Pretending to be an SMT Solver with Vampire (and How We Do Instantiation)

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Introducing Vampire

- Automatic Theorem Prover (ATP) for first-order logic
- Main paradigm: superposition calculus + saturation
- a.k.f.: indexing, incomplete strategies, strategy scheduling

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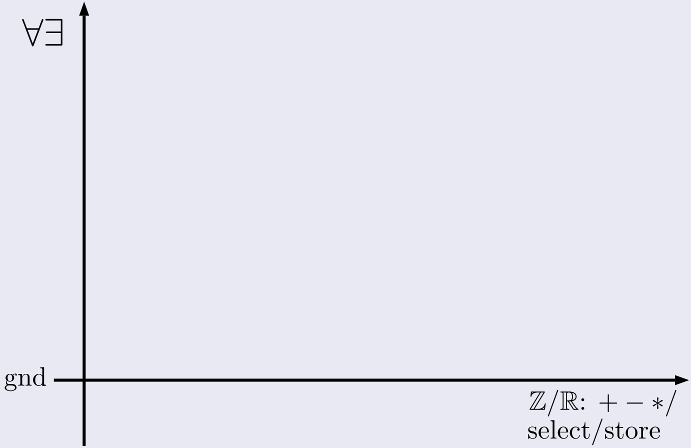
Reasoning with Theories

- since 2010: progressively adding support for theories
- since 2016: participating in SMT-COMP

Reasoning with quantifiers and theories

Two Dimensions of Complexity



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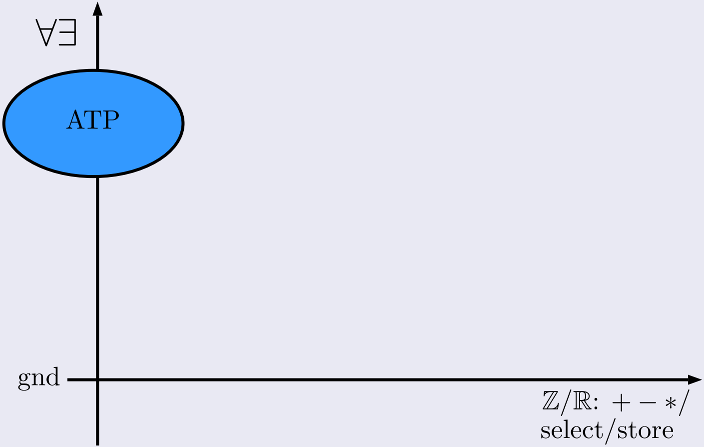

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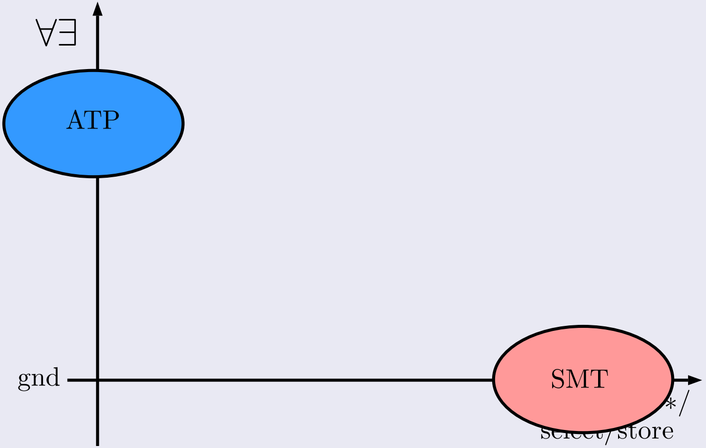

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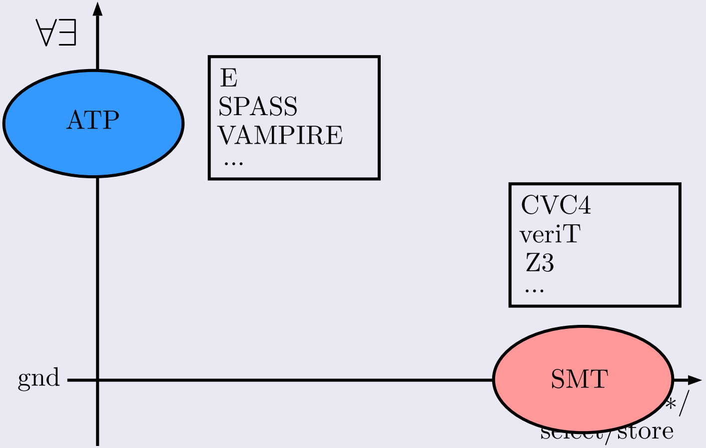

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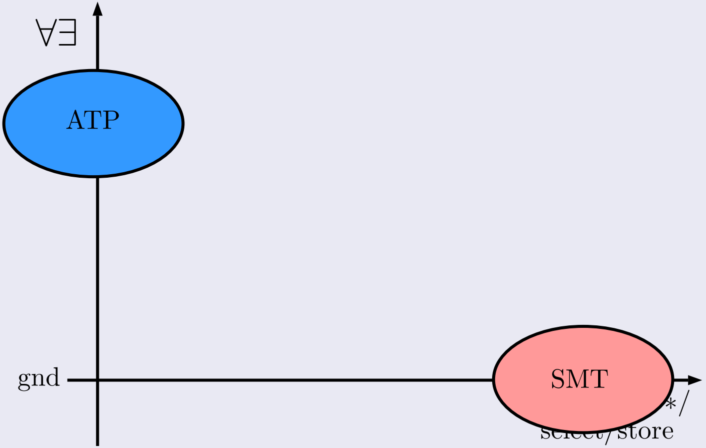

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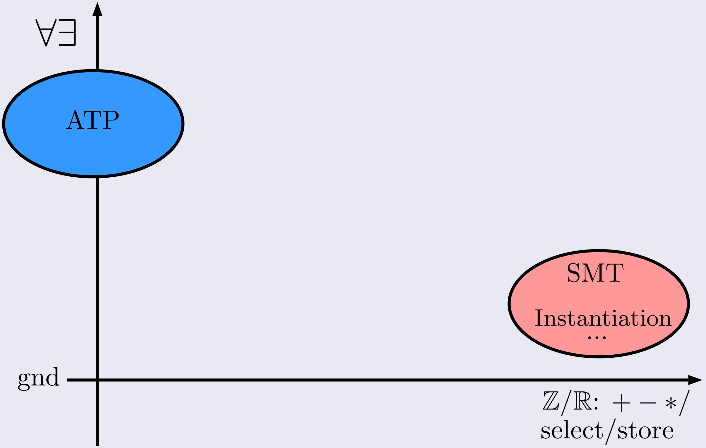

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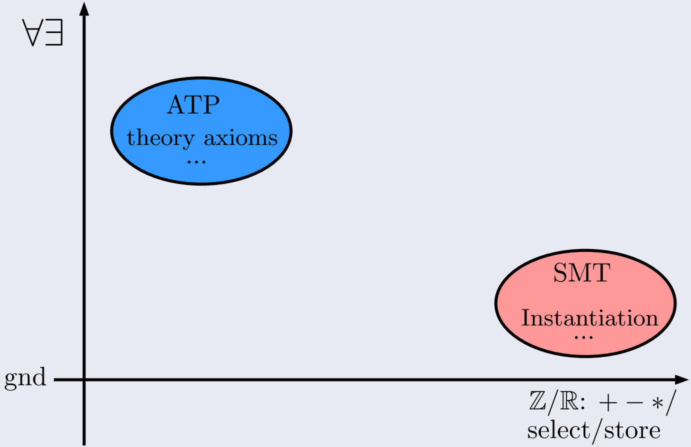

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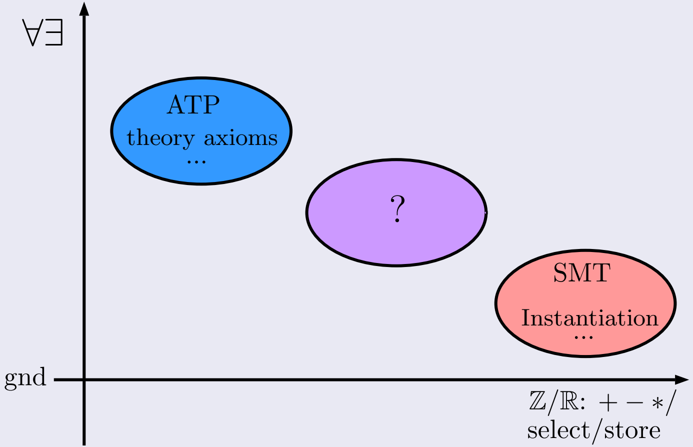

Outline

1. A Brief Introduction to Saturation-Based Proving
2. Theory Reasoning in Vampire
3. Theory Instantiation and Unification with Abstraction
4. Where We Currently Stand

Theorem Proving Pipeline in One Slide

Standard form of the input:

$$F \quad := \quad (Axiom_1 \land \ldots \land Axiom_n) \to Conjecture$$

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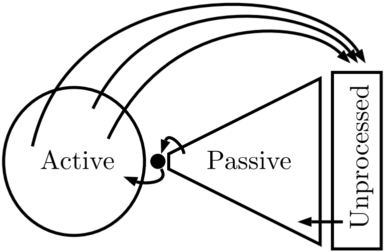
$$\mathcal{S} \quad := \quad \{C_1, \ldots, C_n\}$$

3. saturate $\mathcal{S}$ with respect to the superposition calculus

aiming to derive the obvious contradiction $\perp$

Saturation = fixed-point computation

Given Clause Algorithm:



- set of active clauses is stored in indexing structures
- passive works like a priority queue
- the process is "explosive" in nature

Superposition rule

$$\frac{\underline{l \simeq r} \vee C_1 \quad \underline{L[s]_p} \vee C_2}{(L[r]_p \vee C_1 \vee C_2)\theta} \quad \text{or} \quad \frac{\underline{l \simeq r} \vee C_1 \quad \underline{t[s]_p \otimes t'} \vee C_2}{(t[r]_p \otimes t' \vee C_1 \vee C_2)\theta},$$

where $\theta = \mathrm{mgu}(l, s)$ *and* $r\theta \not\succeq l\theta$ *and, for the left rule* $L[s]$ *is not an equality literal, and for the right rule* $\otimes$ *stands either for* $\simeq$ *or* $\not\simeq$ *and* $t'\theta \not\succeq t[s]\theta$

Controlling the Growth of the Search Space

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Saturation up to Redundancy

- redundant clauses can be safely removed
- subsumption - an example reduction:

remove C in the presence of D such that $D\sigma \subset C$

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Completeness considerations

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Basic Support for Theories

- Normalization of interpreted operations, e.g.

$$t_1 \geq t_2 \rightsquigarrow \neg(t_1 < t_2) \qquad a - b \rightsquigarrow a + (-b)$$

- Evaluation of ground interpreted terms, e.g.

$$f(1+2) \rightsquigarrow f(3) \qquad f(x+0) \rightsquigarrow f(x) \qquad 1+2 < 4 \rightsquigarrow \mathit{true}$$

- Balancing interpreted literals, e.g.

$$4 = 2 \times (x+1) \rightsquigarrow (4 \text{ div } 2) - 1 = x \rightsquigarrow x = 1$$

- Interpreted operations treated specially by ordering

Adding Theory Axioms

$$
\begin{array}{cc}
x + (y + z) = (x + y) + z & x + 0 = x \\
x + y = y + x & -(x + y) = (-x + -y) \\
- - x = x & x + (-x) = 0 \\
x * 0 = 0 & x * (y * z) = (x * y) * z \\
x * 1 = x & x * y = y * x \\
(x * y) + (x * z) = x * (y + z) & \neg(x < y) \vee \neg(y < z) \vee \neg(x < z) \\
x < y \vee y < x \vee x = y & \neg(x < y) \vee \neg(y < x + 1) \\
\neg(x < y) \vee x + z < y + z & \neg(x < x) \\
x < y \vee y < x + 1 \ (for\ ints) & x = 0 \vee (y * x)/x = y \ (for\ reals)
\end{array}
$$

- a handcrafted set
- subsets added based on the signature
- ongoing research on how to tame them [IWIL17]

The AVATAR architecture [Voronkov14]

- modern architecture of first-order theorem provers
- combines saturation with SAT-solving
- efficient realization of the *clause splitting rule*

$$\forall x, z, w.\ \underbrace{s(x) \lor \neg r(x, z)}_{share\ x\ and\ z} \lor \underbrace{\neg q(w)}_{is\ disjoint}$$

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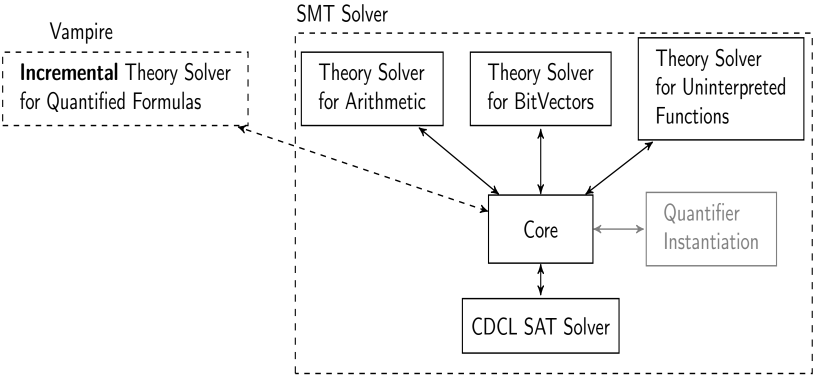
AVATAR modulo Theories

- use an SMT solver instead of the SAT solver
- sub-problems considered are **ground-theory-consistent**
- implemented in Vampire using Z3

One Slightly Imprecise View of AVATAR



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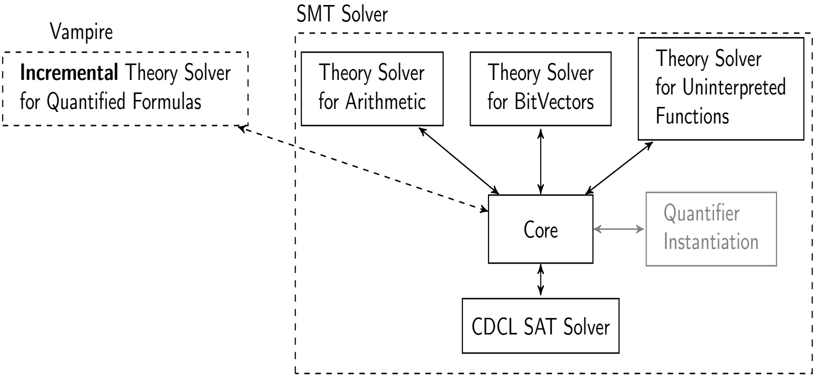

…and please remember: Vampire is the boss here!

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Does Vampire Need Instantiation?

Example

Consider the conjecture $(\exists x)(x + x \simeq 2)$ negated and clausified to

$$x + x \not\simeq 2.$$

It takes Vampire 15 s to solve using theory axioms deriving lemmas such as

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Recall the abstraction rule

$$L[t] \vee C \implies x \not\simeq t \vee L[x] \vee C,$$

where L is a theory literal, t a non-theory term, and x fresh.

The Theory Instantiation

Instantiation which makes some theory literals immediately false

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As an inference rule

$$\frac{C}{(D[\mathsf{x}])\theta}\ \textit{TheoryInst}$$

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Implementation:

- Abstract relevant literals
- Collect relevant pure theory literals $L_1, \ldots, L_n$
- Run an SMT solver on $T[\mathsf{x}] = \neg L_1 \wedge \ldots \wedge \neg L_n$
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Finally, Theory Instantiation could produce

$$p(7)$$

Unification with Abstraction

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Instead of full abstraction ...

- incorporate the abstraction process into unification
- thus abstractions are "on demand" and lazy
- implemented by extending the substitution tree indexing

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SMT-COMP 2017 results – ∀∃ problems

Logic	Vampire	VeriT	CVC4	Z3
ALIA	36	27	**42**	**42**
AUFDTLIA	624	-	**728**	-
AUFLIA	**3**	2	**3**	2
AUFLIRA	19778	19316	19766	**19849**
AUFNIRA	**1072**	-	1052	1031
LIA	229	170	**388**	**388**
LRA	1092	-	2048	**2208**
NIA	5	-	9	**13**
NRA	3803	-	3776	**3805**
UF	**4317**	3242	4125	2846
UFDT	2283	-	**2503**	-
UFDTLIA	**75**	-	73	-
UFIDL	55	55	**60**	59
UFLIA	7559	7518	**7687**	7221
UFLRA	10	10	11	**12**
UFNIA	**2561**	-	2189	2197

Conclusion

Thank you for your attention!