Set of Support for Theory Reasoning

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Theory axioms in proofs

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However, in the meantime, the theory axioms may also yield:

$$\neg(x<y) \vee \neg(y<x)$$

or (perhaps less usefully):

$$\neg(x_0<x_1) \vee \neg(x_2<x_0) \vee \neg(x_1<x_3) \vee \neg(x_4<x_5) \vee \neg(x_3<x_4) \vee \neg(x_5<x_2)$$

Inferences between axioms

Example problem ARI176=1 from TPTP

$$3x + 5y \neq 22$$

can be shown unsatisfiable using axioms

$$x{+}y = y{+}x, \quad x{+}(y{+}z) = (x{+}y){+}z, \quad x{*}1 = x, \quad x{*}(y{+}z) = (x{*}y){+}(x{*}z)$$

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The derivation starts by:

$$\dfrac{\dfrac{x * 1 = x \qquad x * (y + z) = (x * y) + (x * z)}{x * (1 + y) = x + (x * y)} \qquad x + (y + z) = (x + y) + z}{(x * (1 + y)) + z = x + ((x * y) + z)}$$

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The problem cannot be solved in Vampire in reasonable time without first combining axioms among themselves

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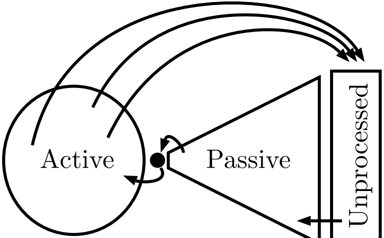
- Set of support is a well known idea to prevent explosion
- Idea 1: apply SOS for theory reasoning
- Idea 2: fine-tune this by allowing limited reasoning among theory axioms
- Preliminary evaluation of the technique

Outline

1. Saturation and Theory Reasoning in Vampire
2. The Set of Support Strategy
3. Set of Support for Theory Reasoning
4. Conclusion

Compute deductive closure of the input N wrt inferences $\mathcal{I}$:



Saturation-based Theorem Proving

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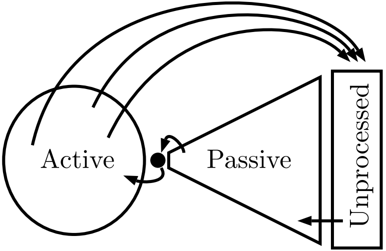

- clause selection schemes
- further aspects: literal selection, ordering restrictions, ...
- completeness considerations

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  - hand-crafted set
  - either all added or none added (based on an option)

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Some axioms

$$x + (y + z) = (x + y) + z \qquad x + 0 = x$$
$$x + y = y + x \qquad -(x + y) = (-x + -y)$$
$$- - x = x \qquad x + (-x) = 0$$
$$x * 0 = 0 \qquad x * (y * z) = (x * y) * z$$
$$x * 1 = x \qquad x * y = y * x$$
$$(x * y) + (x * z) = x * (y + z) \qquad \neg(x < y) \vee \neg(y < z) \vee \neg(x < z)$$
$$x < y \vee y < x \vee x = y \qquad \neg(x < y) \vee \neg(y < x + 1)$$
$$\neg(x < y) \vee x + z < y + z \qquad \neg(x < x)$$
$$x < y \vee y < x + 1 \ (for\ ints) \qquad x = 0 \vee (y * x)/x = y \ (for\ reals)$$

Axioms can be "explosive"

ARI581=1.p

```
tff(mix_quant_ineq_sys_solvable_2,conjecture,(
  ! [X: $int] : ( $less(5,X) =>
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- default strategy with all axioms: not solved in 60 s
- remove commutativity of $+$: solved instantly

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SYN000=2.p

- “test tptp theory syntax” benchmark
- Vampire in default: 223 clauses (90 theory consequences, 1 used in the proof)
- negate the conjecture, run for 10 s:
  456 973 clauses (98 % are consequences of theory axioms)

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The Set of Support Strategy

Basic idea:

- split the input clauses into a set of support and the rest
- restrict inferences to involve at least one premise from SOS
- new clauses are added to SOS

"Every inference must have an ancestor in the initial SOS."

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In practice:

- just put non-SOS clauses directly to active
- define SOS = clauses from the conjecture
  - Note: benchmarks without explicit conjecture SOS-suck

SOS in Vampire

Vampire's `-sos` option values:

- `off`: do not use SOS
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Experiment (relevant TPTP v6.4.0, 300 s)

	competition mode	competition mode with `sos=off`
Solved	11 948	11 613
Uniques	422	87

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SOS and theory axioms

- the whole input problem is the SOS
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Experiment (relevant SMTLIB, default strategy, 60 s)

	default mode	default mode + `sos=theory`
Solved	32 769	32 522
Uniques	641	394

How deep is theory reasoning?

Mining proofs for statistics:

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Experiment (relevant SMTLIB, default strategy, 60 s)

Depth	count
0	31 959
1	209
2	304
3	200
4	49
5	21
6	27

What do useful pure theory consequences look like?

Example (deep pure theory consequences)

$$0 < x \vee x < 4$$

from `UFLIA/sledgehammer/TwoSquares/z3.637729.smt2`

$$\neg((x + (y + ((-x) + 2.0))) < y) \quad \text{and} \quad \neg(2.0 + x < x)$$

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Note that:

- large constants must be obtained by combining the basic axioms
- a clumsy search for a useful instance?

Explicitly liming depth of pure theory consequences

Depth	Count when threshold = 0	1	2	3	5	10	∞
0	32 522	32 253	32 130	32 061	32 162	32 040	31 959
1		552	237	209	216	208	209
2			551	314	310	307	304
3				312	254	212	200
4					69	48	49
5					61	21	21
6						26	27
total	32 522	32 805	32 918	32 896	**33 072**	32 863	32 769

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- decreasing T can also dramatically <u>increase</u> the solution time and length of the found proof

Experiment (relevant SMTLIB, smtcomp mode, 1800 s)

	competition mode	`set sos=theory threshold=5`
Solved	37 009	36 821
Uniques	254	66

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Thank you for your attention!