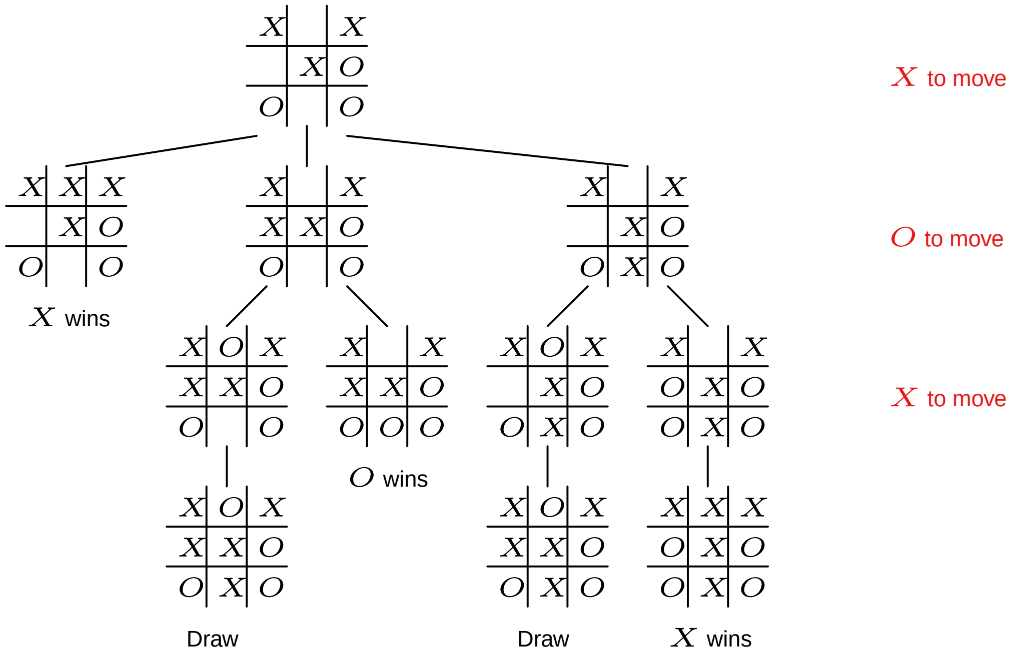
X to move
O to move
X to move
O to move
X to move
X wins
O wins
Draw
Draw
X wins

A game tree



The nodes in such a tree are the positions of the game.

A game tree



At each node it has to be clear whose **turn** it is.

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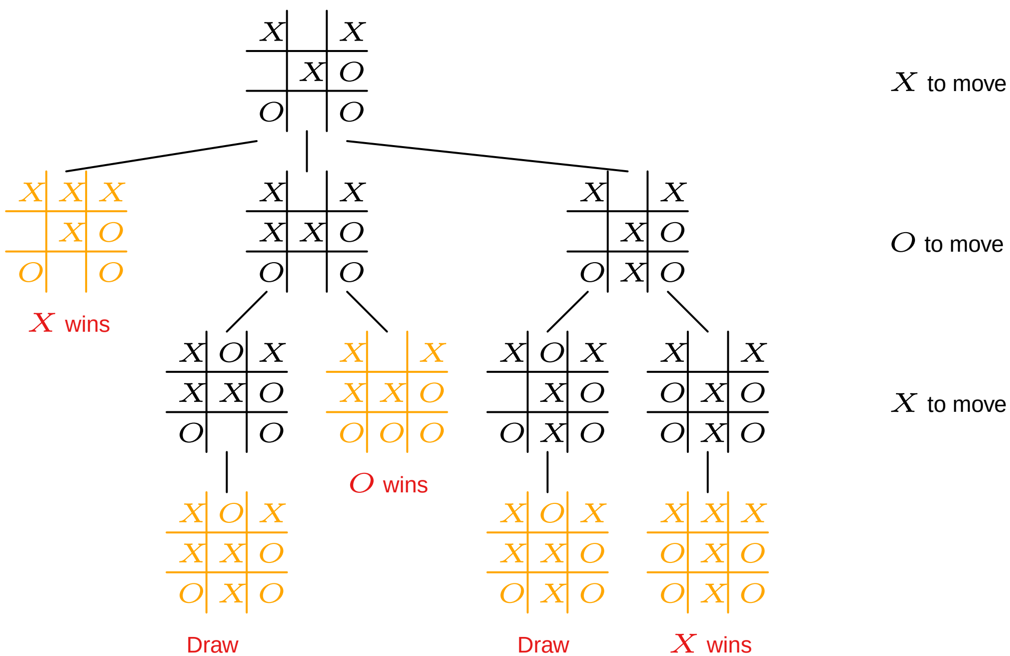

The bottom nodes in the tree, the **leaves**, are the **final positions** of the game. This is where we note the **result**.

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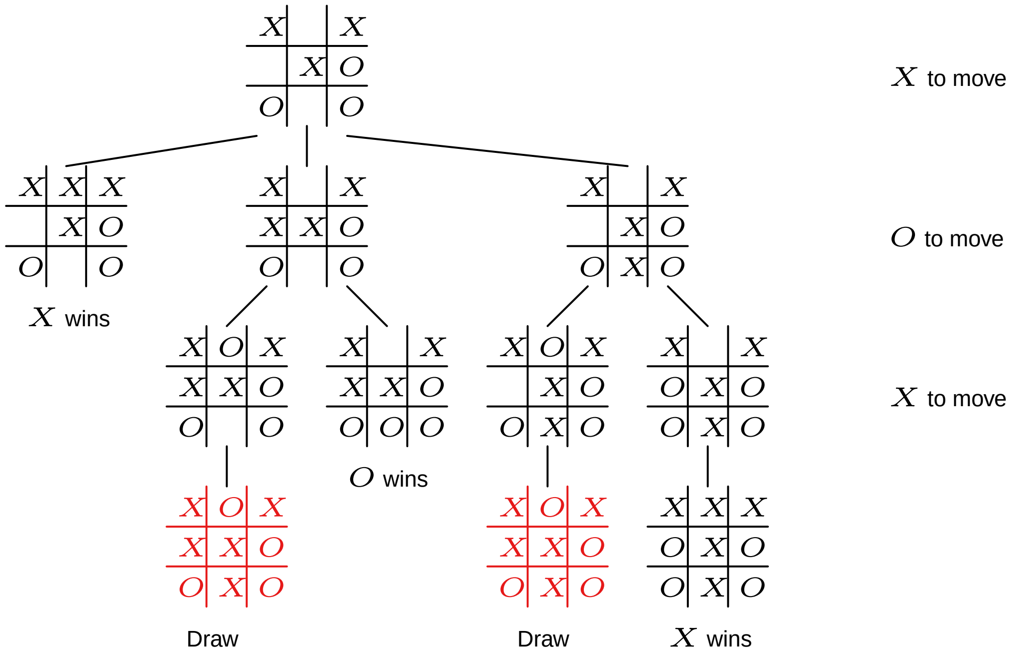

Note that different positions in the tree can correspond to the same constellation on the board.

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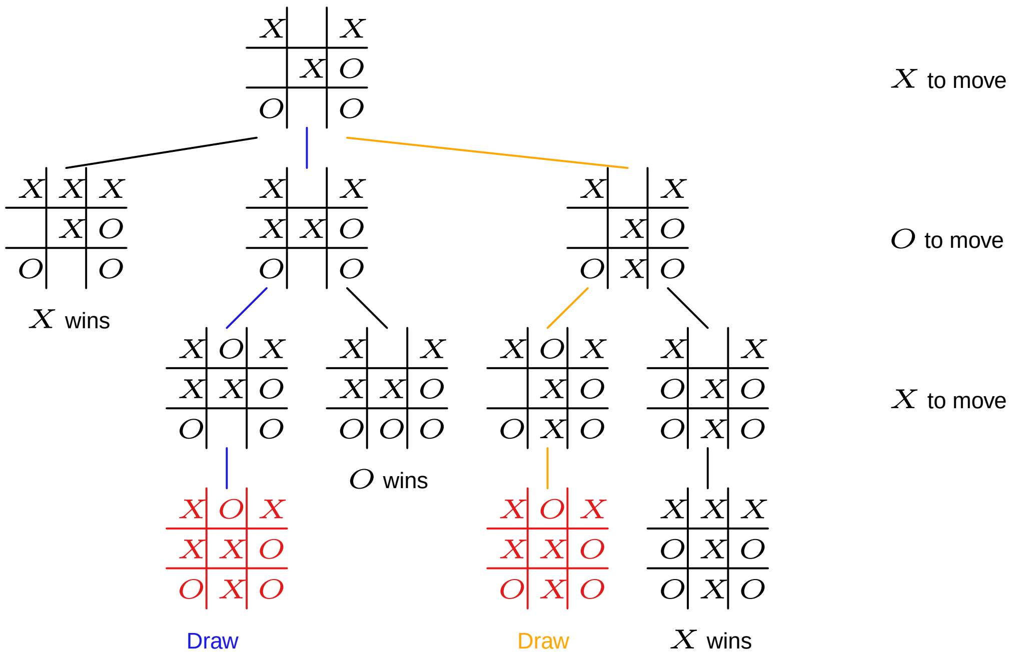

That is because in a tree, given a position we can reconstruct the path from the root leading to it (and thus the play of the game). Another path is shown. Taking trees instead of graphs makes our lives easier.

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(c) Consider the following simple game between two players: Player 1 has a coin which he hides under his hand, having first decided whether it should show head or tail. Player 2 guesses which of these has been chosen. If she guesses correctly, Player 1 pays her 1 quid, otherwise she has to pay the same amount to him. Could you draw a game tree for this game? If not why not?

More general games

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We will see how to deal with these issues later. For now we assume that all our games are of complete (or perfect) information, that is, each player knows precisely where in the game tree they are.

Games–definition

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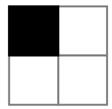

1 to move

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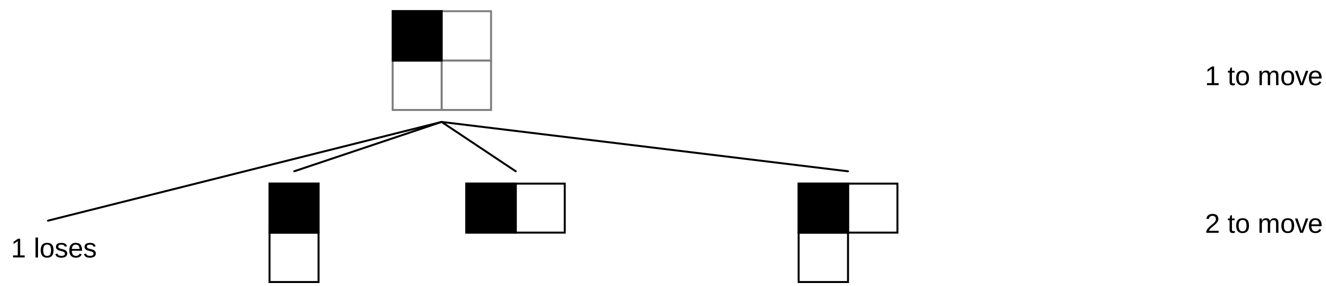

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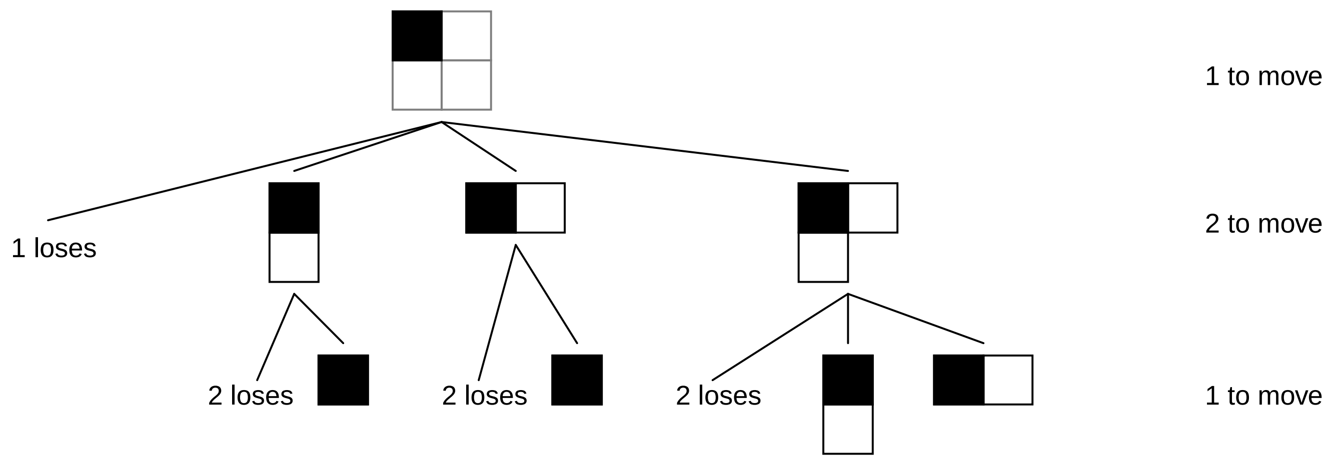

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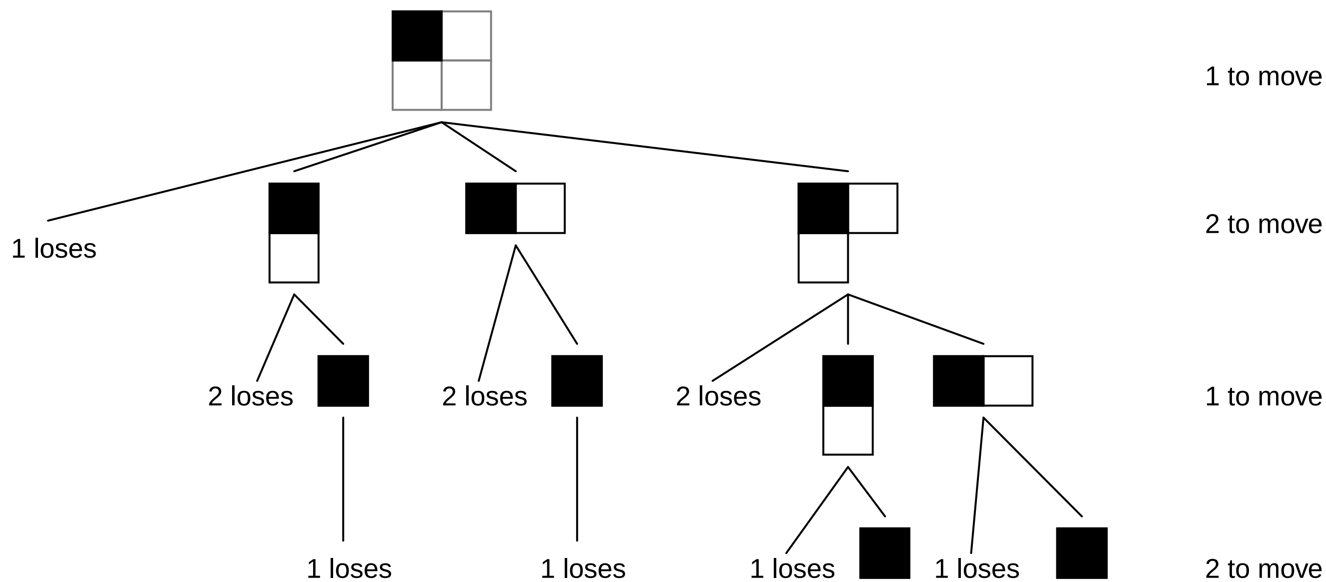

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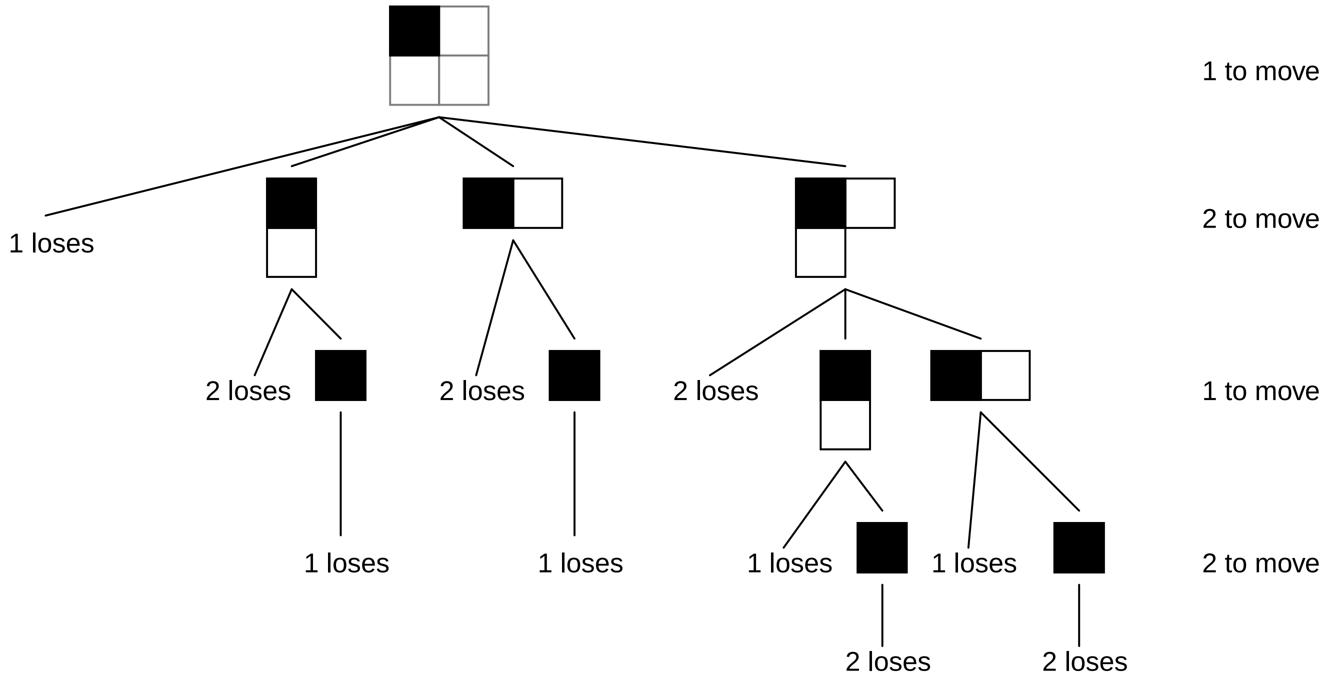

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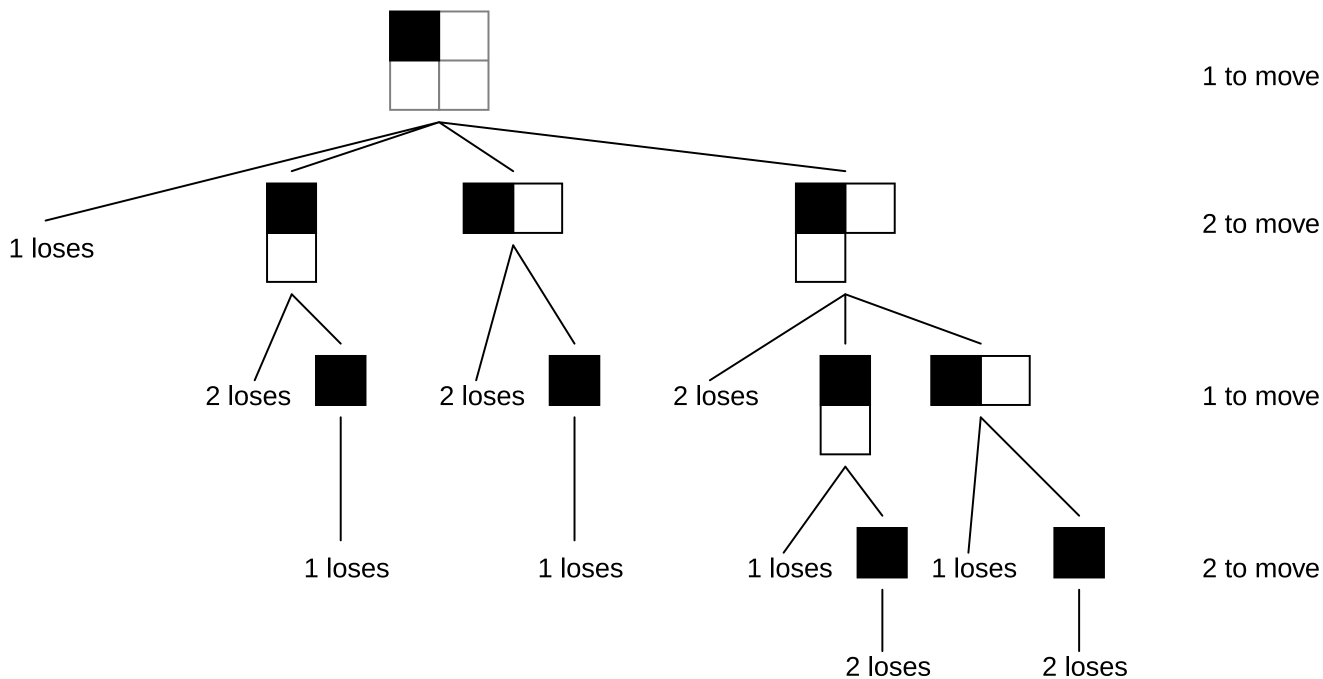

Question. Which player would you rather be, 1 or 2?

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What would make the games more exciting, and why aren't we looking at those issues?

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The answer to this question depends on whether or not we want to consider symmetry consideration. A crude upper bound for the number of plays ignores these. Player 1 has 9 possibilities for placing his first mark, Player 2 has the remaining 8 squares for her first move, then Player 1 can choose among any of the remaining 7 fields, leading to

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$$9 \times 8 \times \cdots \times 2 \times 1 = 9! = 362880$$

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- for each of the branches from such a node there is a label giving the probability of that move occurring.

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To make it simpler, we assume here that the defender has only one army to defend the country with.

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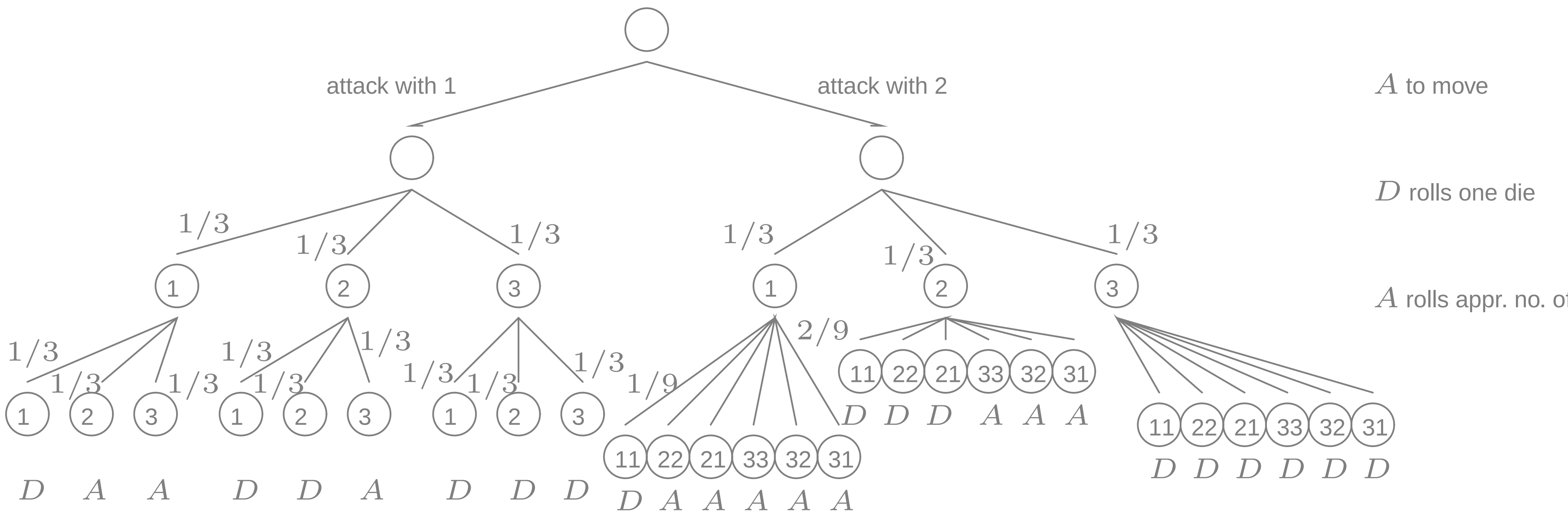

Probabilities for throwing two dice: $1/9$ for each branch where the two numbers agree, $2/9$ where they differ.

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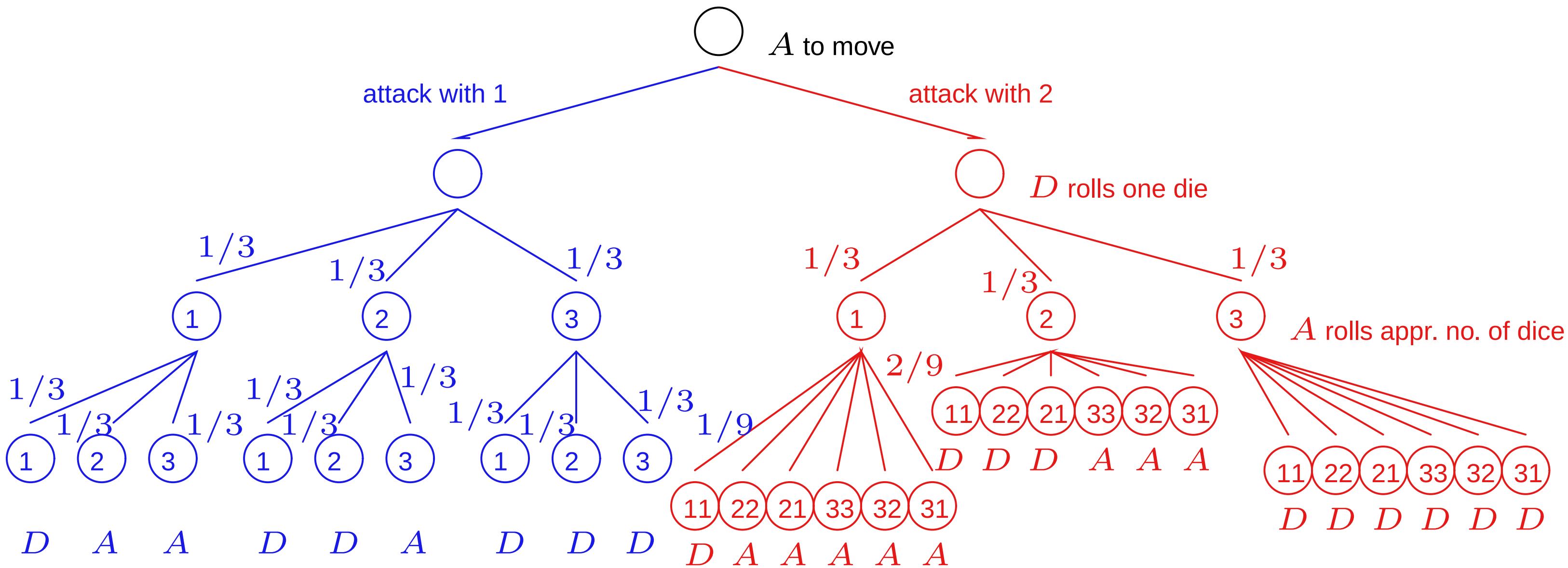

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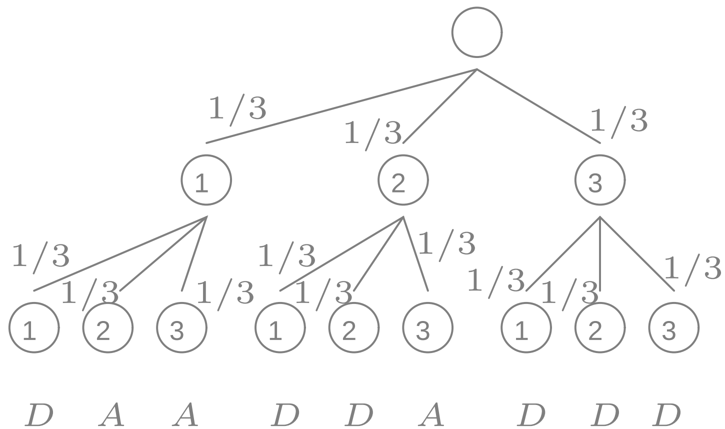

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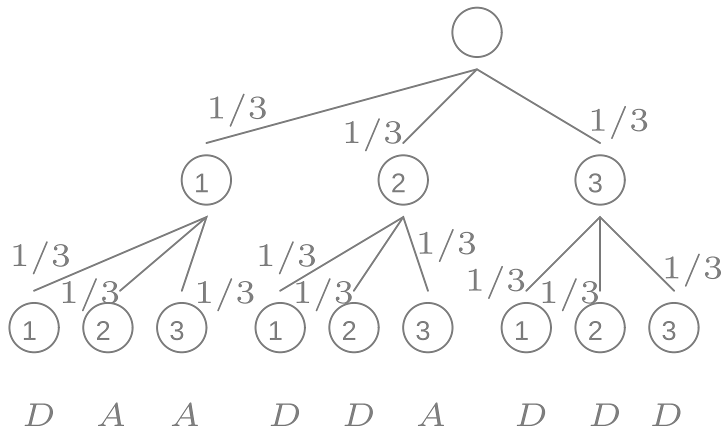

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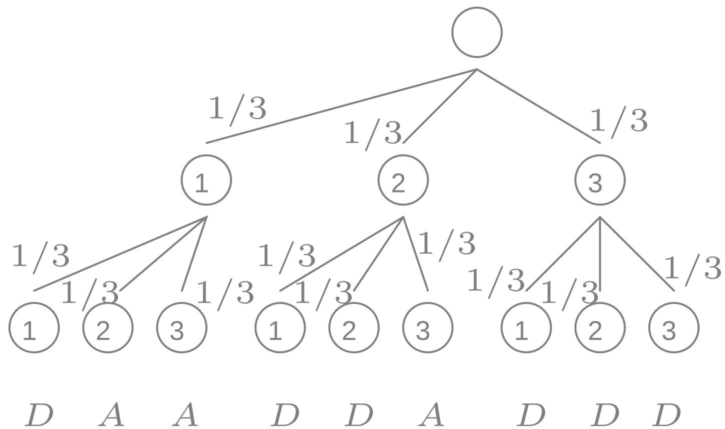

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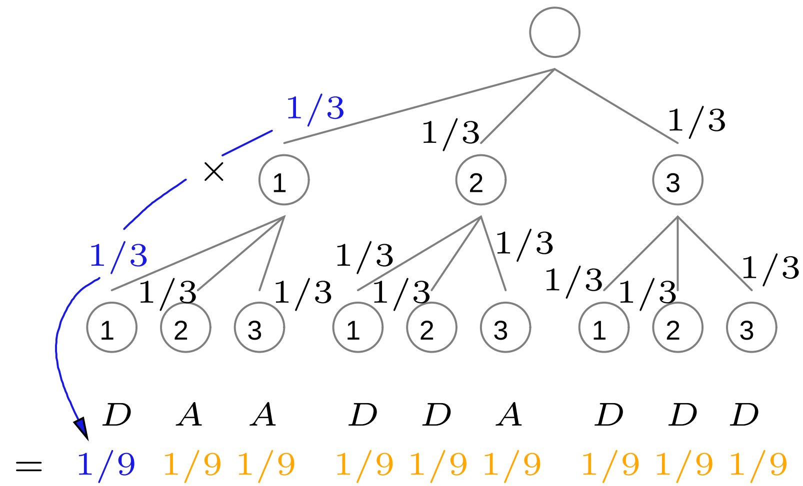

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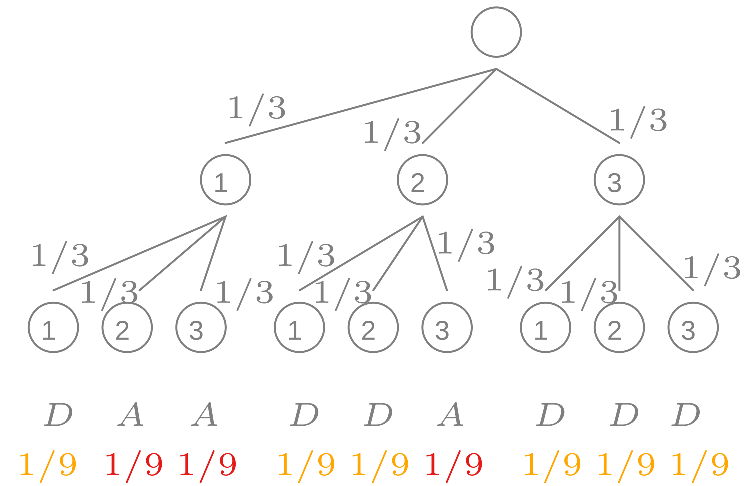

We want to calculate with which probability A will win. For that, we first have to calculate the probability for each outcome (that is, each leaf) of the tree. For that, we have to multiply the probabilities along the path. So every branch has a probability of $1/9$. In order to find the probability for A to win, we have to add the probabilities for all the paths where A wins.

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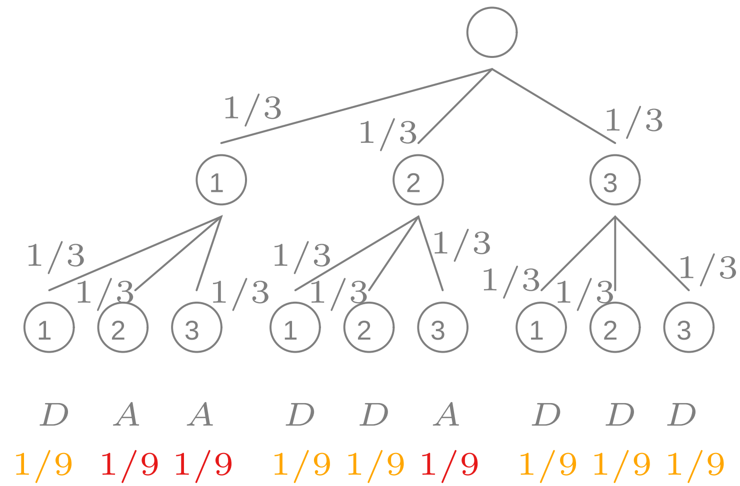
Attack with 1



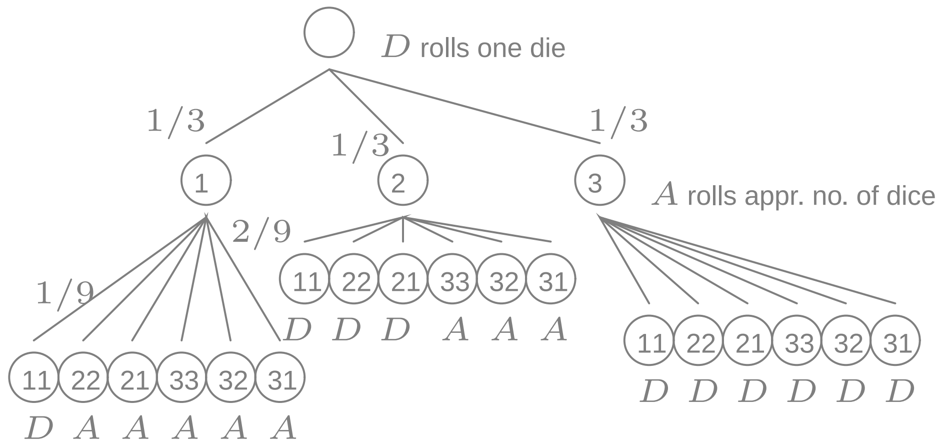
We want to calculate with which probability A will win. For that, we first have to calculate the probability for each outcome (that is, each leaf) of the tree. For that, we have to multiply the probabilities along the path. So every branch has a probability of $1/9$. In order to find the probability for A to win, we have to add the probabilities for all the paths where A wins. This gives

$$1/9 + 1/9 + 1/9 = 3/9 = 1/3,$$

so Attacker wins in $1/3$ of all cases.

Risk–Attack with 2

Attack with 2



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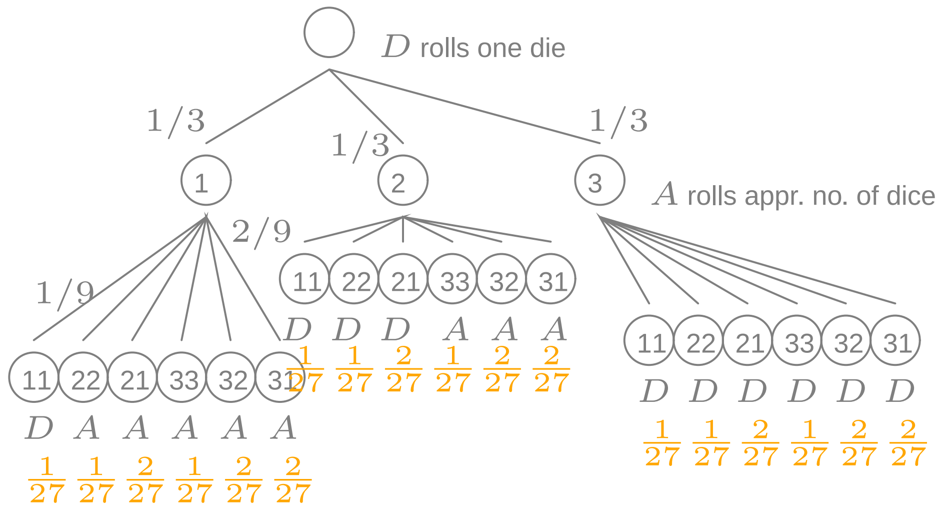

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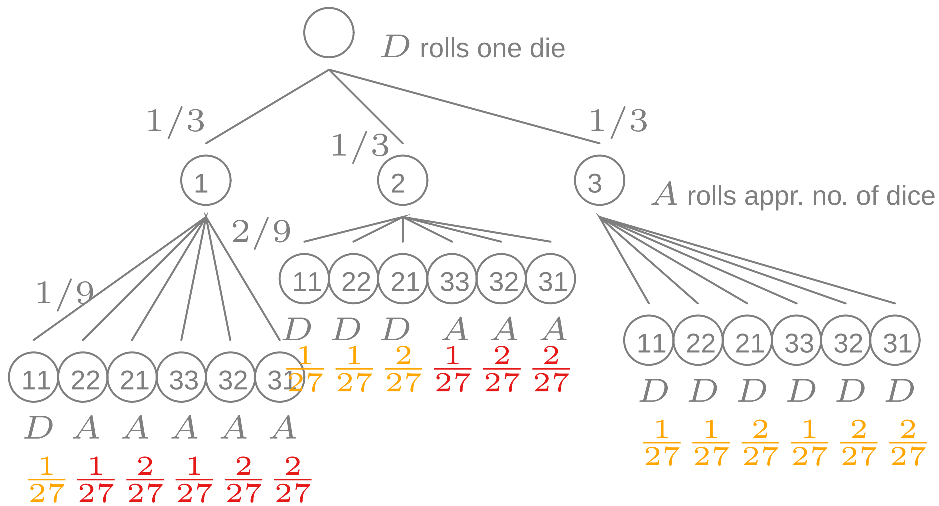

We again calculate the probabilities for the leaves by multiplying the probabilities along the corresponding paths. Again, we add up the probabilities for those occasions where A wins, and get

$$1/27 + 2/27 + 1/27 + 2/27 + 2/27 + 1/27 + 2/27 + 2/27 = 13/27 \approx 0.48.$$

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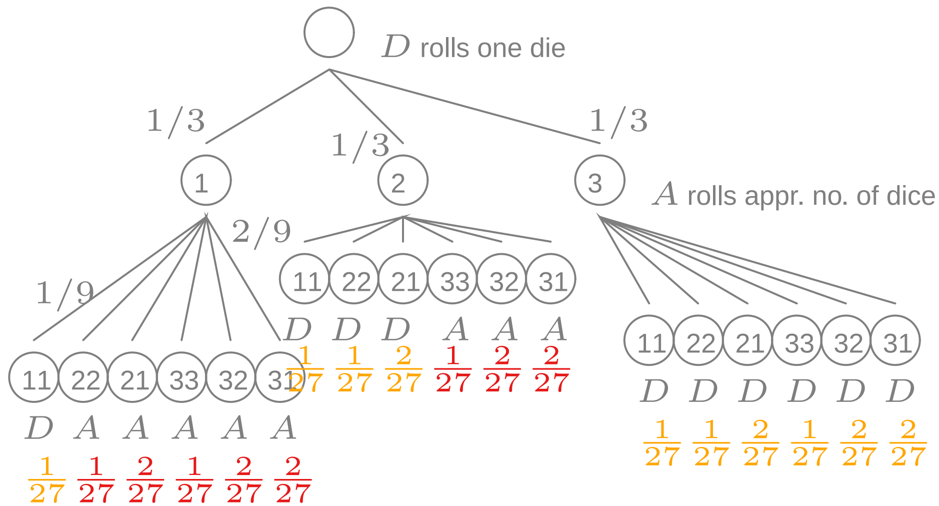

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Since this is bigger than $1/3$, it is better for Attacker to attack with 2 armies. (And, in fact in general for Risk the chance of winning an encounter increases with the number of committed armies.)

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Question. Why do we make that a requirement?

Paper-Stone-Scissors

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In other words, we can cover simultaneous moves by using imperfect information.

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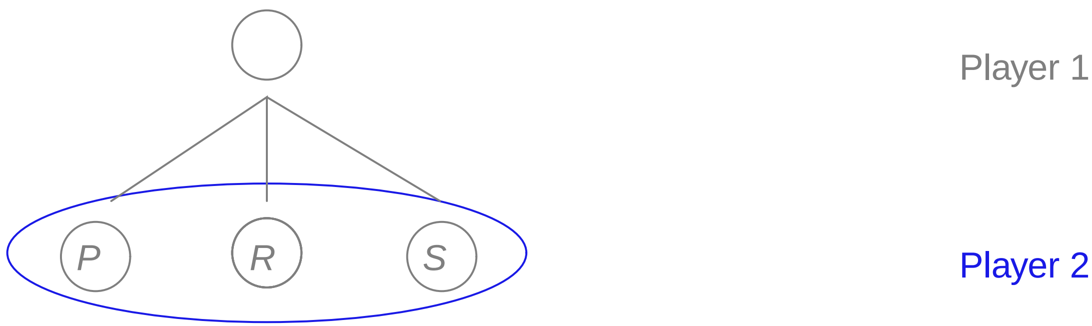

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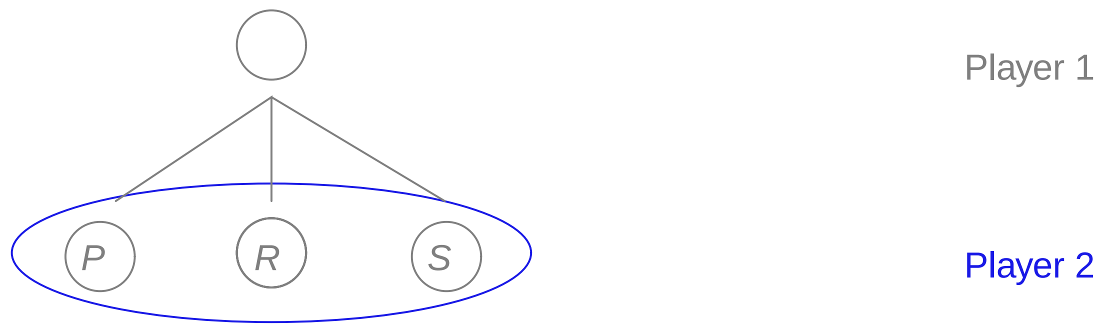

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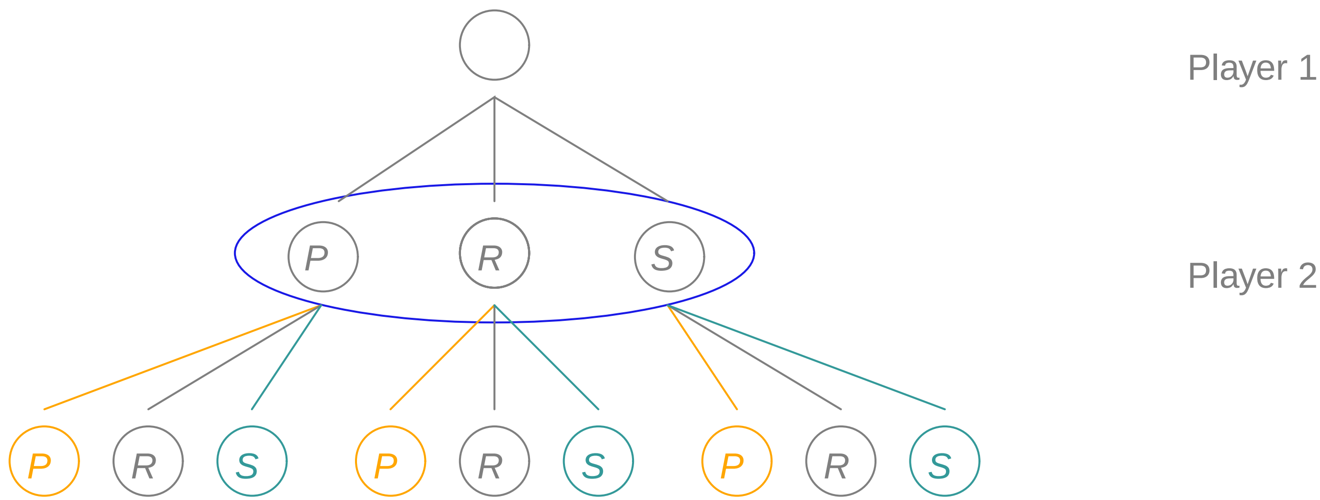

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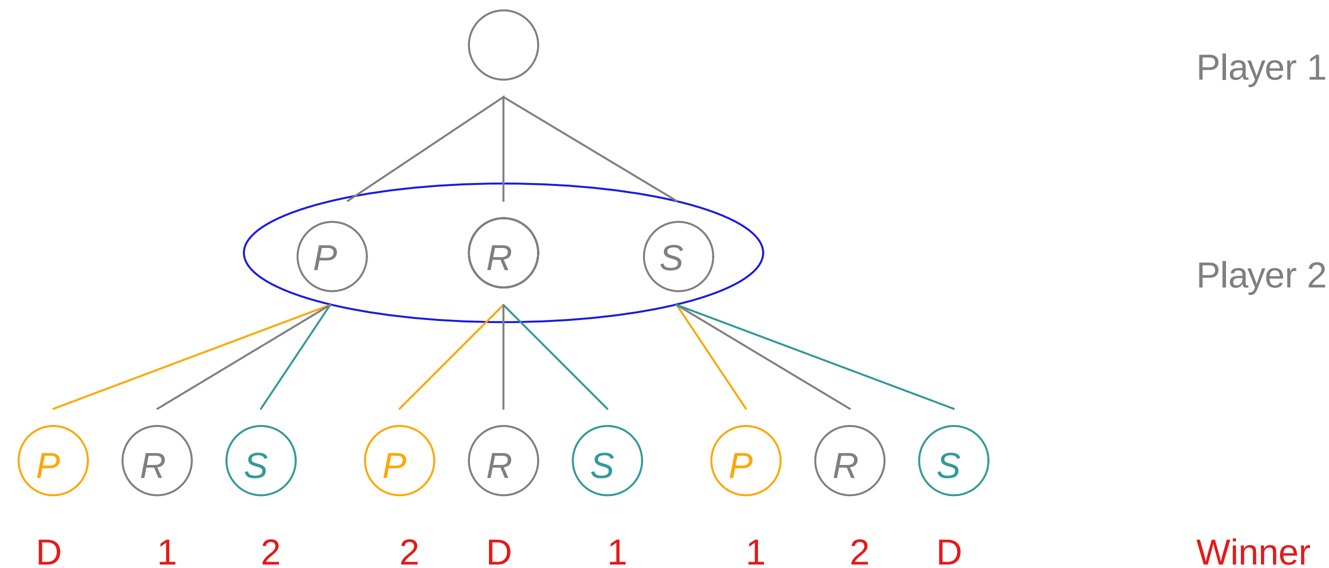

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Games: Addition to definition

Here is a reminder of our definition of game.

Definition 1 *A game is given by*

- *a finite set of players,*
- *a finite game tree,*
- *for each node of the tree, a player whose turn it is in that position and*
- *for each final node and each player a pay-off function. (We will ignore this part of the definition for the moment.)*

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- the moves from one of the nodes in the information set are indistinguishable from the moves from any other such node.

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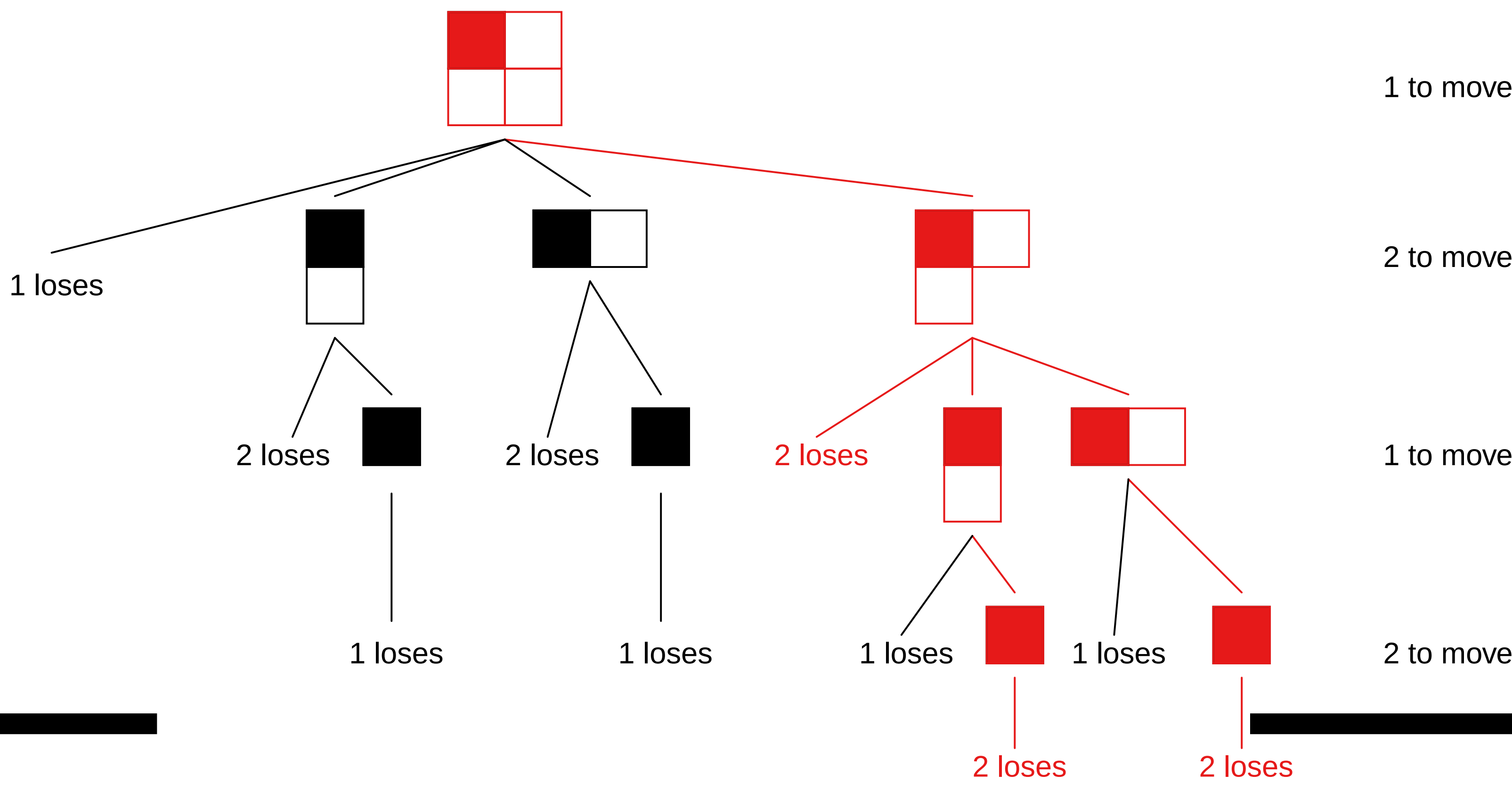

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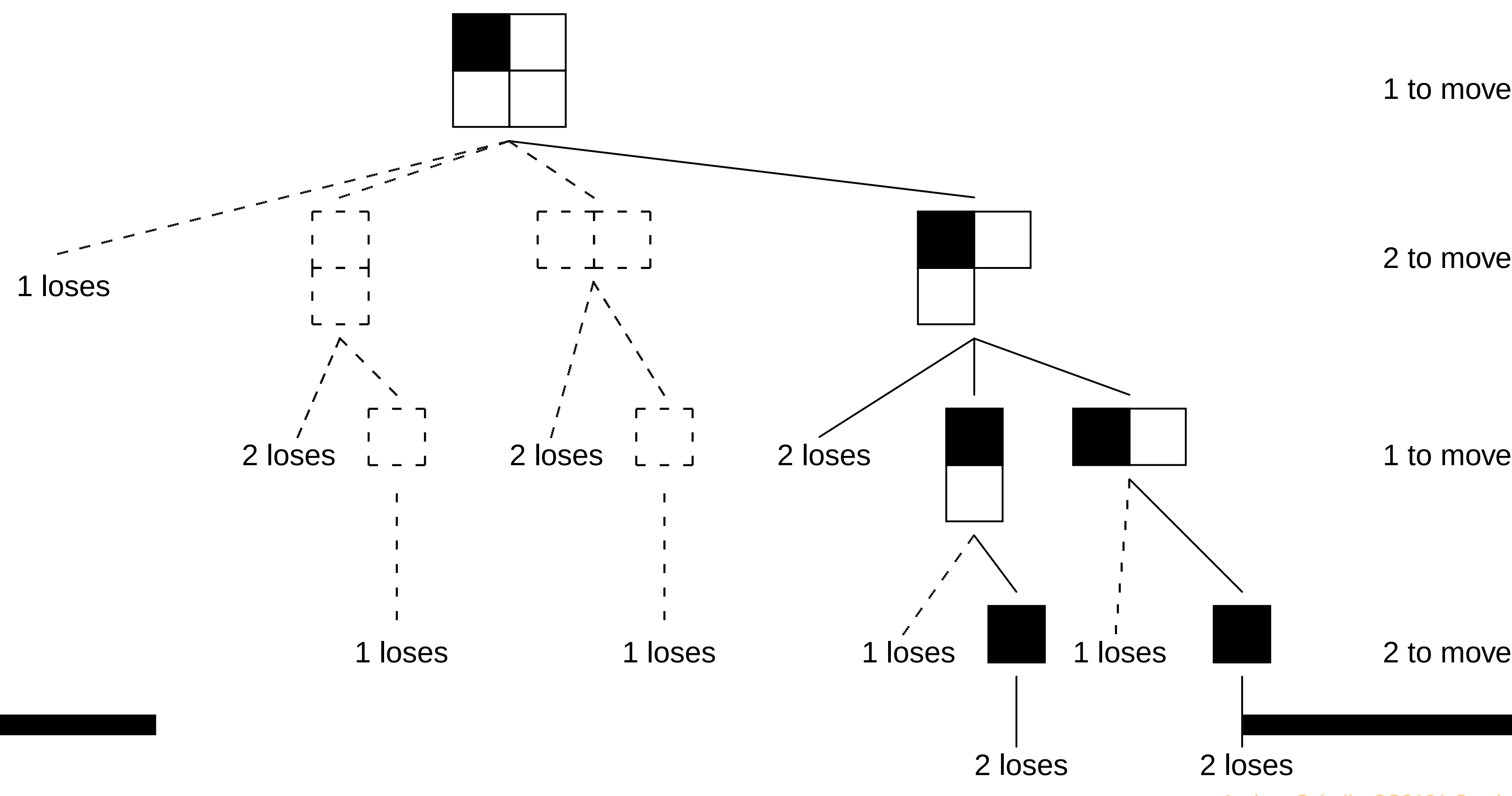

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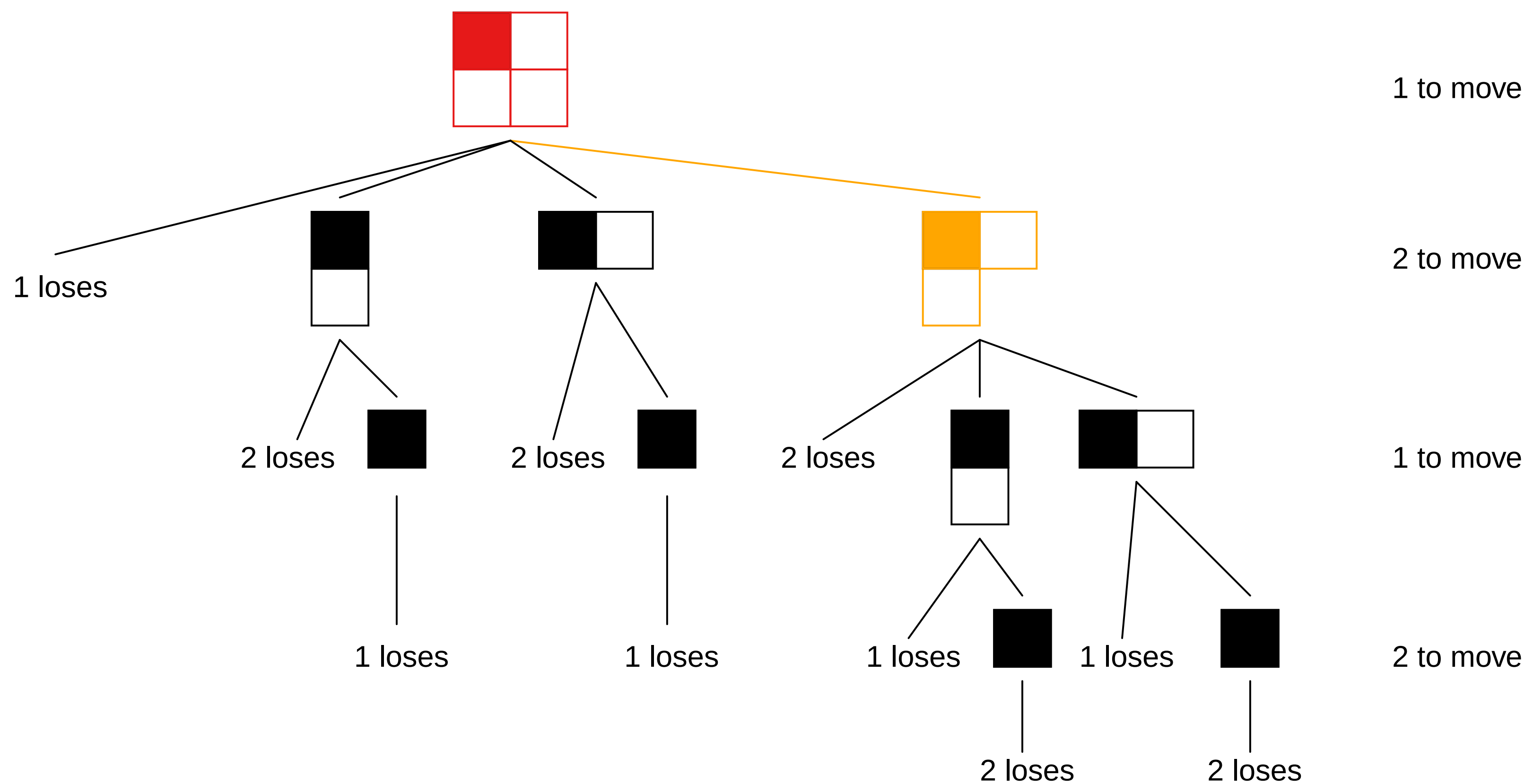

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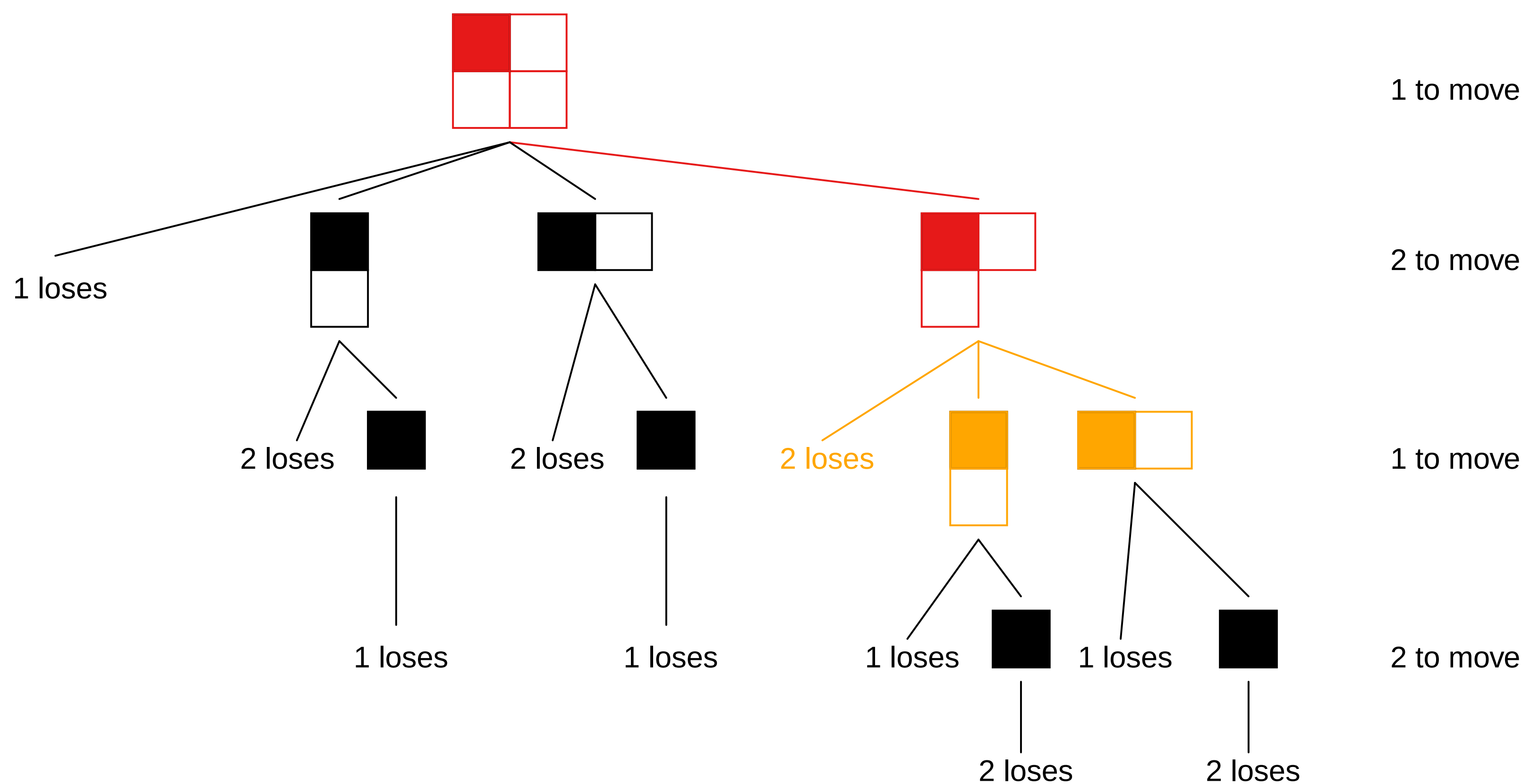

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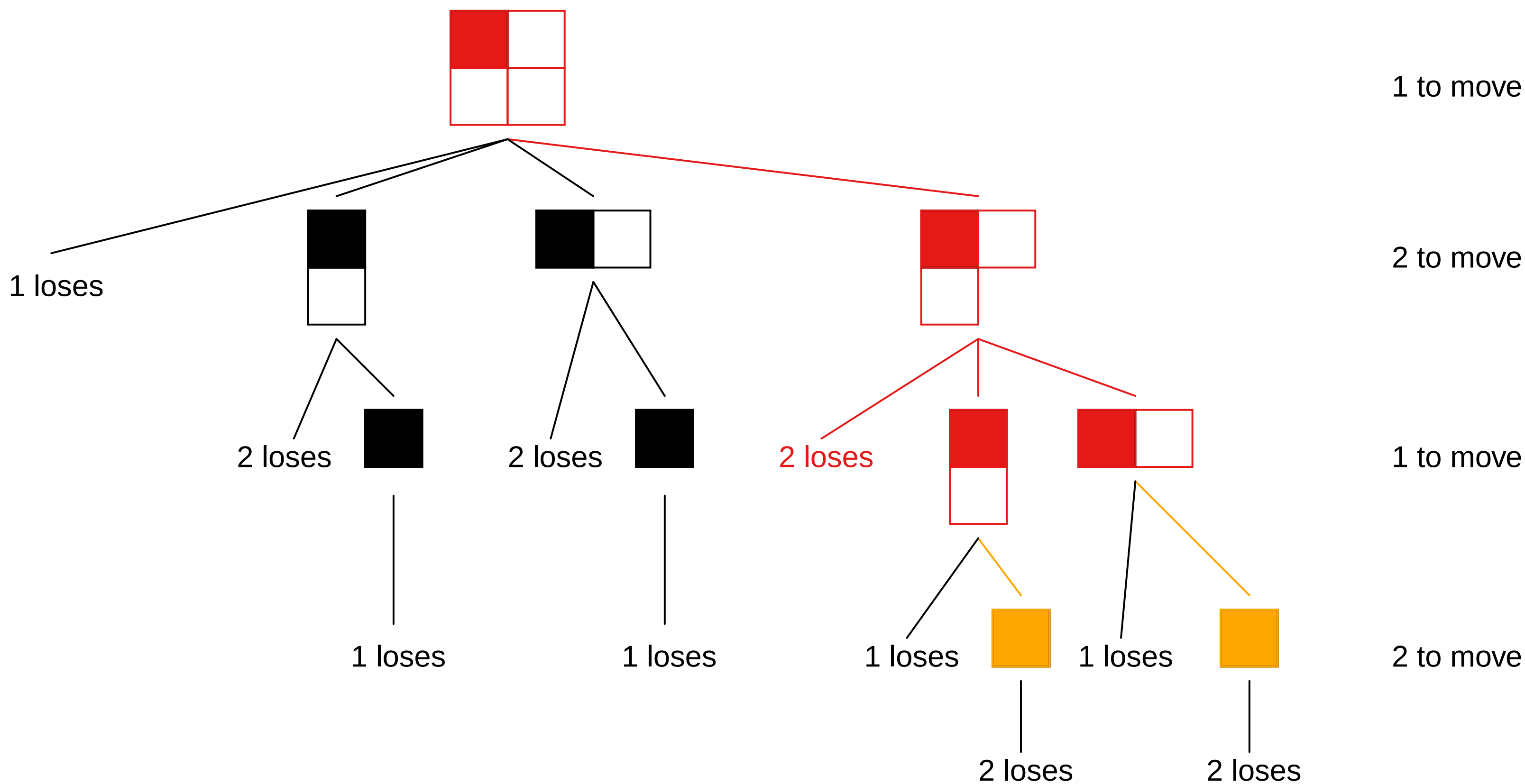

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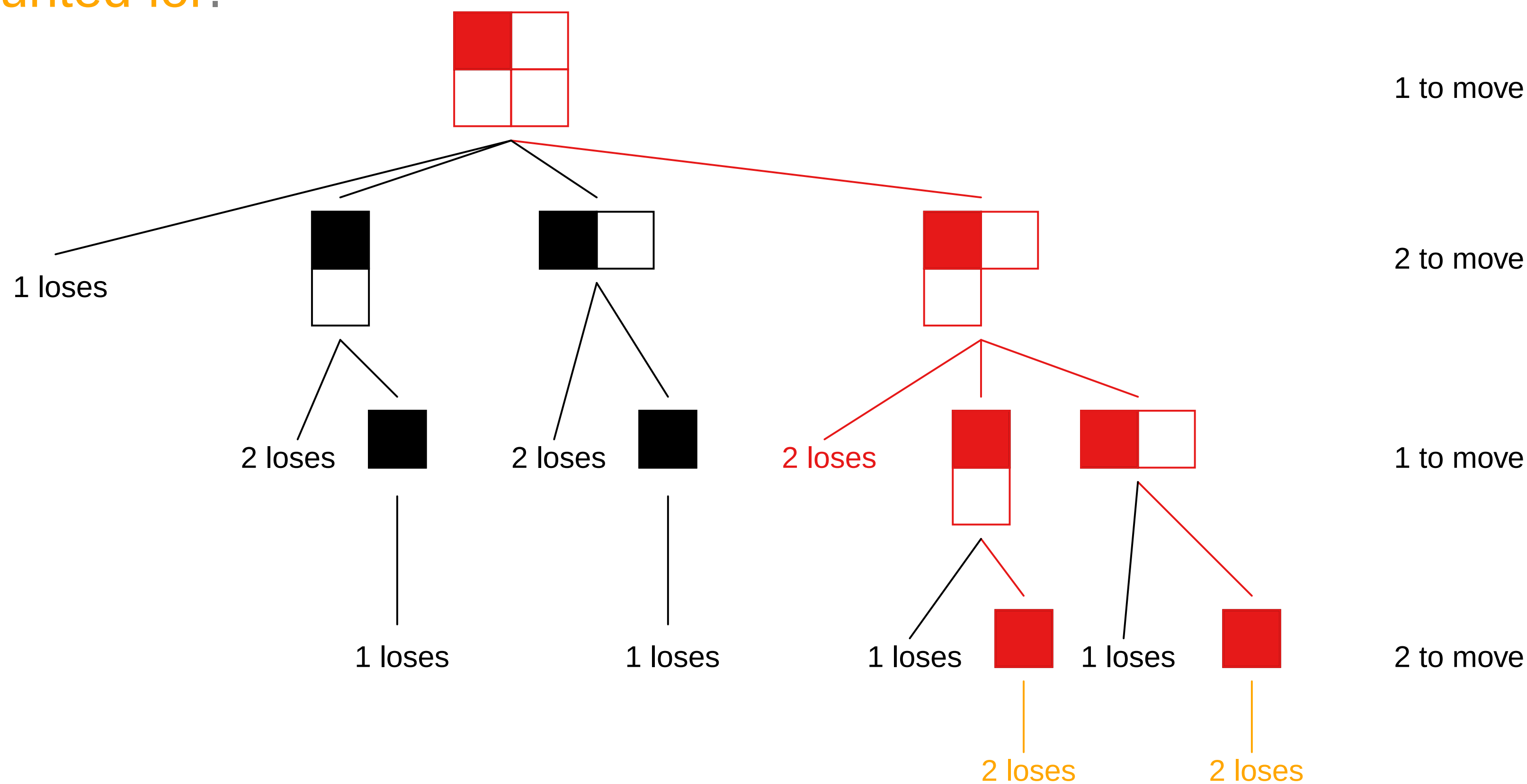

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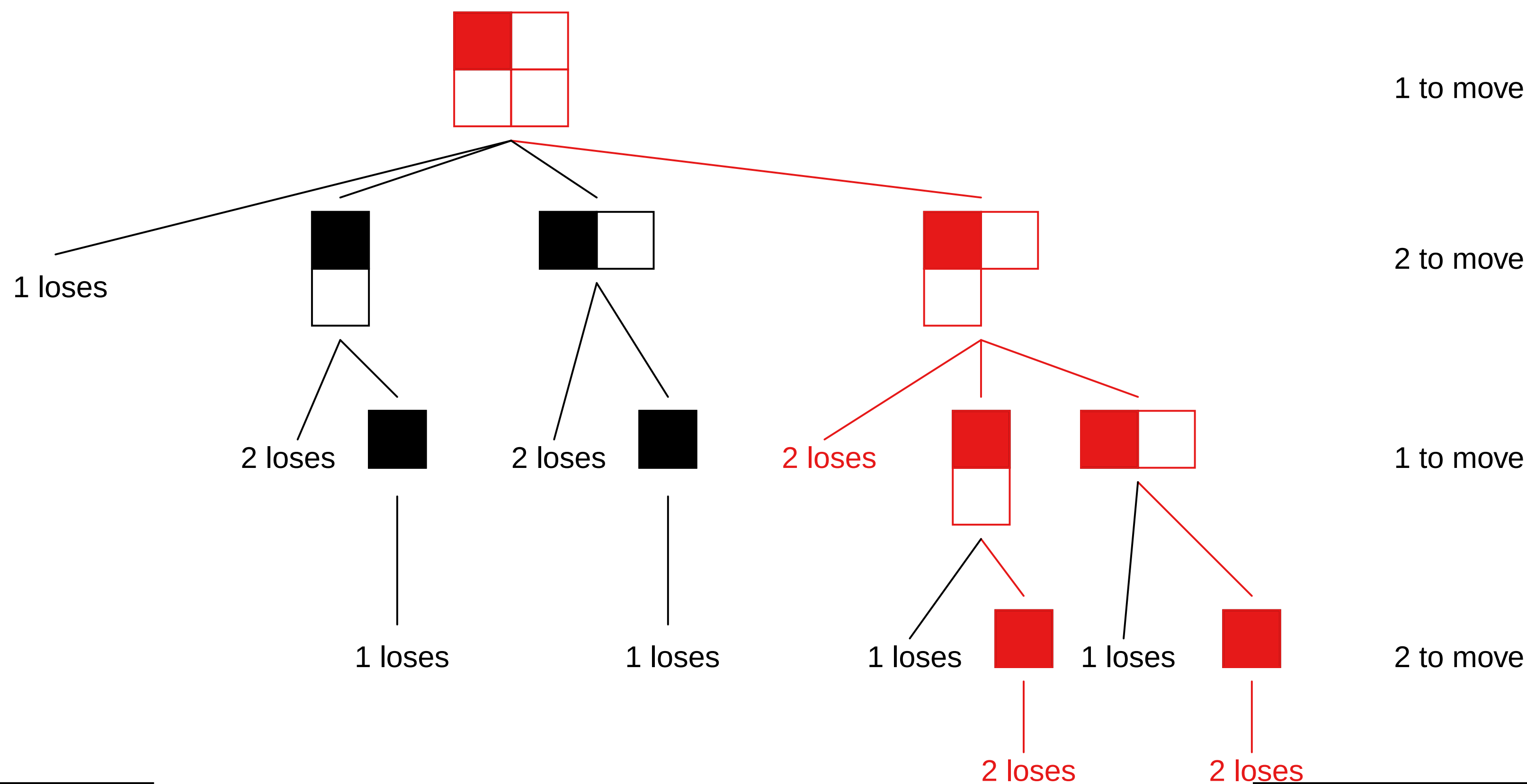

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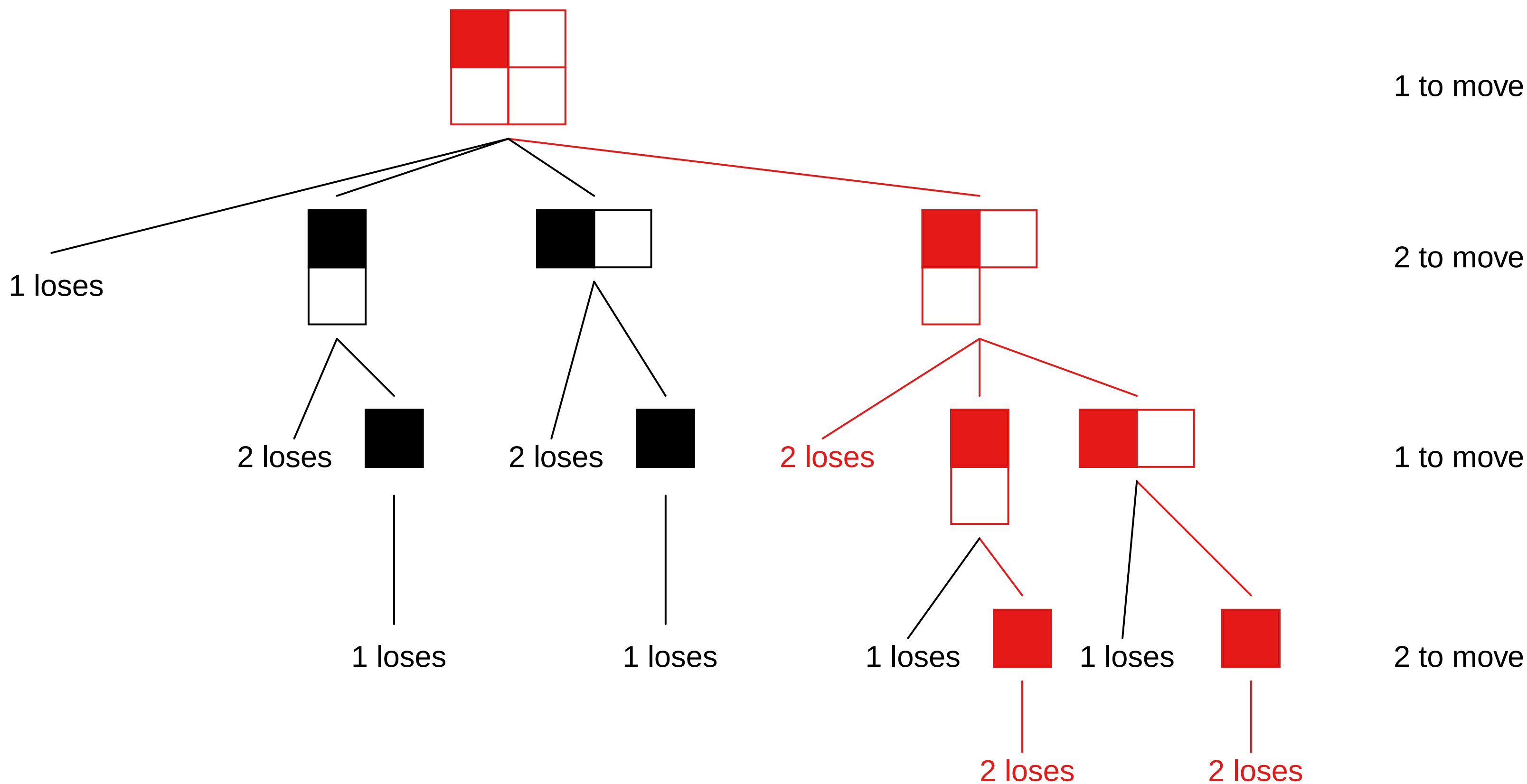
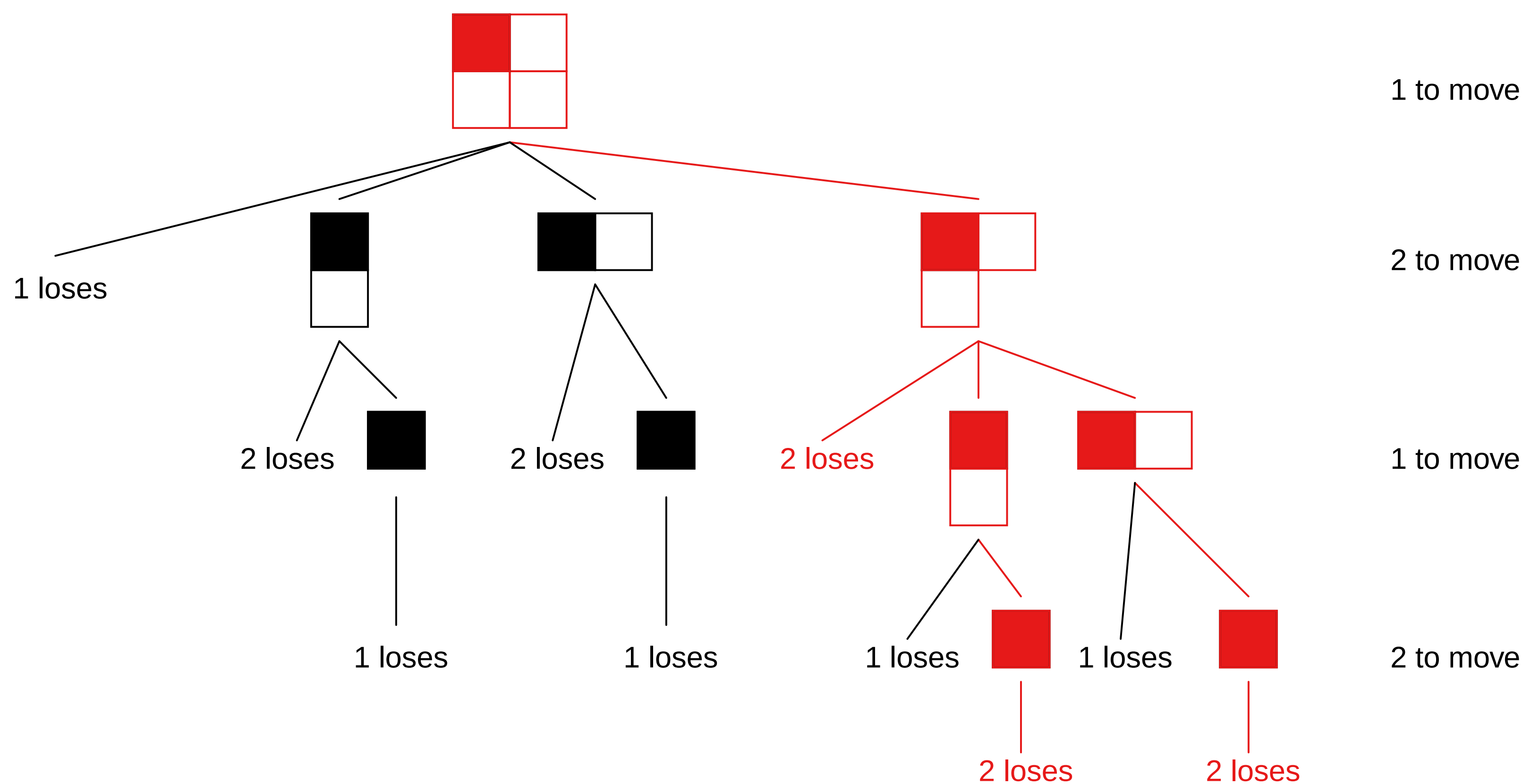


Example

Question. How many possible outcomes (final positions) does playing in accord with this strategy have? How many are advantageous to Player 1?



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Also note that we force a strategy to make a choice of a move if it is the chosen player's turn—we do not allow 'give up by refusing to make a move'. If we want giving up to be an option, we have to make it a move in the game tree.

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We can turn this observation into a definition.

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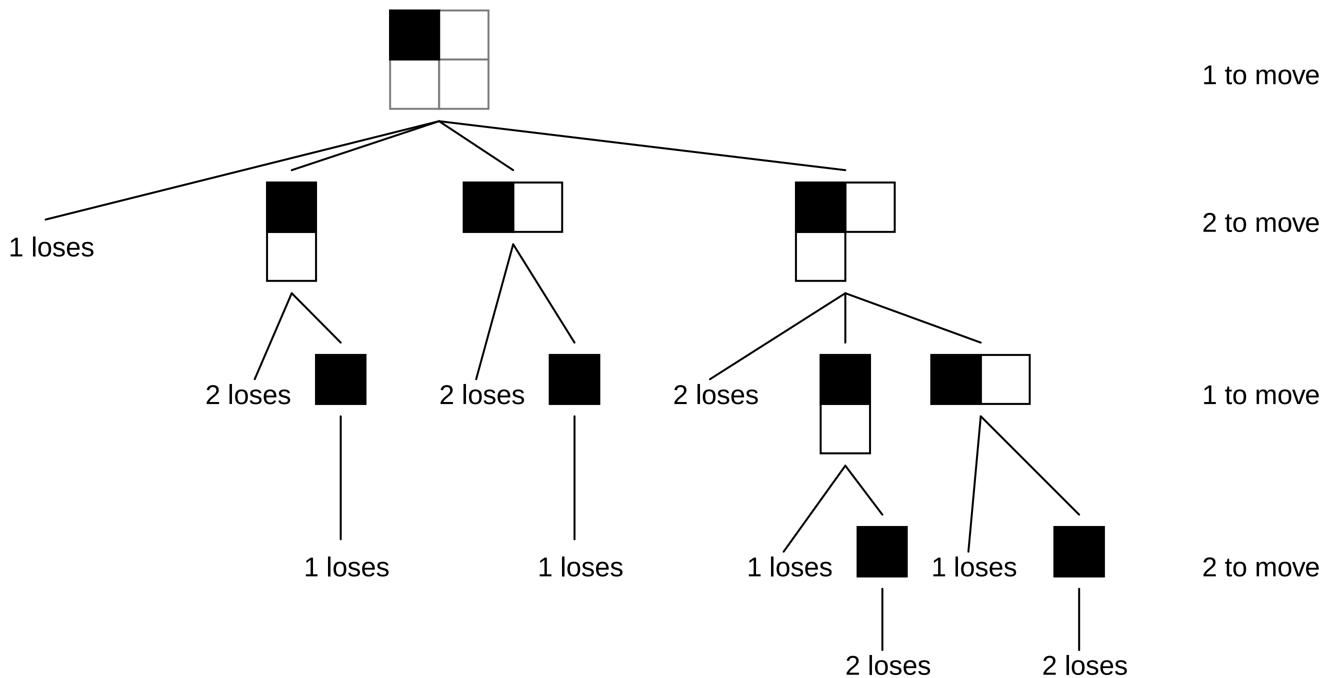
Different strategies

How many strategies are there for Player 1 in (2×2)-Chomp?



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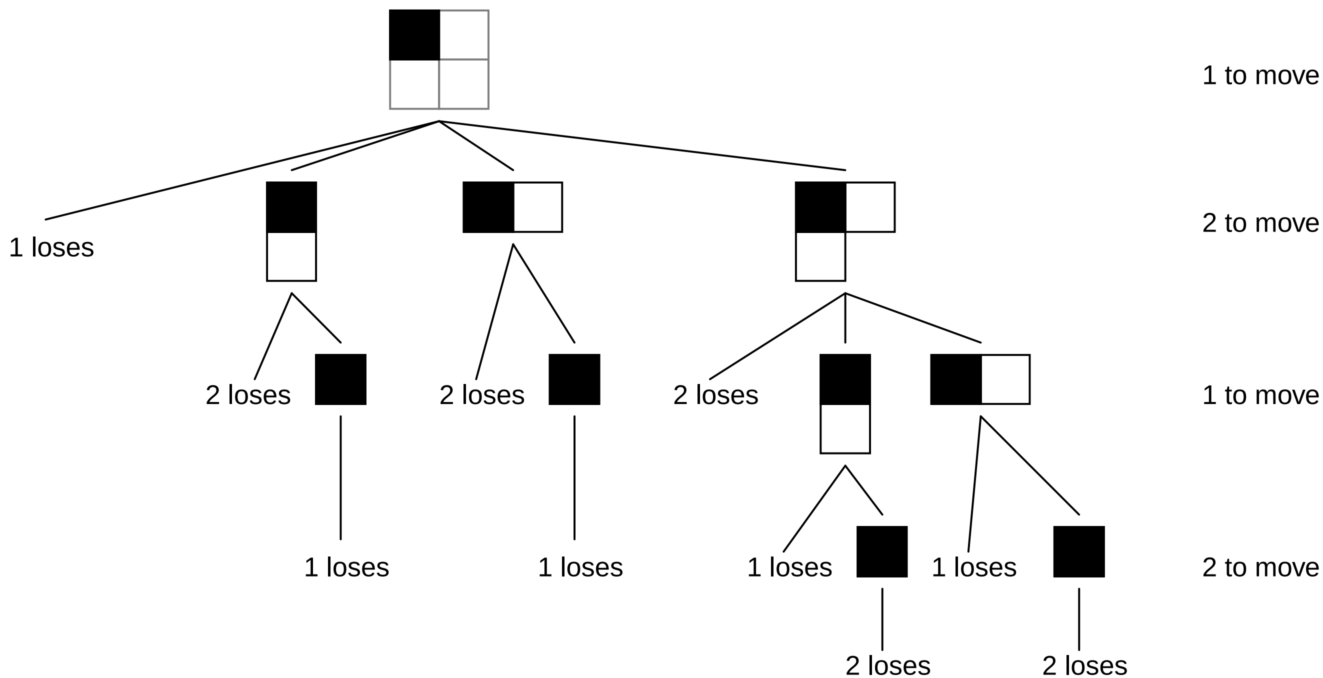

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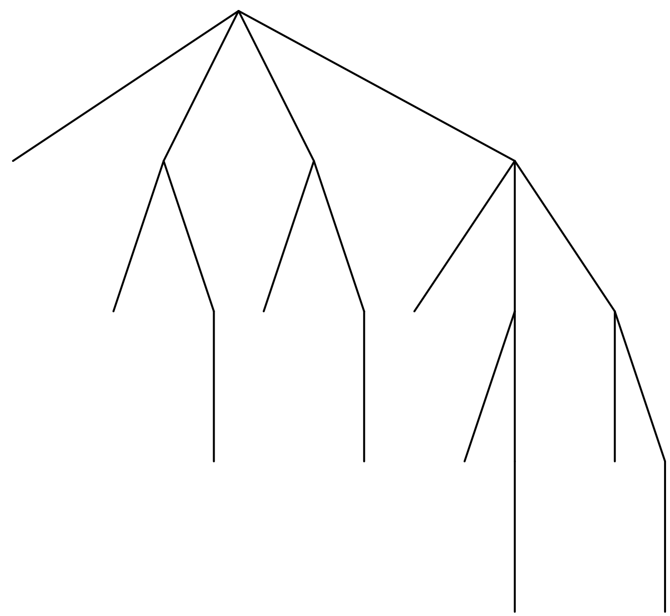
If we remember that Player 1 starts the game and they then move alternatingly, we can think of the game tree for (2×2)-Chomp like this:



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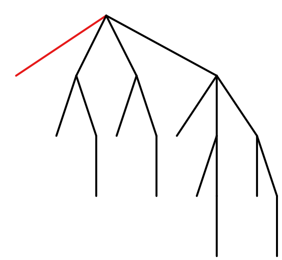
Player 1 chooses the left-most move.



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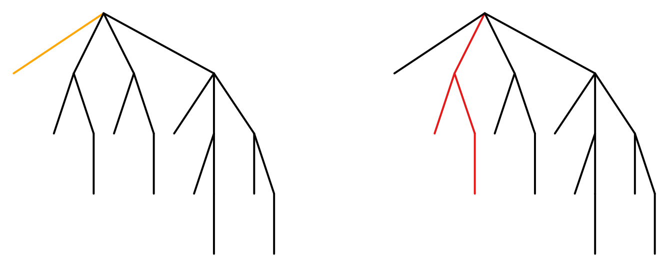

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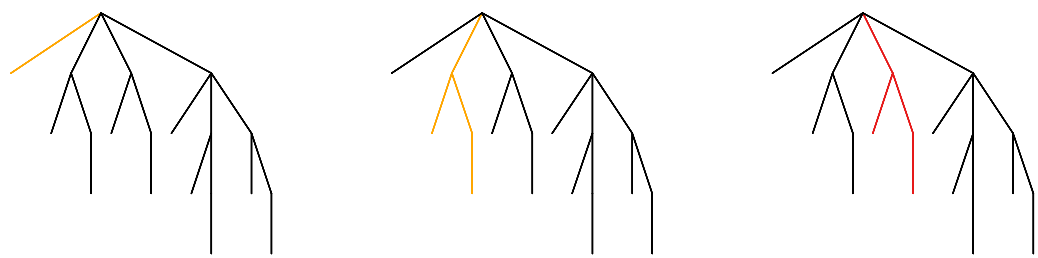

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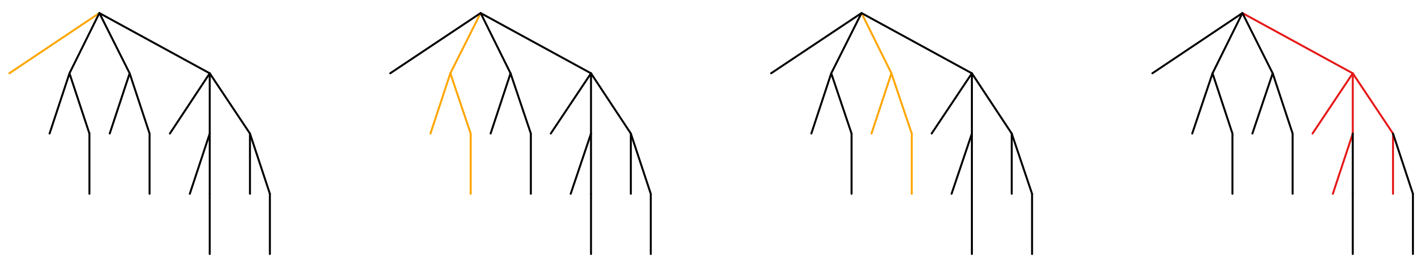
How many strategies are there for Player 1 in (2×2)-Chomp? What are they?

Player 1 chooses the right-most move. But that means there are further choices down the line! Two decision points with two choices each, that gives $2 \times 2 = 4$ strategies.



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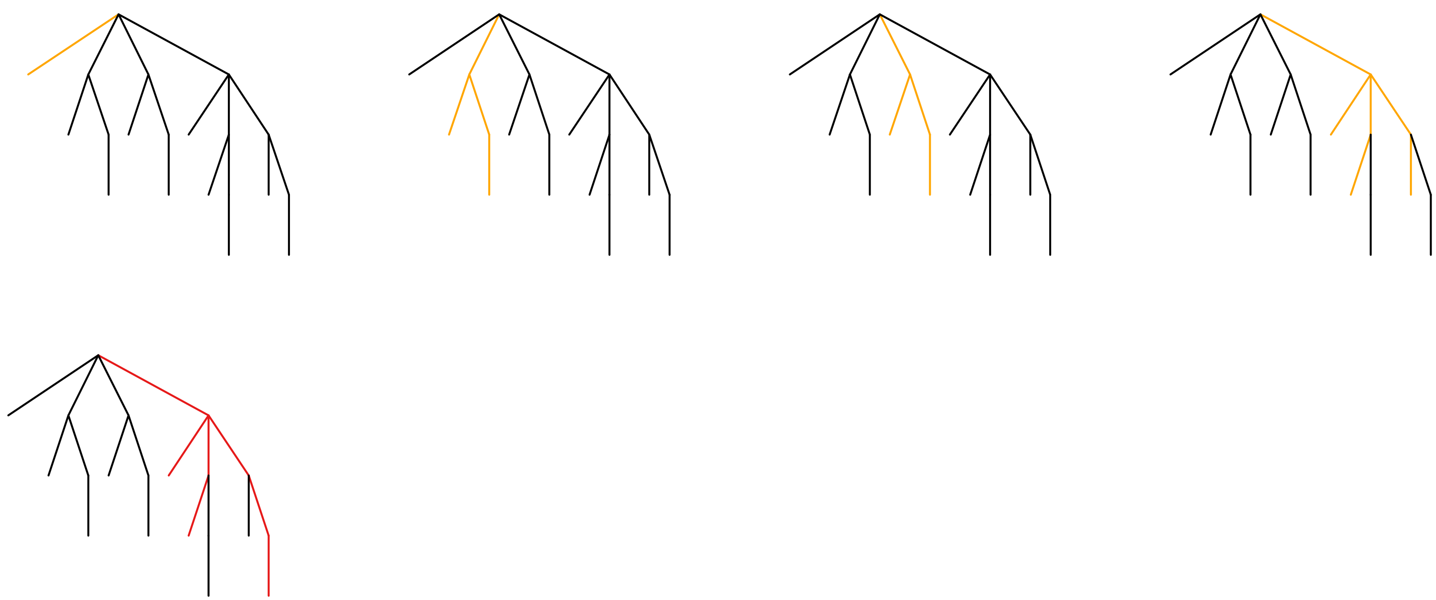

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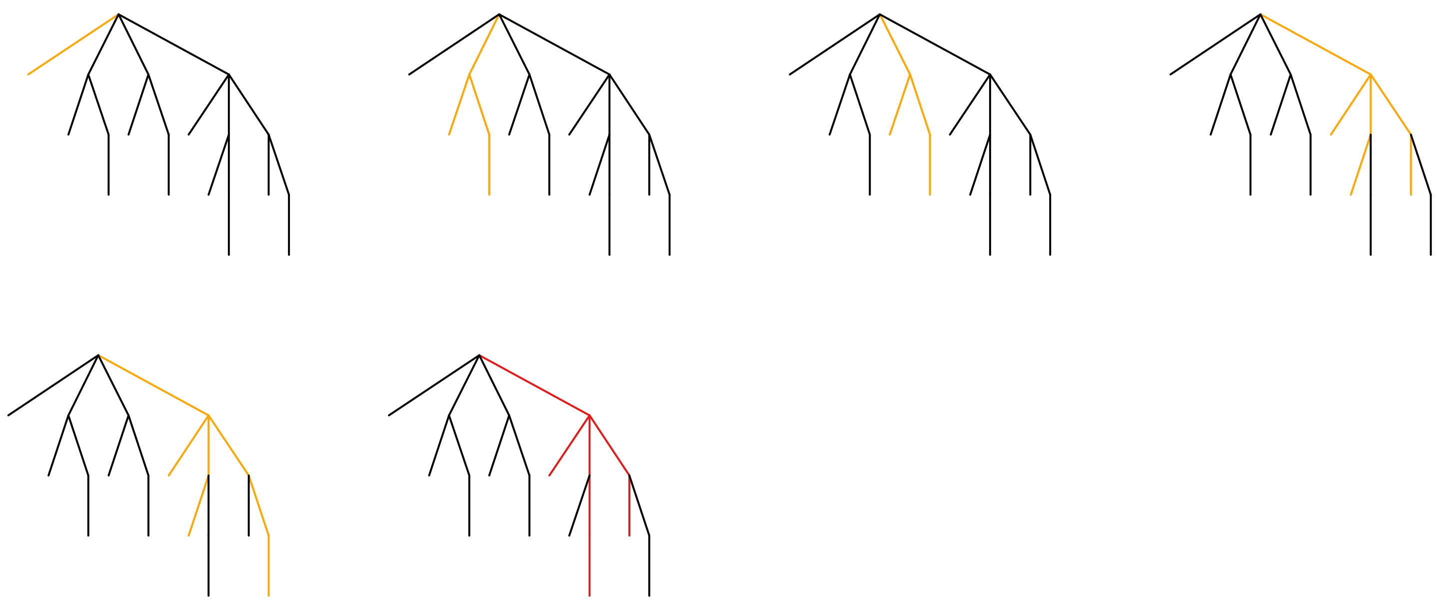

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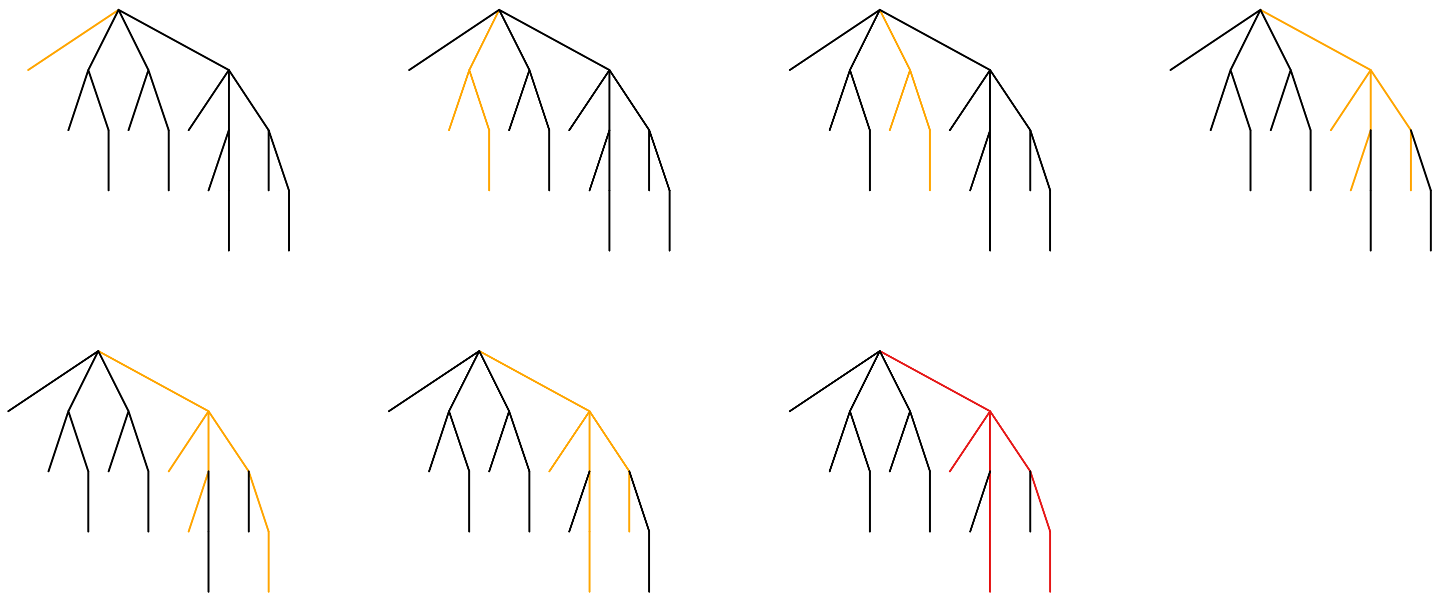

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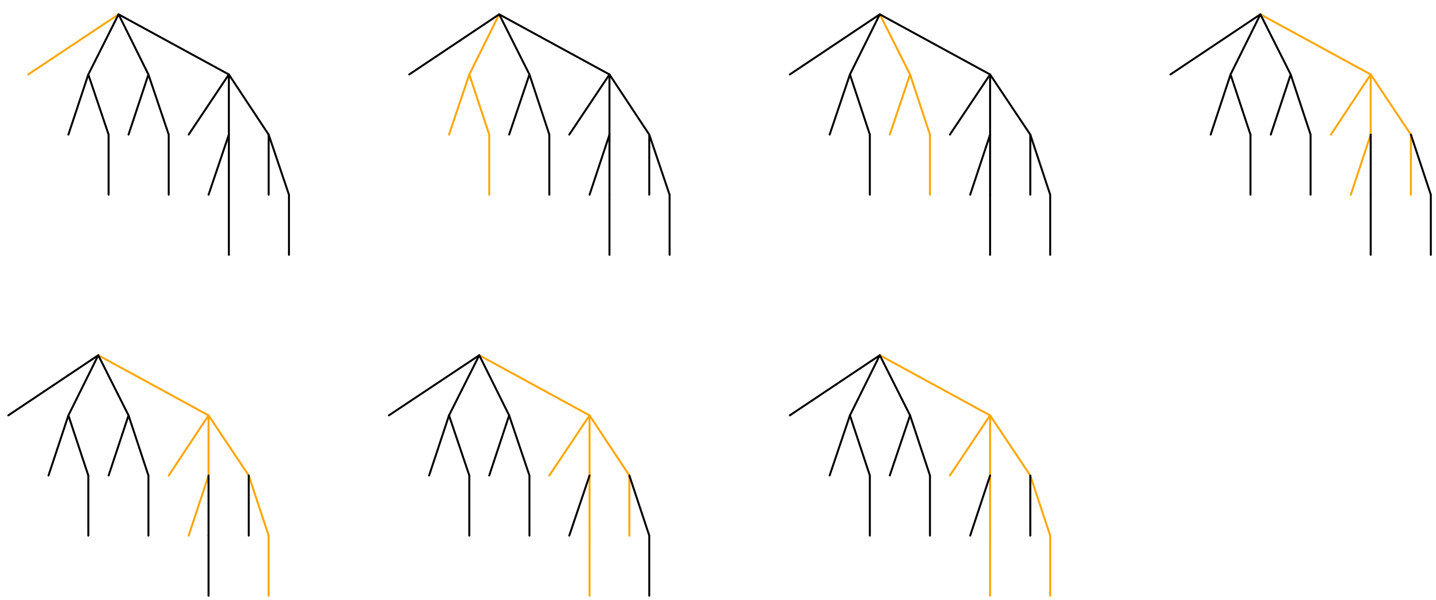

That makes seven strategies altogether.

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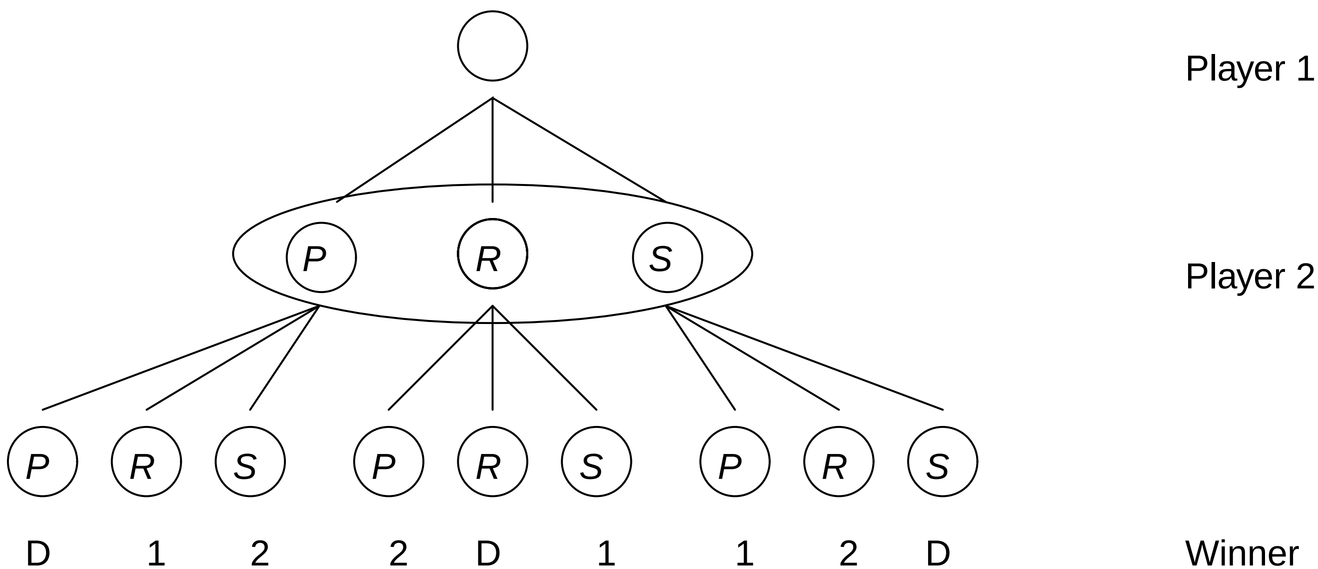
We consider strategies for Player 2 of Paper-Stone-Scissors.



Strategies and imperfect information.

Recall that in our definition of strategy, we stipulated that for all nodes in the same information set for Player X, the move chosen by the strategy is the same.

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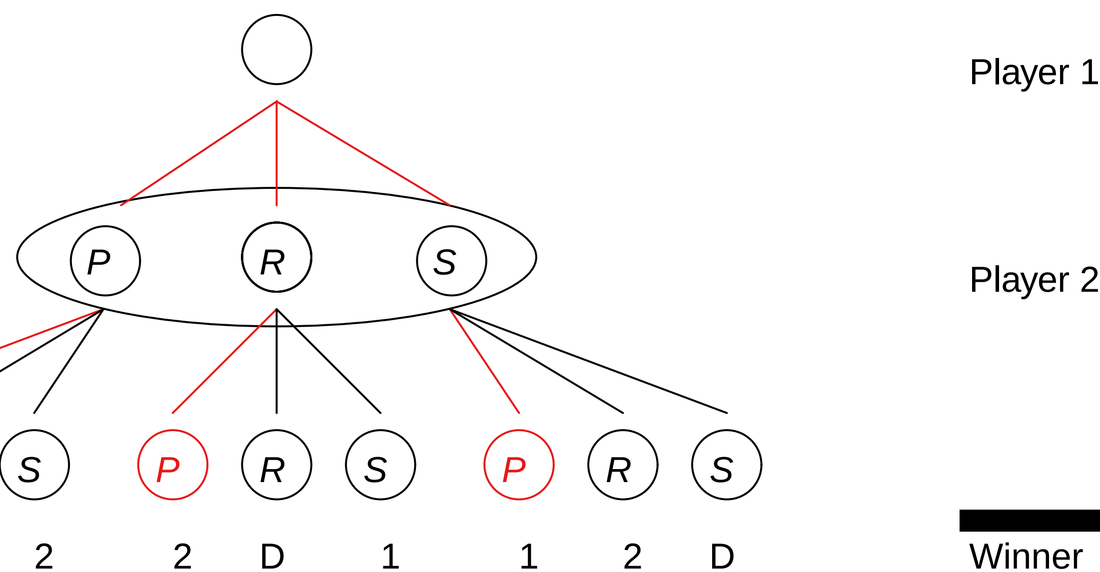
We consider strategies for Player 2 of Paper-Stone-Scissors. Player 2 might decide to show P for paper. She will have to do that for all the nodes in her information set.



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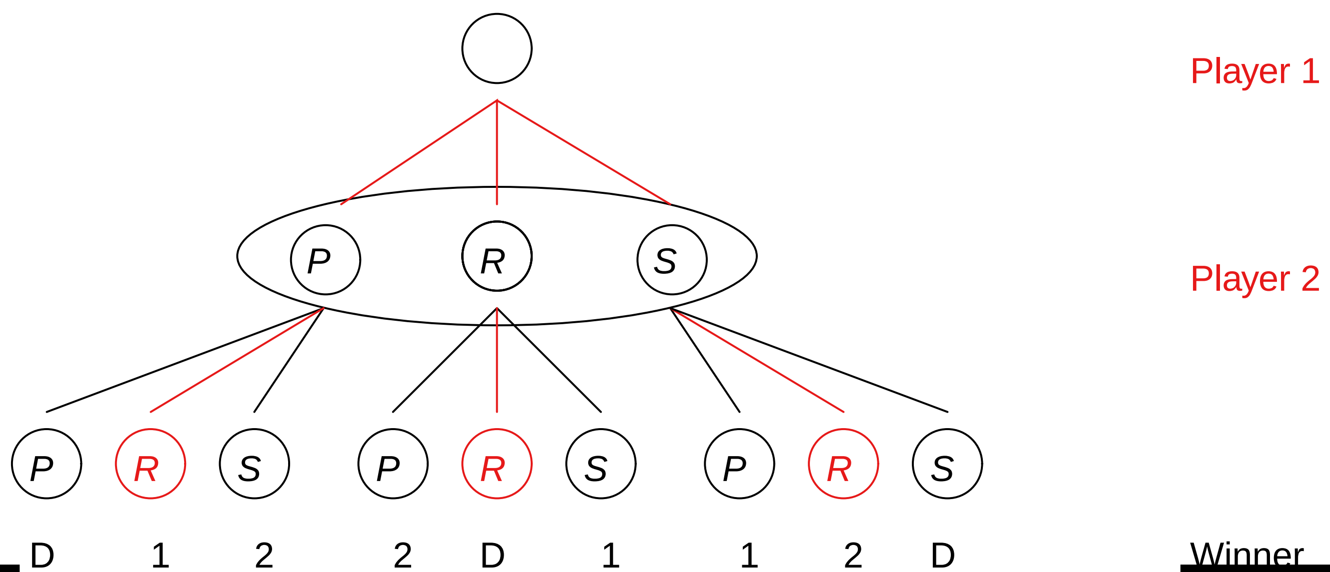

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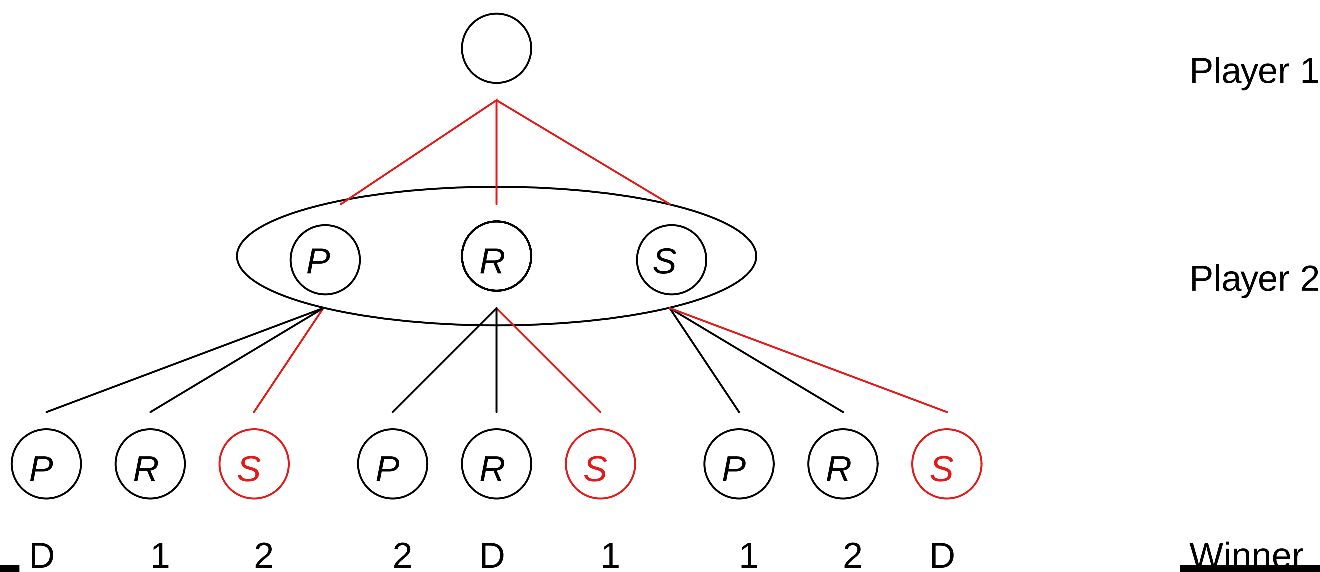

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These are all the strategies for Player 2 in Paper-Stone-Scissors. Again they show that Player 2 cannot use information she does not have; her moves for two nodes in the same information set are the same.

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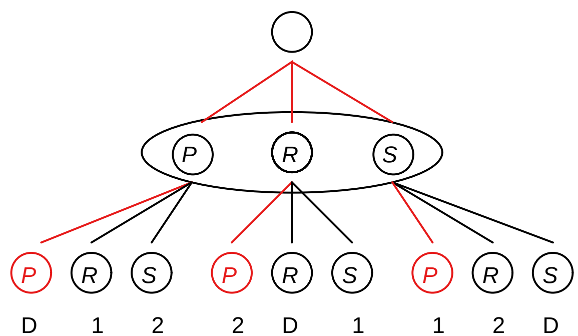

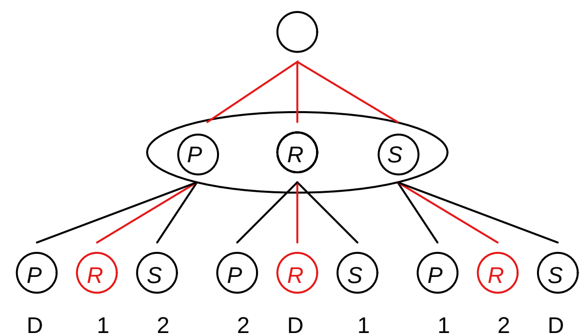

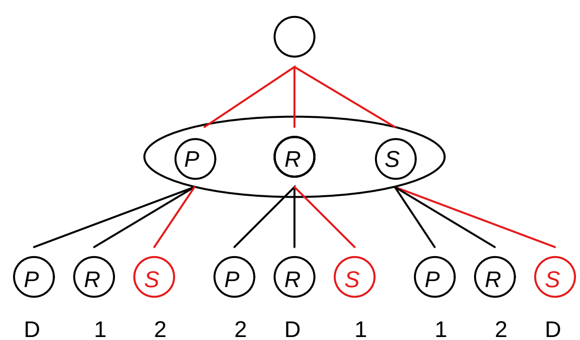

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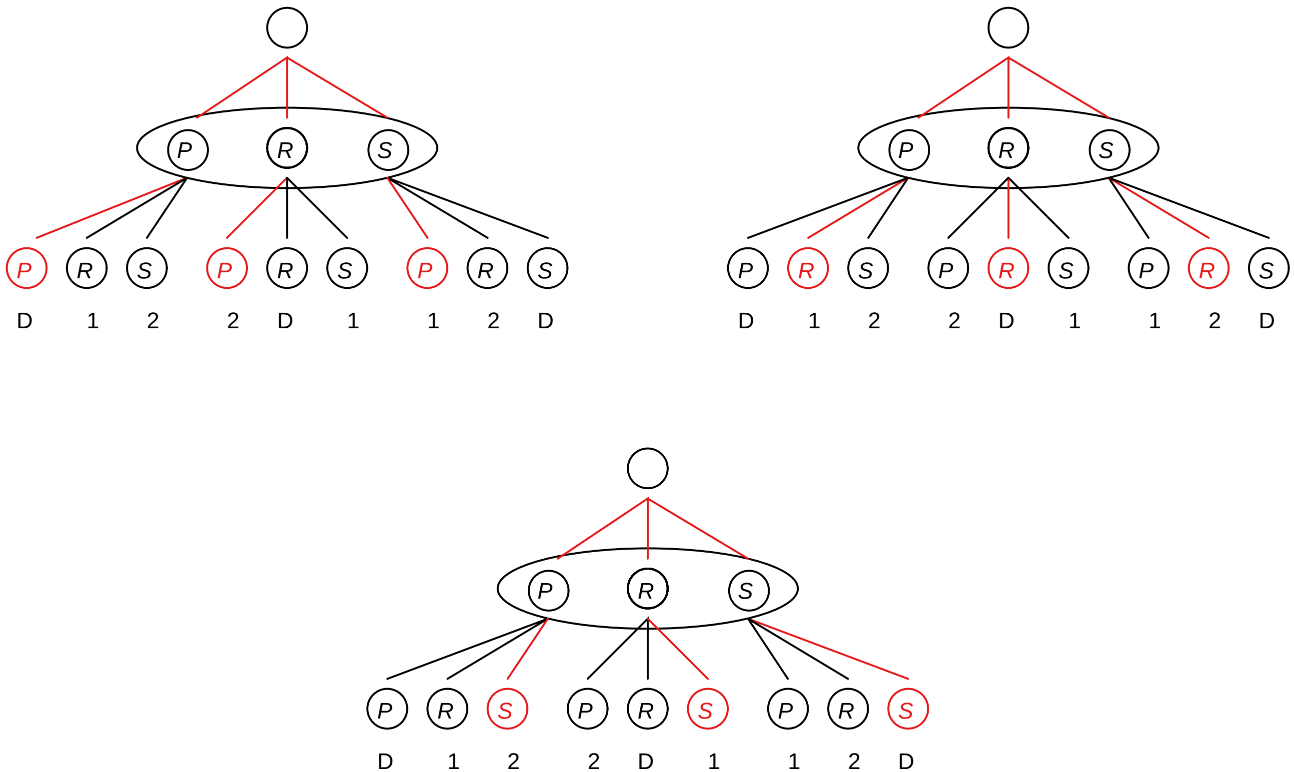

So she has the same choices as Player 1, preserving the symmetry of the game as it is defined informally. So we really have not changed the game by using *via* a game tree.

Generating all strategies

Let's assume we are given a game tree t and are trying to generate all strategies for Player X. This can be done recursively. (Note that when the game is not of perfect information then the following considerations have to be changed accordingly.)

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- There are no moves: There is only the 'do nothing' strategy for each player.

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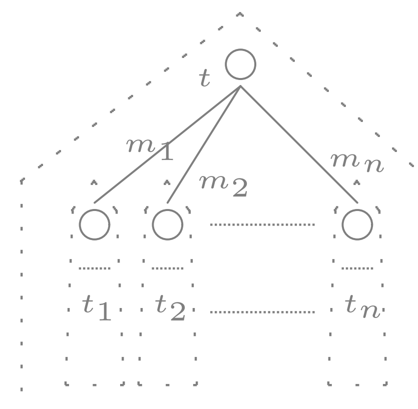
- There are no moves:
- The first move is Player X's: He has an according number of choices (m_1 to m_n). After that move is made, we get a new game tree, one of t_1 to t_n, and Player X can now use any of his strategies for the game following his chosen first move.



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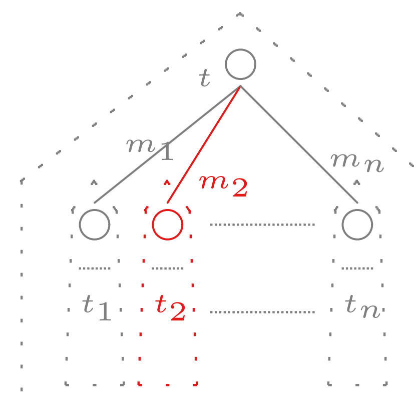

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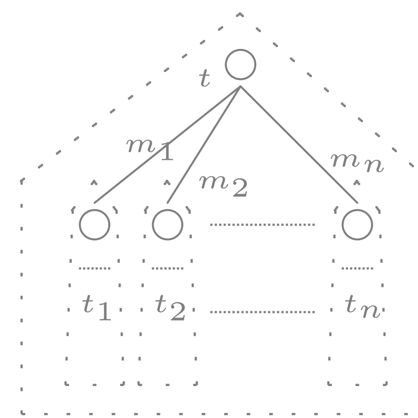

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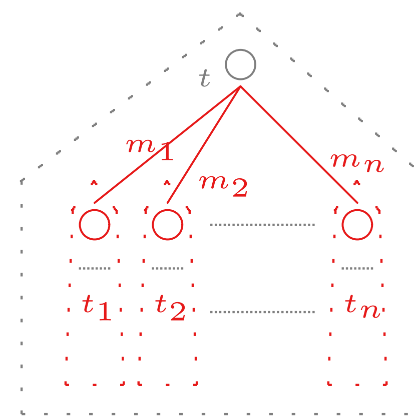

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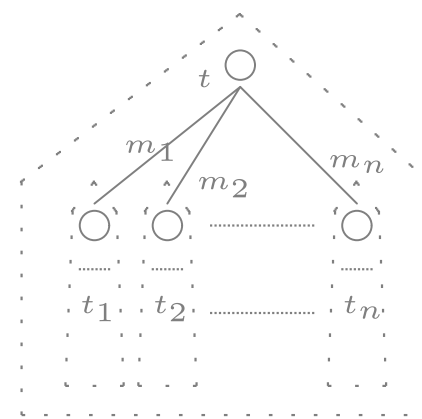

Note that this does account for chance moves.

Counting all strategies

Let's assume we are are now trying to count all strategies for Player X for game t. This can also be done recursively, with the number of strategies for Player X on game t being denoted by $N_X(t)$. (Note that when the game is not of perfect information then once again the following considerations have to be changed accordingly.)

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- There are no moves: It has one strategy for each player.

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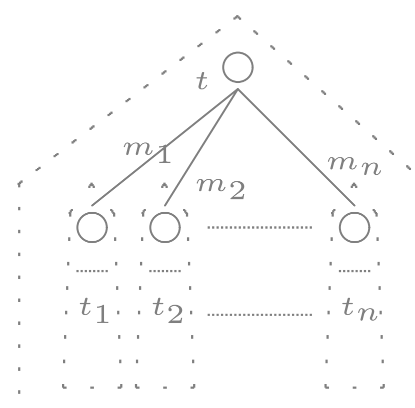

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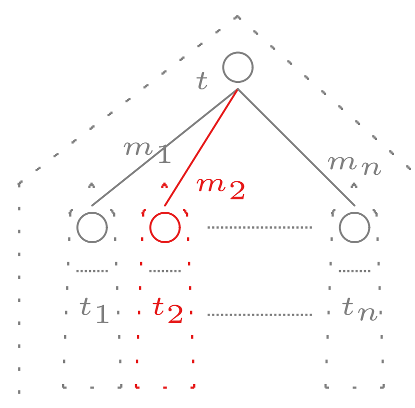
$$N_X(t) = N_X(t_1) + N_X(t_2) + \cdots + N_X(t_n).$$



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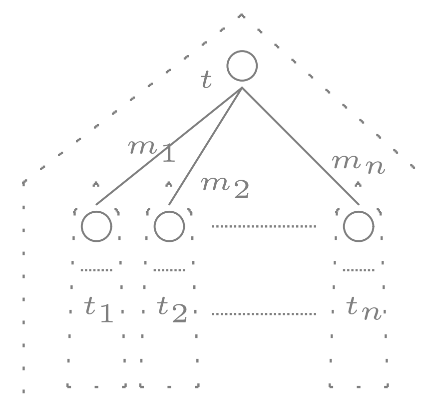

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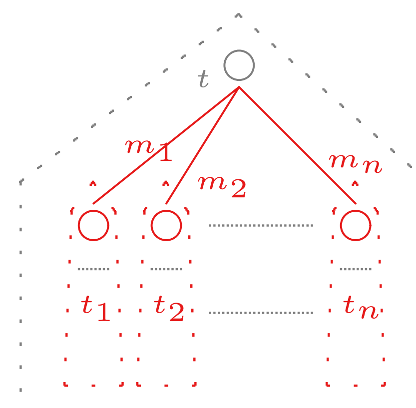

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We then go through the game tree as follows: For every node we reach, starting with the root, we ask the player whose turn it is which move his or her strategy chooses. If there are no chance moves then we thus follow a uniquely determined path through the tree ending at a leaf. That leaf gives us the outcome of playing all these strategies against each other.

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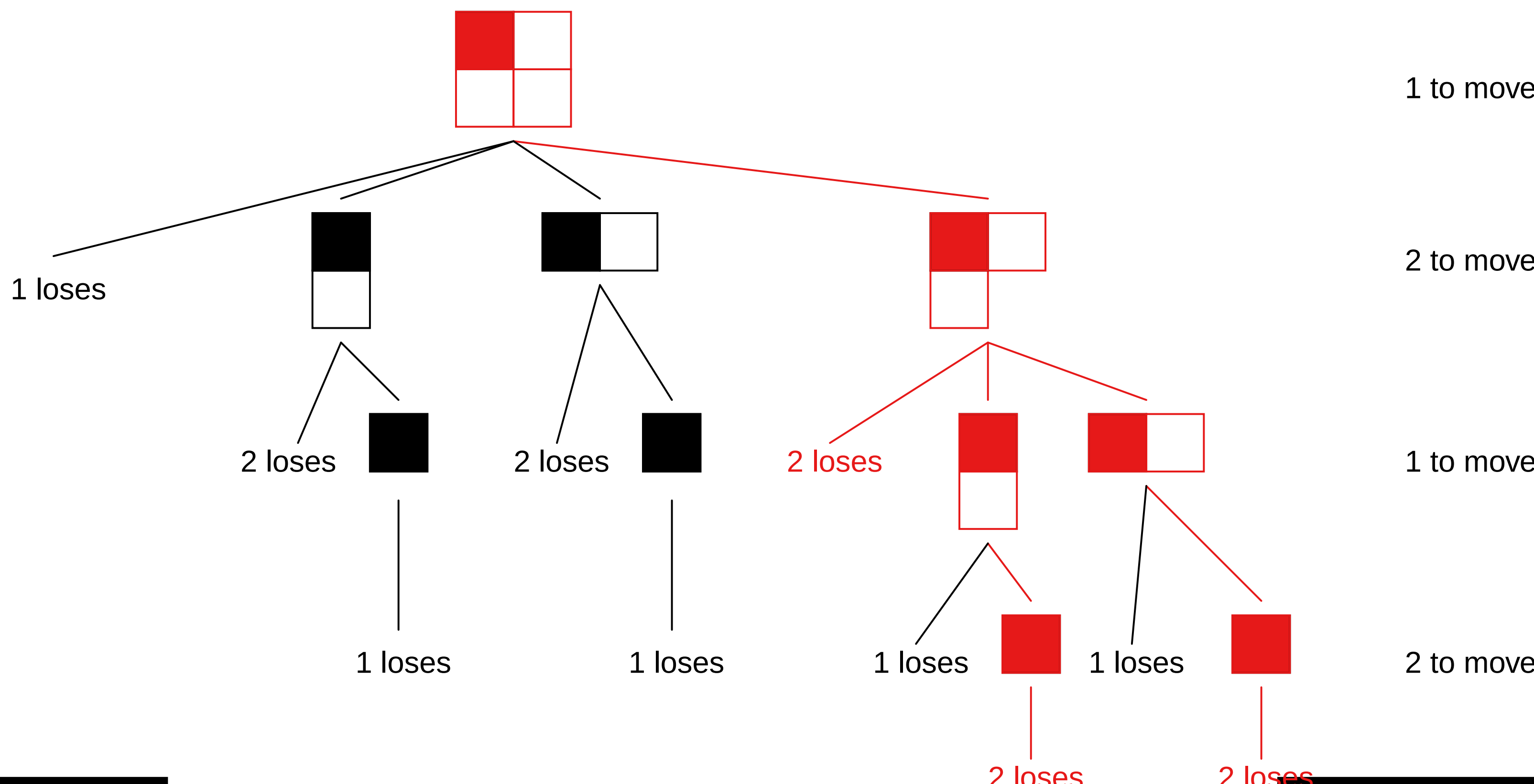
If there are chance moves involved, then we get several paths ending in several leaves, together with a probability for each leaf that it will occur. The probability for a leaf to occur is calculated by multiplying the probabilities occuring along the unique path leading to it. The probabilites for all the possible leaves will add up to one.

Playing strategies against each other

We have a look at an example to show how playing one strategy against another works.

Playing strategies against each other

Here is a strategy for (2×2)-Chomp for Player 1.



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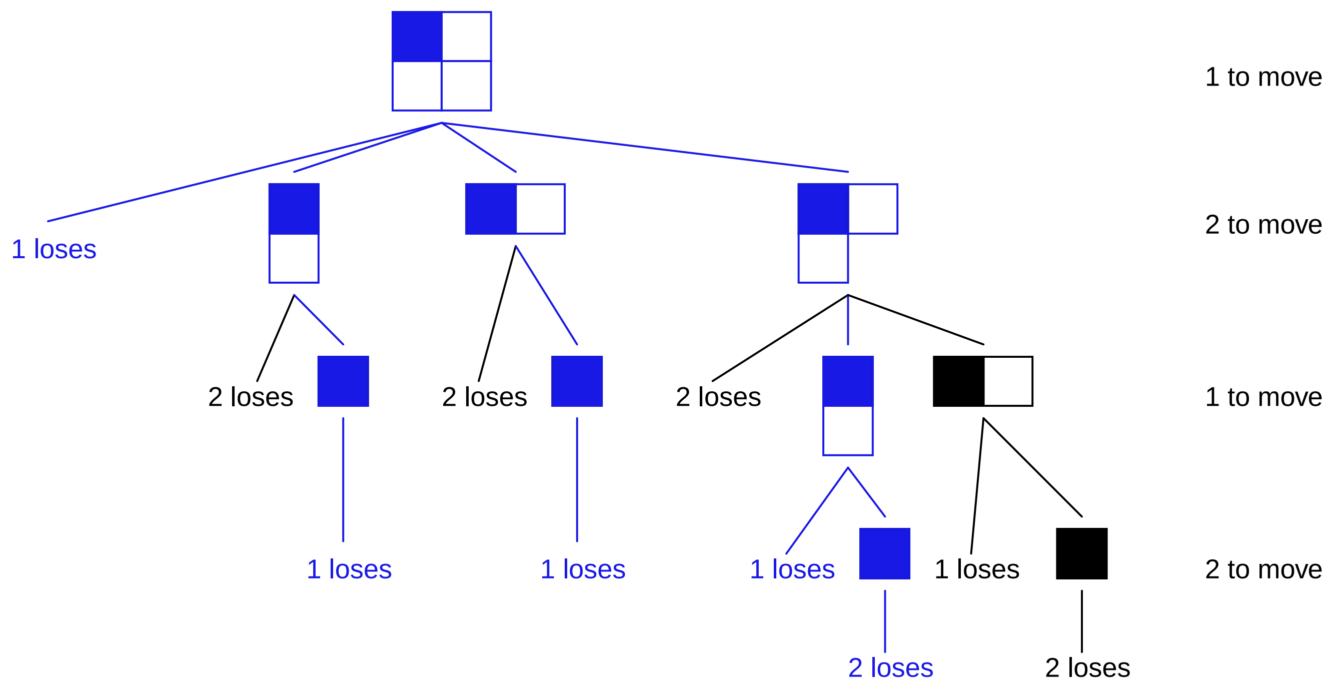
Here is a strategy for (2×2)-Chomp for Player 1. And here is a strategy for the same game for Player 2.



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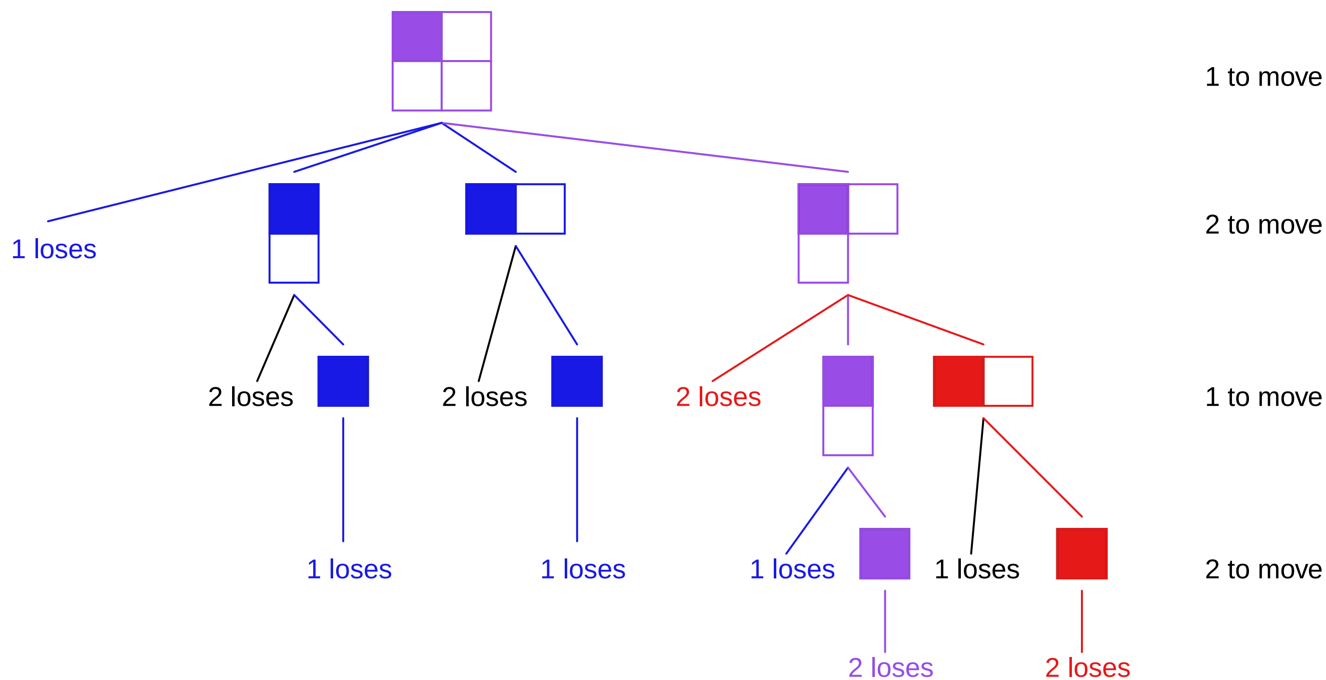
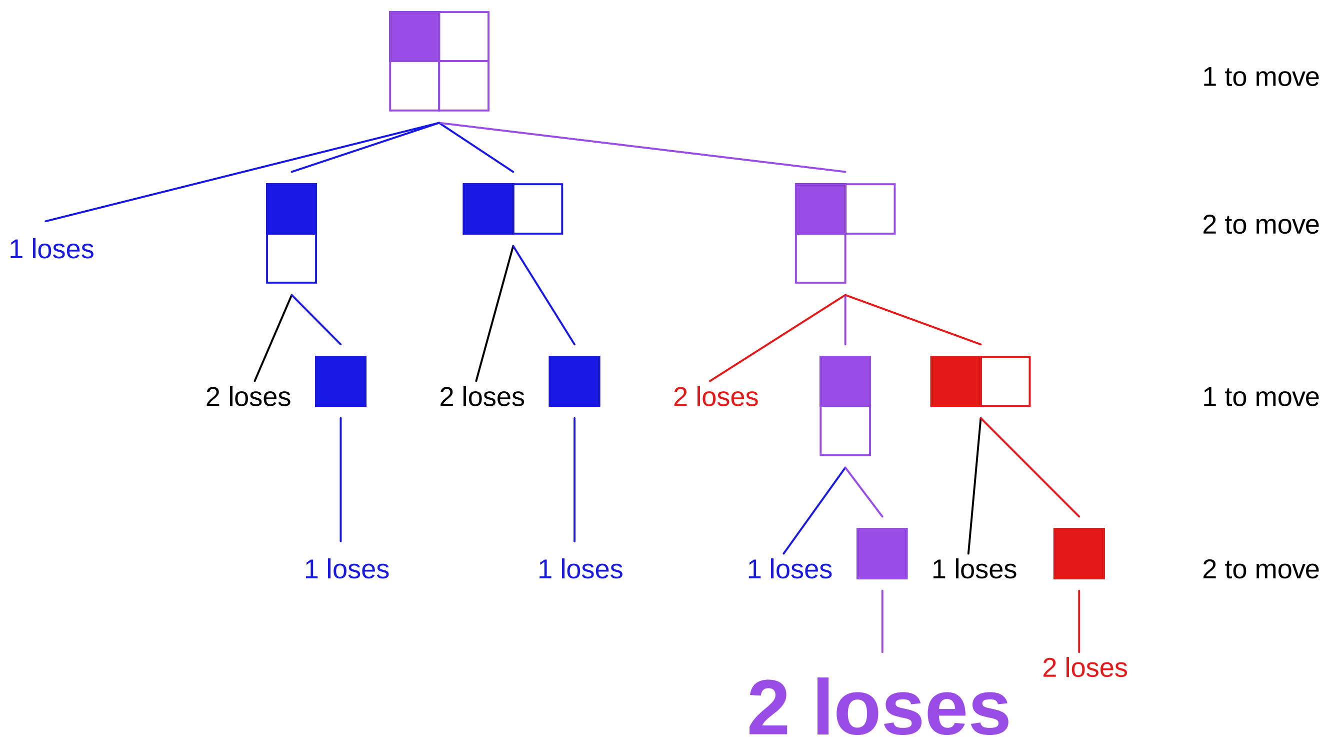


Playing strategies against each other

Here is a strategy for (2×2)-Chomp for Player 1. And here is a strategy for the same game for Player 2. And here they are together. The outcome of playing the one against the other is immediately obvious.



Why bother?

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This is quite hopeless for games like Chess and Go, or even Checkers, simply because the game tree is far too big to hold in a computer—and we've just got some idea of how the number of strategies grows with the size of the game tree!

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Hence when we talk about small games we mean games which are amenable to an analysis *via* strategies. They will keep us busy for quite a while, maybe longer than you'd thought. This is the most mathematical part of the course.

Games *via* strategies

We take the idea of presenting games *via* their strategies to its logical conclusion.

Games *via* strategies

If there are just two players, and no elements of chance involved, then we can present a game in the form of a matrix. We list the strategies for Player 1 along the rows and those for Player 2 along the columns, and record in the centre the outcome of playing the one against the other.

Games *via* strategies

In Paper-Stone-Scissors, Player 1 and 2 have three choices each which we can encode as follows.

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		2		
		P	R	S
	P			
1	R			
	S			

Games *via* strategies

In Paper-Stone-Scissors, Player 1 and 2 have three choices each which we can encode as follows. We record the result as

- 1: win for Player 1
- 2: win for Player 2
- D: draw.

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	P			
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1	R	2	D	1
	S	1	2	D

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	P	D	1	2
1	R	2	D	1
	S	1	2	D

Games given likes this are known as matrix games.

Question

Question. How would you give a matrix version of our miniature Risk game?

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The problem with matrix games as we've described them so far is that as soon as elements as chance are involved, we don't get a unique result from playing strategies against each other.

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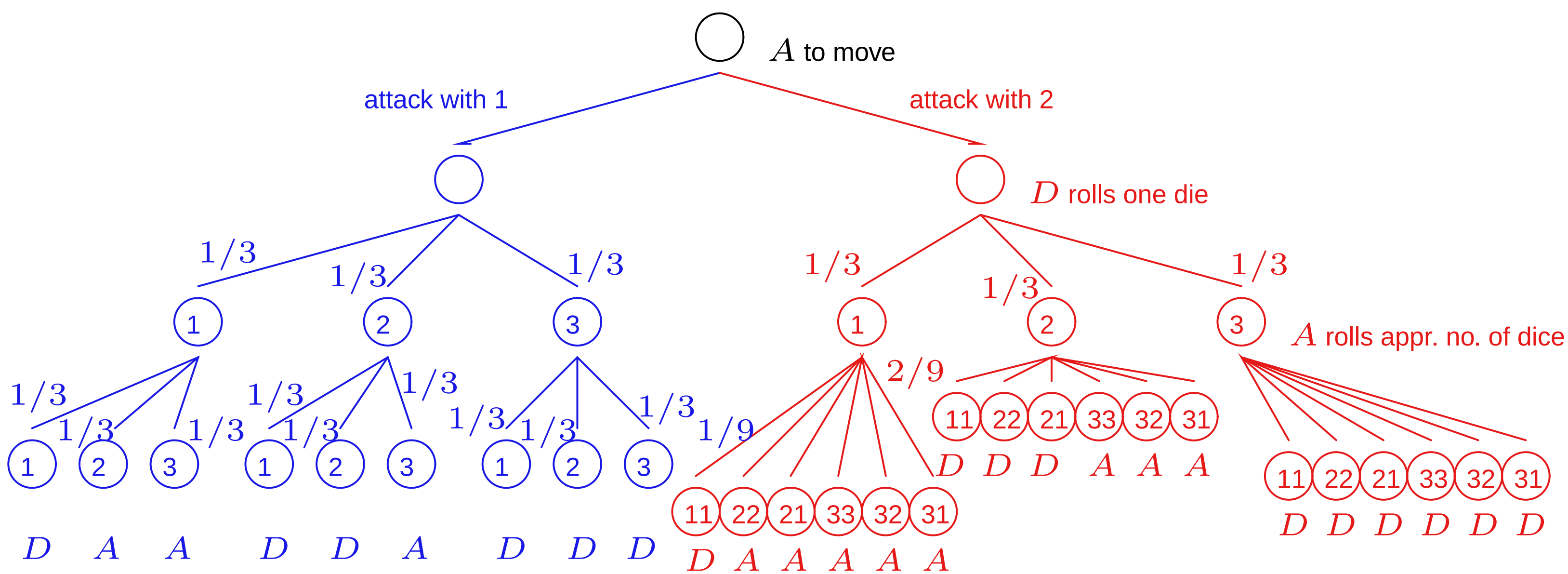
Recall the miniature Risk game we looked at earlier: Player 1 can only choose between the strategies 'attack with 1' and 'attack with 2', while Player 2 has only one strategy. But there are plenty of possible outcomes for either strategy.



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However, if all our outcomes are given as numbers then we can calculate an expected number as the outcome of playing one strategy against another.

If, for each player and each outcome of the game, we have a number to indicate what the outcome means for this player, then we can turn every game into a matrix.

Pay-off functions

It is for this reason that our definition of ‘game’ talks about something called a ‘pay-off function’.

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Definition 1 *A game is given by*

- *a finite set of players,*
- *a finite game tree,*
- *for each node of the tree, a player whose turn it is in that position and*
- *for each final node and each player a pay-off function.*

Example

Assume that for Paper-Stone-Scissors the loser pays the winner one (pound, point, chocolate bar), and the pay-off is 0 for both if the game ends in a draw. Then the pay-offs for this game for Player 1 are as follows.

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	P	0	1	-1
1	R	-1	0	1
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		2		
		P	R	S
	P	0	1	-1
1	R	-1	0	1
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		2		
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	P	0	-1	1
1	R	1	0	-1
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		2		
		P	R	S
	P	$(0,0)$	$(1,-1)$	$(-1,1)$
1	R	$(-1,1)$	$(0,0)$	$(1,-1)$
	S	$(1,-1)$	$(-1,1)$	$(0,0)$

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Question. What would happen if there were more than 2 players?

Example

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1	R	1	0	-1
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Note that if we add the entries in the left hand table to the corresponding ones in the right hand table we get a table consisting entirely of 0s. This is because all payments to Player 1 are pay-outs by Player 2 and *vice versa*.

Zero-sum games

It turns out that games with this property are important enough to deserve their own name.

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		2		
		P	R	S
	P	0	1	-1
1	R	-1	0	1
	S	1	-1	0

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Hence in such a case it is sufficient to give just the one matrix, which normally is that for Player 1. In fact, whenever we give a game by just such a matrix, the implicit assumption is that we are describing a 2-person zero-sum game.

Non zero-sum games

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Some of the most interesting games are non zero-sum. We will see plenty of examples in this course, in particular since most of the known techniques do not work as well for these kinds of games.

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Consider the following game between two players. Player 1 rolls a a three-faced die. If he throws 1 he pays two units to Player 2. If he throws 2 or 3, Player 2 has a choice. She can either choose to pay one unit to Player 1 (she stops the game) or she can throw the die. If she repeats Player 1's throw, he has to pay her two units. Otherwise she pays him one unit. The game tree is given below, with the pay-off being given for Player 1.

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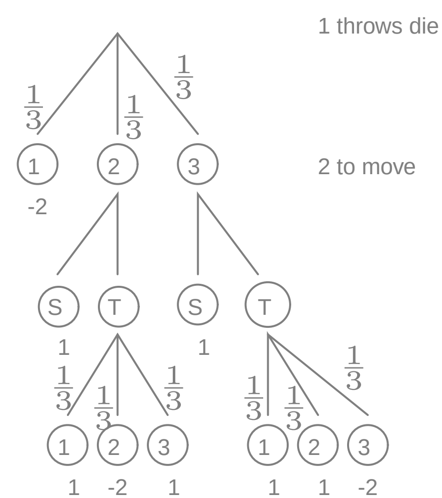
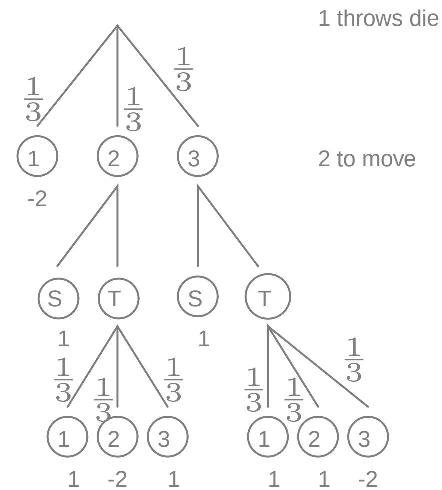


Example



Player 1 has only one strategy (he never gets a choice). Player 2 has four strategies (she can choose to throw the die or not, and is allowed to make that dependent on Player 1's throw). We can encode her strategies by saying what she will do when Player 1 throws a 2, and what she will do when Player 1 throws a 3, stop (S) or throw the die (T). So $S|T$ means that she will stop if he throws a 2, but throw if he throws a 3. The matrix will look something like this:

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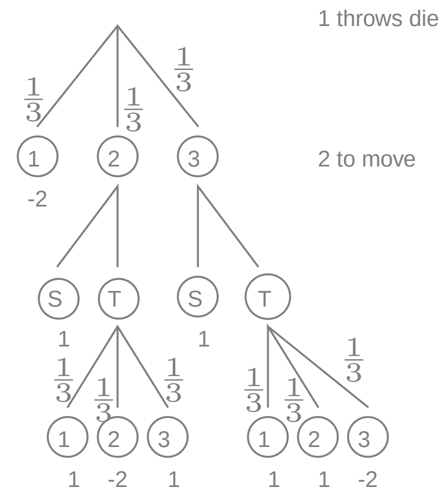

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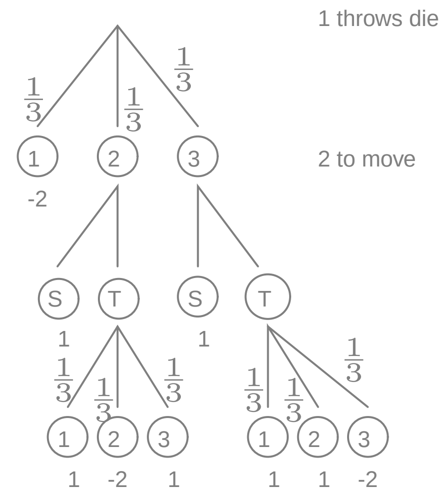
	2			
	$S\|S$	$S\|T$	$T\|S$	$T\|T$
1				

Example



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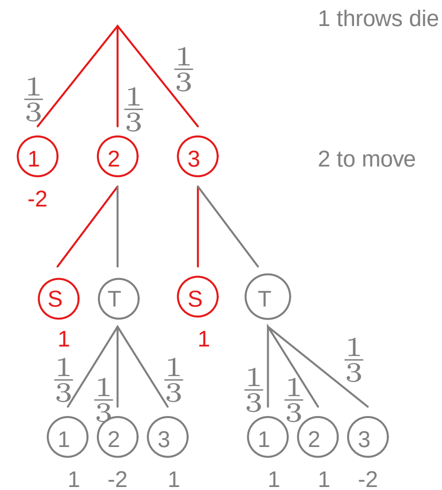
	$S\|S$	$S\|T$	$T\|S$	$T\|T$
1				

What are the expected pay-offs for the outcome of these strategies?

Example



	2			
	$S\|S$	$S\|T$	$T\|S$	$T\|T$
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What are the expected pay-offs for the outcome of these strategies?

Case $S|S$. For each of the possible outcomes of playing this strategy, take the probability that it will occur and multiply it with the pay-off, then add all these up.

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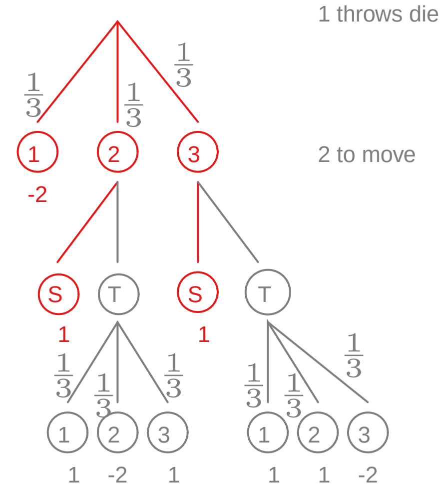
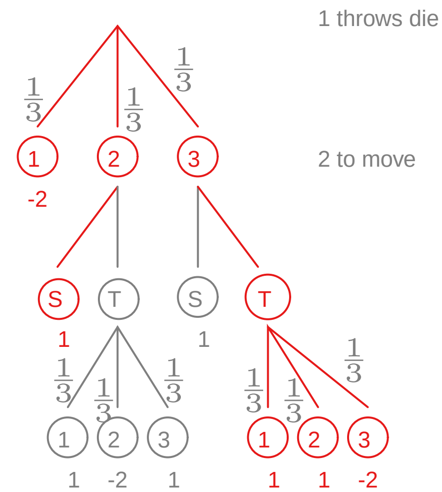


	2			
	$S\|S$	$S\|T$	$T\|S$	$T\|T$
1	0			

What are the expected pay-offs for the outcome of these strategies?
Case $S|S$. For each of the possible outcomes of playing this strategy, take the probability that it will occur and multiply it with the pay-off, then add all these up. Hence we get

$$(1/3 \times (-2)) + (1/3 \times 1) + (1/3 \times 1) = 0.$$

Example

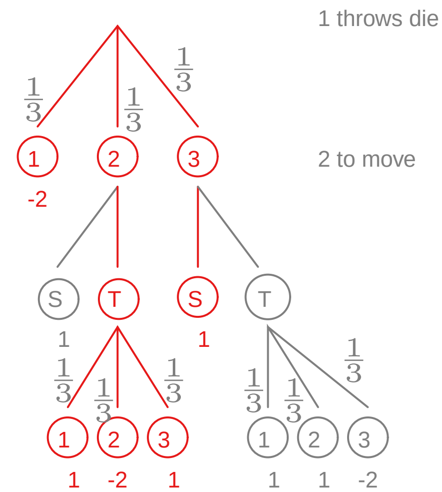


	2			
	$S\|S$	$S\|T$	$T\|S$	$T\|T$
1	0	$-1/3$		

What are the expected pay-offs for the outcome of these strategies?
Case $S|T$.

$$(1/3\times(-2))+(1/3\times 1)+(1/9\times 1)+(1/9\times 1)+(1/9\times(-2)) = -3/9 = -1/3.$$

Example

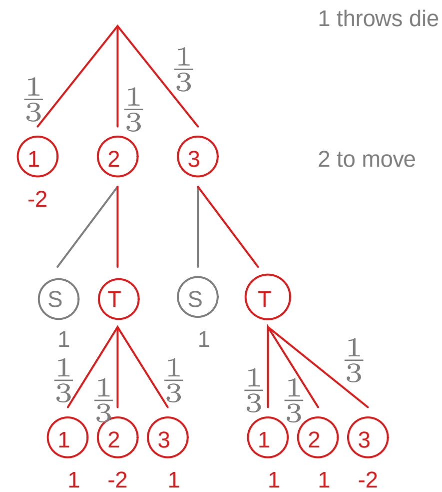


	2			
	$S\|S$	$S\|T$	$T\|S$	$T\|T$
1	0	$-1/3$	$-1/3$	

What are the expected pay-offs for the outcome of these strategies?

Case $T|S$. This is a symmetric variation of case $S|T$, and the pay-off is the same.

Example



	2			
	$S\|S$	$S\|T$	$T\|S$	$T\|T$
1	0	$-1/3$	$-1/3$	$-2/3$

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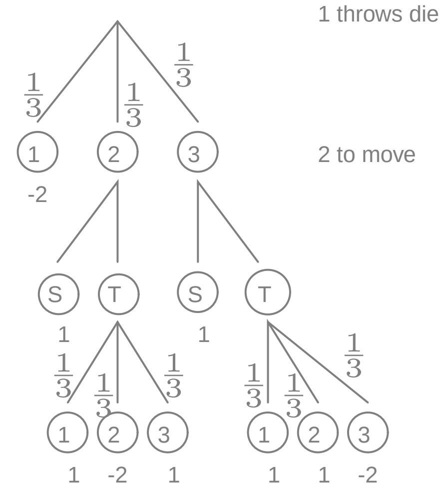
Case $T|T$.

$$(1/3 \times (-2)) + (1/9 \times 1) + (1/9 \times 1) + (1/9 \times (-2)) + \\ (1/9 \times 1) + (1/9 \times 1) + (1/9 \times (-2)) \\ = -2/3.$$

Example



	2			
	$S\|S$	$S\|T$	$T\|S$	$T\|T$
1	0	$-1/3$	$-1/3$	$-2/3$

Question. Which player would you rather be in this game?

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Theorem 1.1 *Consider a game with two players, 1 and 2, of perfect information without chance, which can only have three different outcomes: Player 1 wins, Player 2 wins, or they draw. Then one of the following must be true.*

(i) *Player 1 has a strategy which allows him always to win;*

(ii) *Player 2 has a strategy which allows him always to win;*

(iii) *Player 1 and Player 2 both have strategies which ensure that they will not lose (which means that either side can enforce a draw).*

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If a player has a strategy which allows him to always win, no matter what the other player does, we call that strategy a winning strategy.

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If a player has a strategy which allows him to always win, no matter what the other player does, we call that strategy a winning strategy. Then we can restate the theorem by saying that in such a game, it is the case that either one of the players has a winning strategy or that they can both enforce a draw.

Proof

Induction over the height of the game tree.

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Base case: The height is 0. Then the result must be stated with the only node, win for Player 1, win for Player 2 or draw, and the Theorem is obviously true.

Proof

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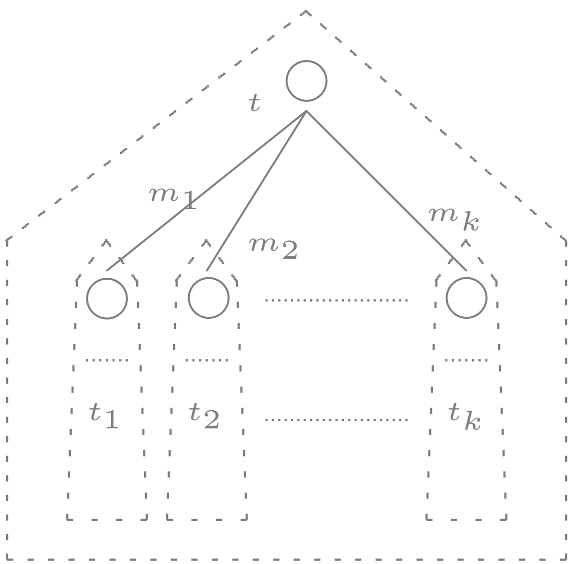
Induction hypothesis: We can label the root of a game tree of height at most n with a number which indicates which case occurs. We use 1 if Player 1 can force a win, -1 if Player 2 can force a win and 0 if both sides can enforce a draw.

Proof

Induction over the height of the game tree.

Induction step.



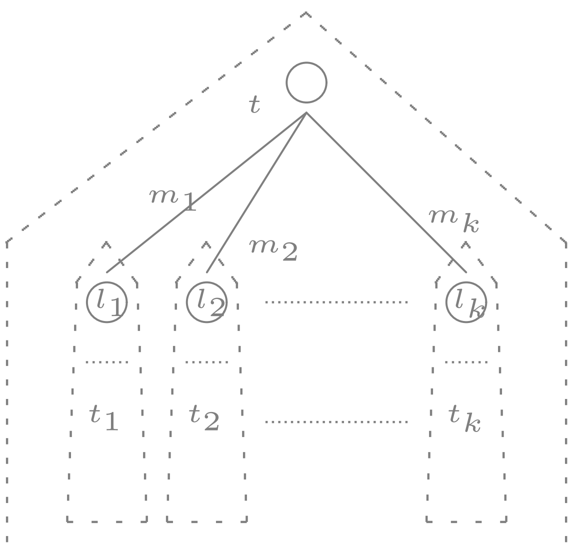
Let t be a tree of height $n + 1$.

Proof

Induction over the height of the game tree.
Induction step.

Let t be a tree of height $n+1$. By the induction hypothesis we can label the roots of the game trees reached by making a first move of t with a number (say l_i for tree t_i) as follows:

- it bears label $l_i = 1$ if Player 1 wins the game rooted there;
- it bears label $l_i = -1$ if Player 2 wins the game rooted there;
- it bears label $l_i = 0$ if both sides can enforce a draw in the game rooted there.



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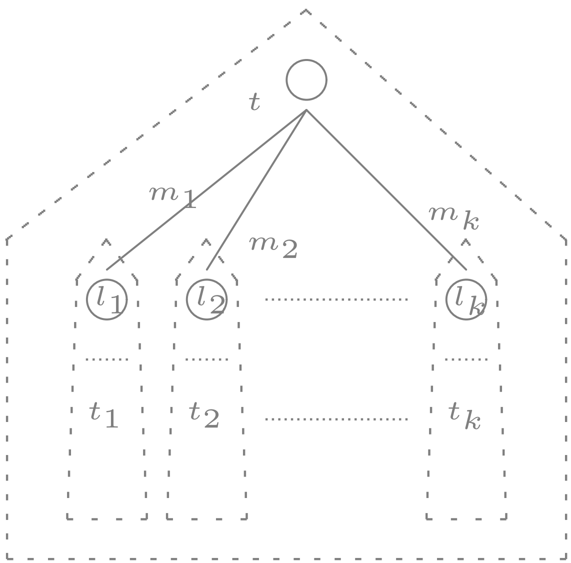

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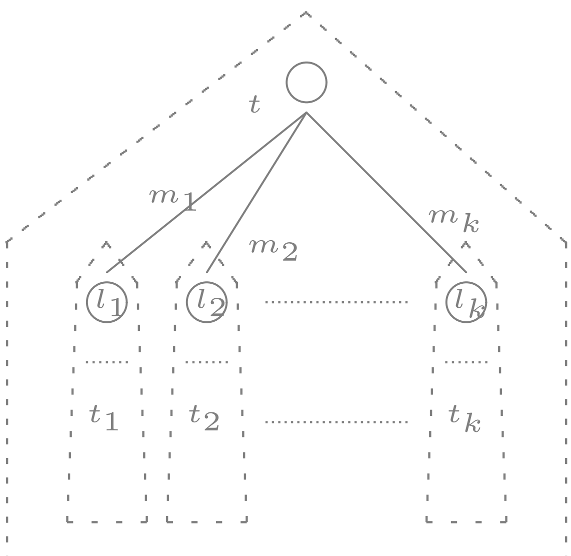

Let t be a tree of height $n+1$. Assume the first move is made by Player 1. The following cases can arise.

There is a tree labelled 1. So there is a $1 \leq i \leq k$ with $l_i = 1$. Then by making move m_i Player 1 can ensure that he will win the subsequent game t_i (because he has a winning strategy there), and thus the overall game t. Hence t gets label 1.

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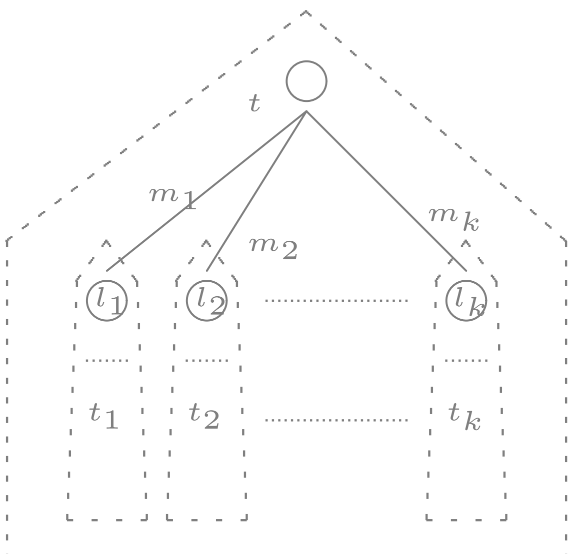

Let t be a tree of height $n+1$. Assume the first move is made by Player 1. The following cases can arise.

There is a tree labelled 1.

There is no tree labelled 1, but one labelled 0. So $l_i \leq 0$ is true for all $1 \leq i \leq n$ and $l_i = 0$ for one particular i. Then by making the first move m_i Player 1 transforms the game into t_i, where he can ensure a draw.

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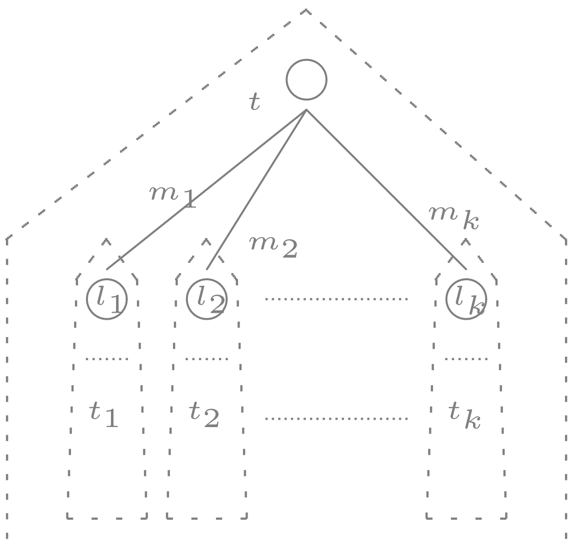

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There is no tree labelled 1, but one labelled 0. So $l_i \leq 0$ is true for all $1 \leq i \leq n$ and $l_i = 0$ for one particular i. Then by making the first move m_i Player 1 transforms the game into t_i, where he can ensure a draw. But the fact that all the trees have a label of at most 0 means that Player 2 can enforce at least a draw in all games $t_1, \dots, t_k$, and thus in the game t no matter what Player 1's first move is. So t gets label 0.

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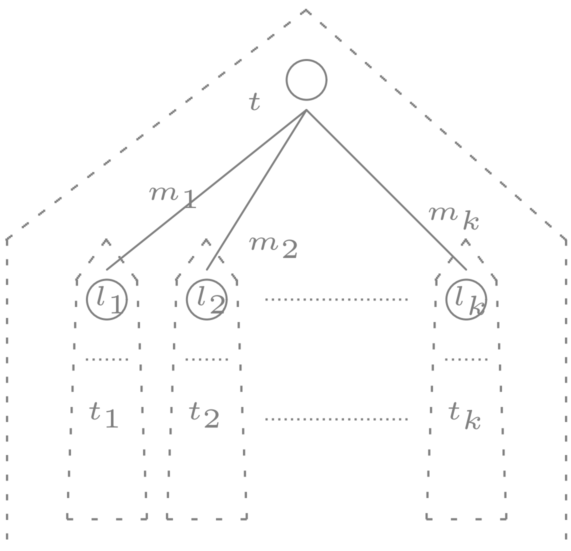

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All trees are labelled -1. Then no matter which first move m_i Player 1 makes, Player 2 can enforce a win in the subsequent game t_i. Hence she can enforce a win in t and it gets label -1.

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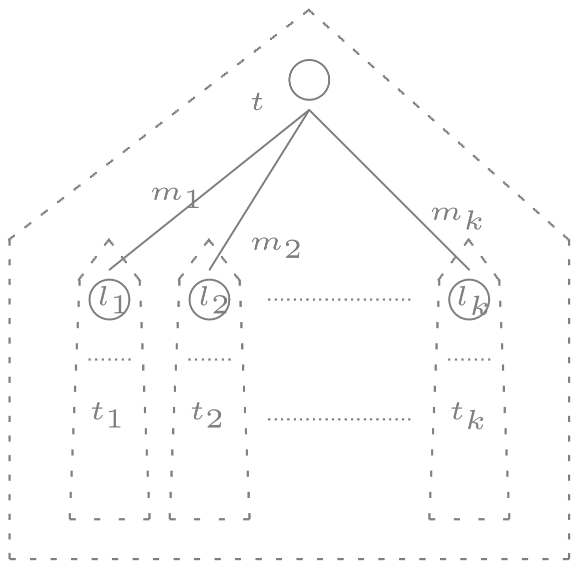
Induction step.



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The case where Player 2 makes the first move is analogous.

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