

Two hours

**UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE**

Mathematical Techniques for Computer Science

Time:

Please answer all THREE Questions

Use a SEPARATE Answerbook for each SECTION

Note that the last two pages contain inference rules for natural deduction

This is a CLOSED book examination

The use of electronic calculators is permitted provided they are not programmable and do not store text or formulas.

Section A

1. a) Show that for all i, j and k in \mathbb{Z} , (3 marks)

i divides j and i divides k implies i divides $j+k$.

- b) Consider the following function.

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C} \\ a+bi &\longmapsto ab+bi. \end{aligned}$$

Is this function injective? Is it surjective? Justify your answers. (5 marks)

- c) Let X be a set. Consider the binary operation on the powerset $\mathcal{P}X$ of X given by

$$\begin{aligned} \otimes: \mathcal{P}X \times \mathcal{P}X &\longrightarrow \mathcal{P}X \\ (S, T) &\longmapsto (S \cup T) \setminus (S \cap T). \end{aligned}$$

- i) Is the given operation commutative? Give a reason for your answer. (3 marks)
- ii) Is there a unit for the given operation? Justify your answer. (3 marks)
- d) Consider the following two functions from \mathbb{N} to \mathbb{N} :

$$f: x \longrightarrow x^2 \quad \text{and} \quad g: x \longrightarrow 100x + 100.$$

Does one of these functions eventually dominate the other? Justify your answer. (4 marks)

- e) Assume you know that you have a function f that has an inverse function. What can you say about the properties of f without making reference to its inverse function? (2 marks)

2. Assume you have a device that shows each of the following numbers with equal probability:

0, 2, 4, 8.

We will refer to it as a 'four-sided die', or just 'die' for the duration of this question.

- a) Give a probability space that describes the four-sided die being rolled. (2 marks)
- b) What is the probability that the number thrown is greater than or equal to 1? (1 mark)
- c) Assume two four-sided dice are rolled. What is the probability that the numbers shown, multiplied with each other, give a result greater than or equal to 30? (2 marks)
- d) What is the expected value of the sum of the numbers shown by rolling a four-sided die twice in succession? Justify your answer. (1 mark)
- e) What is the expected value of the product of the numbers shown by rolling a four-sided die twice in succession? Justify your answer. (2 marks)
- f) Let X be the random variable given by rolling the four-sided die once. Assume you can see the top of the number rolled, and you can see that this is a rounded shape $\hat{\cdot}$. What is the expected value of X given this information? (2 marks)
- g) Assume you are given a four-sided die but told that it gives one of the possible numbers with probability $2/5$, and the remaining three each with probability $1/5$.
You want to carry out Bayesian updating to work out which number the die is most likely to show. Carry out Bayesian updating, assuming that you roll the die and get 4, and that you roll it again and get 8. (10 marks)

Section B

3. a) Construct the truth table for the formula: (2 marks)

$$(P \leftrightarrow Q) \rightarrow (Q \wedge \neg P).$$

- b) Give a brief explanation of **two** of the following. (4 marks)

- i) propositional formula
- ii) Boolean valuation
- iii) equivalent replacement
- iv) natural deduction proof

- c) Consider the propositional formula

$$\neg((P \rightarrow Q) \wedge \neg R) \vee (\neg P \wedge \neg Q)$$

- i) Give a CNF for the given propositional formula. (3 marks)
 - ii) Simplify your previous result further if you can. (2 marks)
 - iii) Give a reason that the given formula is not a tautology. (1 mark)
- d) Give a natural deduction proof to show this formula is a tautology. Justify every step in your proof. (4 marks)

$$\neg A \rightarrow (A \rightarrow B)$$

Use the inference rules of our natural deduction system, which are given on the last pages of this exam paper.

- e) Consider the first-order language with the three binary predicate symbols K, D, E , one unary predicate symbol S , two constants a, b and a supply of variables x, y, z, \dots

Express each of the following sentences as formulas. (4 marks)

Use

$K(x, y)$	for x knows y	$S(x)$	for x is a sportscar
$D(x, y)$	for x drives y	a	for Adam
$E(x, y)$	for $x = y$	b	for Ben

- i) Ben knows someone who drives a sportscar.
- ii) Not everyone knows someone who drives a sportscar.
- iii) The only one Adam knows who drives a sportscar is Ben.

Rules of inference of our propositional natural deduction system

Conjunction elimination:

If $A \wedge B$ is derivable from a set of formulas, then so is A , and also B .

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

Conjunction introduction:

If A is derivable from a set of formulas, and B is derivable from the same set, then $A \wedge B$ is derivable from this set as well.

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

Disjunction introduction:

If A is derivable from a set, then so is $A \vee B$, and also $B \vee A$.

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash B \vee A}$$

Disjunction elimination (proof by cases):

If $A \vee B$ is derivable from a set and C is derivable from the set along with A , and also from the set along with B , then C is derivable from the set alone.

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}$$

Implication introduction:

If B is derivable from A and a set, then $A \rightarrow B$ is derivable from the set .

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$$

Implication elimination:

If A is derivable from a set, and $A \rightarrow B$ is derivable from the same set, then B is derivable from this set.

$$\frac{\Gamma \vdash A \quad \Gamma \vdash A \rightarrow B}{\Gamma \vdash B}$$

Negation introduction (reductio ad absurdum):

If A and a set leads to a contradiction, then $\neg A$ can be inferred from the set.

$$\frac{\Gamma, A \vdash \perp}{\Gamma \vdash \neg A}$$

Negation elimination:

If A is derivable from a set, and also $\neg A$ is derivable from the set, then anything (including \perp) is derivable from the set.

$$\frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash B}$$

Double negation introduction:

If A is derivable from a set, then $\neg\neg A$ is derivable from the same set.

$$\frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A}$$

Double negation elimination:

If $\neg\neg A$ is derivable from a set, then A is derivable from the same set.

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

Axiom (starting point):

A can always be inferred from A and a set of formulas.

$$\overline{\Gamma, A \vdash A}$$

Weakening:

New assumptions may be introduced at any point in a derivation.

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$