Two hours

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

Time:

Please answer all THREE Questions

Use a SEPARATE Answerbook for each SECTION
Note that the last two pages contain inference rules for natural deduction

This is a CLOSED book examination

The use of electronic calculators is permitted provided they are not programmable and do not store text.

Section A

1. a) Prove the following statement by induction: (4 marks)

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

b) Recall the recursively defined notion of a list over a set S, and the length function on such lists given by

Base case len. $\text{len}\,[] = 0$
Step case len. $\text{len}(s : l) = 1 + \text{len}\, l.$

i) Give a recursive definition for a function rpt that takes as input a list over $\mathbb{Z}$ and returns another such list, where each element of the list is repeated. The expected output of this function for the list $[2,1,3]$, for example, is (3 marks)

$$[2,2,1,1,3,3].$$

ii) In the notes there is a Java class List which is the direct counterpart of a recursively defined list. For this class give a method that implements the function described in part (i) by filling in the two return cases in the following Java program. (3 marks)

```
public static List rpt (List l)
{
if (l == null)
    return
else
    return
}
```

iii) Prove by induction that for all lists l over S we have (5 marks)

$$\text{len}(\text{rpt}\, l) = 2\,\text{len}\, l.$$

c) Show by induction that for all $n \in \mathbb{N}$, (5 marks)

$$8^n - 3^n \qquad \text{is divisible by 5.}$$

2. a) Answer the following questions regarding modular arithmetic.

   i) What can you say about the existence of a multiplicative inverse for $6 \pmod 9$? Justify your answer. (1 mark)

   ii) Calculate the multiplicative inverse of $7 \pmod{15}$. (2 marks)

b) This part of the question is concerned with equivalence relations. (6 marks)

   i) For the following relations on the set $\mathsf{Lists}_{\mathbb{N}}$ of lists over the set $\mathbb{N}$ check whether they are equivalence relations. Justify your answer.

   - Firstly, the relation

   $$l \sim l' \qquad \text{iff} \qquad l \text{ and } l' \text{ have at least one element in common.}$$

   - Secondly, the relation

   $$l \sim l'$$

   if and only if

   – either both l and l' are empty,

   – or they both have a most recently added element, and these are equal. In other words in this case we have two lists of the form

   $$s : l \qquad \text{and} \qquad s : l'.$$

   ii) For any equivalence relations identified above, describe the equivalence class of [1].

c) Which of the following relations are partial orders? Justify your answer. (5 marks)

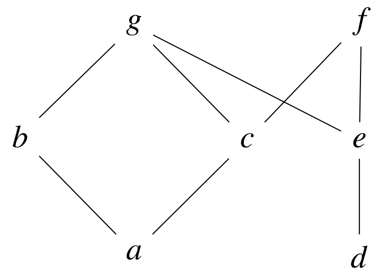
   i) The relation on $\mathbb{C}$ where

   $$z \sqsubseteq z' \qquad \text{iff} \qquad |z| \leq |z'| \text{ and the argument of } z \text{ is less than or equal to the argument of } z'.$$

   ii) The relation on $\mathbb{N}$ where

   $$m \sqsubseteq n \qquad \text{iff} \qquad m^2 \leq n.$$

d) Consider the poset P given by the following Hasse diagram.



i) What are the minimal elements of P? Is there a least element? (1 mark)

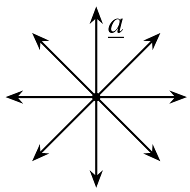
ii) What are the maximal elements of P? Is there a greatest element? (1 mark)

iii) What are the upper bounds of $\{c, e\}$? Is there a least upper bound for this set? (2 marks)

iv) What are the lower bounds of $\{b, f\}$? Is there a greatest lower bound for this set? (2 marks)

Section B

3. a) Consider the vectors in the following diagram.



i) State what is the sum of the vectors in the diagram? (1 mark)

ii) What is the sum of the vectors in the diagram, if the vector named $\underline{a}$ (the upward pointing vector) is removed? (1 mark)

b) Consider the following matrices.

$$\underline{A} = \begin{pmatrix} -1 & 5 & \pi \\ 0 & -2 & 6 \end{pmatrix} \qquad \underline{B} = \begin{pmatrix} 4 & -1 \\ 1 & 5 \\ 0 & 3 \end{pmatrix}$$

Compute the following or say why it is not defined. (4 marks)

i) $\underline{A} + \underline{B}$
ii) $\underline{A}^T + \underline{B}$
iii) $2 - \underline{AB}$

c) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ -y \end{pmatrix}.$$

i) Give the matrix representation of T. (1 mark)
ii) Define a linear transformation $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

$$F \circ T(\underline{v}) = \underline{v},$$

for any vector $\underline{v} \in \mathbb{R}^2$. (3 marks)

d) A square matrix is a diagonal matrix, if all the entries off the main diagonal are zero.

i) Which of these matrices are diagonal and which are not? (2 marks)

$$\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$$

ii) Let $\underline{D}$ be an $n \times n$ diagonal matrix and suppose $k_1, k_2, \ldots, k_n$ are the entries on the main diagonal of $\underline{D}$, i.e., if $\underline{D} = (d_{ij})$ then $d_{ii} = k_i$.
Describe the effect of pre-multiplying a matrix by a diagonal matrix $\underline{D}$. Use an example as illustration in your description. (3 marks)

e) Consider the the following system of linear equations.

$$\begin{aligned} 7x_1 + 5x_2 + 6x_3 &= 0 \\ 7x_1 + 2x_3 &= 0 \\ 5x_2 + 4x_3 &= 0 \end{aligned}$$

i) Write the augmented matrix for this system of equations. (1 mark)

ii) Compute the set of all solutions of this system. (3 marks)

iii) Modify the augmented matrix in your answer in i) so that the solution set of the associated system of linear equations is empty. (1 mark)

END OF EXAMINATION