Notes on Constructive Set Theory
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1 Introduction

The general topic of Constructive Set Theory originated in the seminal 1975 paper of John Myhill, where a specific axiom system CST was introduced. Constructive Set Theory provides a standard set theoretical framework for the development of constructive mathematics in the style of Errett Bishop[1] and is one of several such frameworks for constructive mathematics that have been considered. It is distinctive in that it uses the standard first order language of classical axiomatic set theory [2] and makes no explicit use of specifically constructive ideas. Of course its logic is intuitionistic, but there is no special notion of construction or constructive object. There are just the sets, as in classical set theory. This means that mathematics in constructive set theory can look very much like ordinary classical mathematics. The advantage of this is that the ideas, conventions and practise of the set theoretical presentation of ordinary mathematics can be used also in the set theoretical development of constructive mathematics, provided that a suitable discipline is adhered to. In the first place only the methods of logical reasoning available in intuitionistic logic should be used. In addition only the set theoretical axioms allowed in constructive set theory can be used. With some practise it is not dificult for the constructive mathematician to adhere to this discipline.

Of course the constructive mathematician is concerned to know that the axiom system she is being asked to use as a framework for presenting her mathematics makes good constructive sense. What is the constructive notion of set that constructive set theory claims to be about? I believe that I have answered this question in a series of three papers on the Type Theoretic Interpretation of Constructive Set Theory. These papers are based on taking Martin-Lof's Constructive Type Theory as the most acceptable foundational framework of ideas that make precise the constructive approach to mathematics. I show how a particular type of the type theory can be used as the type of sets forming a universe of objects to interprete constructive set theory so that by using the Curry-Howard 'propositions as types' idea the axioms of constructive set theory get interpreted as provable propositions.

Why not present constructive mathematics directly in the type theory? This is an obvious option for the constructive mathematician. It has the drawback that there is no extensive tradition of presenting mathematics in a type theoretic setting. So, many techniques for representing mathemat-

[1] See *Constructive Analysis*, by Bishop and Bridges

[2] Myhill's original paper used some other primitives in CST besides the notion of set. But this was inessential and I prefer to keep to the standard language in the axiom systems that I use.

ical ideas in a set theoretical language have to be reconsidered for a type theoretical language. This can be avoided by keeping to the set theoretical language.

Surprisingly there is still no extensive presentation of an approach to constructive mathematics that is based on a completely explicitly described axiom system - neither in constructive set theory, constructive type theory or any other axiom system.

One of the aims of these notes is to initiate an account of how constructive mathematics can be developed on the basis of a set theoretical axiom system. At first we will be concerned to prove each basic result relying on as weak an axiom system as possible. But later we will be content to explore the consequences of stronger axiom systems provided that they can still be justified on the basis of the type theoretic interpretation. Because of the open ended nature of constructive type theory we also think of constructive set theory as an open ended discipline in which it may always be possible to consider adding new axioms to any given axiom system.

In particular there is current interest in the formulation of stronger and stronger notions of type universes and hierarchies of type universes in type theory. This activity is analogous to the pursuit of ever larger large cardinal principles by classical set theorists. In the context of constructive set theory we are led to consider set theoretical notions of universe. As an example there is the notion of inaccessible set of Griffor and Rathjen.[3] An aim of these notes is to lay the basis for a thorough study of the notion of inaccessible set and other notions of largeness in constructive set theory.

A further motivation for these notes is the current interest in the development of a 'formal topology' in constructive mathematics. It would seem that constructive set theory may make a good setting to represent formal topology. We wish to explore the extent to which this is indeed the case.

These notes represent work in progress and are necessarily very incomplete and open to change.

[3]'See 'Inaccessibilty in Constructive Set Theory and Type Theory'.

2 Some Axiom Systems

Constructive Set Theory is a variant of Classical Set Theory which uses intuitionistic logic. It differs from another such variant called Intuitionistic Set Theory because of its avoidance of the full impredicativity that Intuitionistic Set Theory has. Constructive Set Theory does not have the Powerset axiom or the full Separation Axiom Scheme. We introduce constructive set theory here by contrasting it with the other two theories. Note that we consider each theory as a framework and consider representative axiom systems for each of them, ZF and IZF for the Classical and Intuitionistic set Theories and CZF_0, CZF and $BCST$ for Constructive Set Theory.

2.1 Classical Set Theory

The classical Zermelo-Fraenkel axiomatic set theory, ZF, is formulated in first order logic with equality, using a binary predicate symbol $\in$ as its only non-logical symbol. We will use $\mathbb{S}y\theta(y)$ to abbreviate

$$\exists b\forall y[y \in b \ \leftrightarrow \ \theta(y).$$

The axiom system ZF uses the axioms and axiom schemes

Pairing

$$\forall a\forall b\mathbb{S}y[y = a \ \vee \ y = b]$$

Union

$$\forall a\mathbb{S}y\exists x{\in}a[y \in x]$$

Powerset

$$\forall a\mathbb{S}y\forall x{\in}y[x \in a]$$

Infinity

$$\exists a\ [\exists x\ x \in a \ \wedge \ \forall x{\in}a\exists y{\in}a\ x \in y]$$

Extensionality

$$\forall a\forall b[\forall x[x \in a \ \leftrightarrow \ x \in b] \rightarrow a = b]$$

Foundation

$$\forall a[\exists x[x \in a] \ \rightarrow \ \exists x{\in}a\forall y{\in}a[y \notin x]]$$

Separation

$$\forall \cdots \forall a \mathbb{S} x\, [x \in a \wedge \phi(x, \ldots)]$$

for all formulae $\phi(x, \ldots)$.

Replacement

$$\forall \cdots \forall a [\forall x{\in}a \exists ! y \phi(x, y, \ldots) \ \rightarrow \ \mathbb{S} y \exists x{\in}a \phi(x, y, \ldots)]$$

for all formulae $\phi(x, y, \ldots)$.

2.2 Intuitionistic Set Theory

A natural Intuitionistic version of ZF is Intuitionistic Zermelo-Fraenkel (IZF). It is like ZF except that the following changes are made.

1. It uses Intuitionistic logic instead of Classical logic.
2. It uses the Set Induction Scheme instead of the Foundation axiom.
3. It uses the Collection Scheme instead of the Replacement Scheme.

Set Induction Scheme:

$$\forall \cdots \forall a\, [\forall x{\in}a \phi(x, \ldots) \rightarrow \ \phi(a, \ldots)] \ \ \rightarrow \ \ \forall a \phi(a, \ldots)$$

Collection Scheme:

$$\forall \cdots \forall a [\forall x{\in}a \exists y \theta(x, y, \ldots) \ \ \rightarrow \ \ \exists b\, \forall x{\in}a \exists y{\in}b\ \theta(x, y, \ldots)]$$

2.3 Constructive Set Theory

2.3.1 CZF_0

Our first axiom system for Constructive Set Theory, CZF_0, is like IZF except for the following changes.

1. It uses the Replacement Scheme instead of the Collection Scheme.
2. It drops the Powerset Axiom.
3. It uses the Restricted Separation Scheme instead of the full Separation Scheme.

Restricted Separation Scheme

$$\forall \cdots \forall a \mathbb{S} x\, [x \in a \wedge \phi(x, \ldots)]$$

for all restricted formulae $\phi(x, \ldots)$. A formula is restricted if all its quantifiers are restricted; i.e. occur only in one of the forms $\exists x{\in}y$ or $\forall x{\in}y$.

2.3.2 Constructive Zermelo Fraenkel, CZF

The constructive Set Theory CZF is obtained from CZF_0 as follows.

1. Add the Subset Collection Scheme.
2. Use the Strong Collection Scheme instead of the Replacement Scheme.

In stating these schemes we shall use $\mathbb{B}(x{\in}a, y{\in}b)\phi$ as an abbreviation for

$$\forall x{\in}a\exists y{\in}b\phi \ \wedge \forall y{\in}b\exists x{\in}a\phi.$$

Subset Collection Scheme:

$$\forall a\forall b\exists c\ [\ \forall u\ \forall x{\in}a\exists y{\in}b\ \theta(x,y,u) \rightarrow\ \exists z{\in}c\ \mathbb{B}(x{\in}a, y{\in}z)\theta(x,y,u)\ ]$$

Strong Collection Scheme:

$$\forall\cdots\forall a[\ \forall x{\in}a\exists y\theta(x,y,\ldots)\ \rightarrow\ \exists b\ \mathbb{B}(x{\in}a, y{\in}b)\ \theta(x,y,\ldots)\ ]$$

2.3.3 Basic Constructive Set Theory, $BCST$

The subsystem $BCST$ of CZF_0 is obtained by leaving out Infinity and Restricted Separation, but putting in two axioms, Emptyset and Infimum. These two axioms are consequences of Restricted Separation and we show that Restricted Separation can be proved in $BCST$, so that $BCST$ has the same theorems as CZF_0−Infinity. The axioms and schemes of $BCST$ are given below. We use $Empty(y)$ to abbreviate $\neg\exists y[y \in x]$.

Extensionality

$$\forall a\forall b[\forall x[x \in a\ \leftrightarrow\ x \in b] \rightarrow a = b]$$

Set Induction

$$\forall\cdots\forall a[\forall x{\in}a\ \phi(x,\cdots) \rightarrow \phi(a)] \rightarrow \forall a\phi(a,\cdots)$$

Pairing

$$\forall a\forall b\mathbb{S}y[y = a\ \vee\ y = b]$$

Union-Replacement

$$\forall\cdots\forall a[\forall x{\in}a\mathbb{S}y\phi(x,y,\cdots)\ \rightarrow\ \mathbb{S}y\exists x{\in}a\phi(x,y,\cdots)]$$

Emptyset

$$\mathbb{S}y\perp$$

Infimum

$$\forall a[\forall x{\in}a\forall y{\in}x\ Empty(y)\ \rightarrow\ \mathbb{S}y[Empty(y)\ \wedge\ \forall x{\in}a\ y \in x]]$$

3 Elementary Mathematics in Constructive Set Theory

We show how to develop some of the standard apparatus for representing mathematical ideas in weak axiom systems for constructive set theory.

3.1 Class Notation

In doing mathematics in Constructive Set Theory we shall exploit the use of class notation and terminology, just as in Classical Set Theory. So, if $\phi(x)$ is a first order formula in the language of set theory with one free variable x, that may have parameters for sets, then we may define a class A.

$$A = \{x \mid \phi(x)\}$$

with defining axiom

$$\forall x[\ x \in A \ \ \leftrightarrow \ \ \phi(x)\ ].$$

In particular each set a is identified with the class $\{x \mid x \in a\}$. Classes A, B are defined to be equal if

$$\forall x[x \in A \ \leftrightarrow \ x \in B].$$

Also, A is a subclass of B, written $A \subseteq B$, $\forall x \in A\ x \in B$. So, without assuming any non-logical axioms we may form the following classes, where A, B, C are classes and $a, a_1, \ldots, a_n$ are sets.

1. $\{a_1, \ldots, a_n\} = \{x \mid x = a_1 \vee \cdots \vee x = a_n\}$. When $n = 0$ this is the empty class $\emptyset$.
2. $\bigcup A = \{x \mid \exists y{\in}A\ x \in y\}$.
3. $A \cup B = \{x \mid x \in A \vee x \in B\}$.
4. $a^+ = a \cup \{a\}$.
5. $Pow(A) = \{x \mid x \subseteq A\}$.
6. $V = \{x \mid x = x\}$.

If A is a class and $\theta(x, y)$ is a formula in the language of set theory, that may have parameters for sets and has at most the variables x, y occuring free, then we may form a **family of classes** $(B_a)_{a\in A}$, where for each $a \in A$

$$B_a = \{y \mid \theta(a, y)\}.$$

If $(B_a)_{a\in A}$ is a family of classes then we may form the class

$$\bigcup_{a\in A} B_a = \{y \mid \exists a{\in}A\ y \in B_a\}.$$

3.2 Pairing

Here we introduce unordered and ordered pairs, cartesian products of classes and relations and function between classes. We will only use the axioms of Extensionality and Pairing.

The Pairing axiom asserts that for all sets a, b the class $\{a, b\}$ is a set, the unordered pair of a and b. In particular, when $a = b$ we get that the singleton $\{a\}$ is a set.

3.2.1 Ordered Pairs

We define ordered pairs as usual. For sets a, b let

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

Proposition: 3.1

$$(a, b) = (c, d) \Rightarrow [a = c \ \wedge \ b = d].$$

Proof: The usual classical proof argues by cases depending, for example, whether or not $a = b$. This method is not available here as we cannot assume that instance of the classical law of excluded middle. Instead we can argue as follows. Assume that $(a, b) = (c, d)$.

As $\{a\}$ is an element of the left hand side it is also an element of the right hand side and so either $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$. In either case $a = c$.

As $\{a, b\}$ is an element of the left hand side it is also an element of the right hand side and so either $\{a, b\} = \{c\}$ or $\{a, b\} = \{c, d\}$. In either case $b = c$ or $b = d$. If $b = c$ then $a = c = b$ so that the two sets in (a, b) are equal and hence $\{c\} = \{c, d\}$ giving $c = d$ and hence $b = d$. So in either case $b = d$.

■

3.2.2 Cartesian Products of Classes

For classes A, B let $A \times B$ be the class given by

$$A \times B = \{z \mid \exists a{\in}A \exists b{\in}B\ z = (a, b)\}.$$

3.2.3 Relations and Functions between Classes

If R is a class of ordered pairs then we use aRb for $(a,b) \in R$. If A, B are classes and $R \subseteq A \times B$ such that

$$\forall x{\in}A \exists y{\in}B \; xRy$$

then we will write

$$R : A \rightarrowtail B$$

and if also

$$\forall y{\in}B \exists x{\in}A \; xRy$$

then we write

$$R : A \rightarrowtail\!\!< B.$$

If

$$\forall x{\in}A \exists ! y{\in}B \; xRy$$

then we use the standard notation

$$R : A \to B,$$

and for each $a \in A$ we write $R(a)$ for the unique $b \in B$ such that aRb.

3.3 The class of Natural Numbers

In this part we need only assume the axioms of Extensionality, Emptyset and Successor and the Set Induction Scheme.

By the Emptyset axiom the class $\emptyset$ is a set. The Successor Axiom asserts that for all sets a, b the class $a \cup \{b\}$ is a set.

Successor

$$\forall a \forall b \mathbb{S} y [y \in a \lor y = b]$$

Note that Pairing is an easy consequence of Emptyset and Successor. Also Successor is an easy consequence of the Union and Pairing Axioms and implies that the class $a^+ = a \cup \{a\}$ is a set for each set a. We will define the natural numbers as sets in the standard von Neumann manner so that $0 = \emptyset$, $1 = 0^+ = \{0\}$, etc. Let $Trans(X)$ if X is a transitive set; i.e. $\forall a{\in}X \; a \subseteq X$. We define

$$\mathbb{N} = \{y \mid \exists X [Trans(X) \land y \in X \land \forall a{\in}X (a = 0 \lor \exists b (a = b^+))]\}.$$

Proposition: 3.2 $\mathbb{N}$ *is the smallest class* Y *such that*

$$(*) \qquad 0 \in Y \ \wedge \ \forall a{\in}Y\ a^+ \in Y.$$

Proof: Let $Z = \{a \mid a = 0 \vee \exists b(a = b^+)\}$. So

$$\mathbb{N} = \{y \mid \exists X[Trans(X) \wedge y \in X \wedge X \subseteq Z]\}.$$

Using the transitive set $X = \{0\} \subseteq Z$ we see that $0 \in \mathbb{N}$. If $a \in \mathbb{N}$ then $a \in X$ for some transitive set $X \subseteq Z$. Then $a^+ \in X'$ where $X' = X \cup \{a^+\}$ is a transitive subset of Z so that $a^+ \in \mathbb{N}$. Thus $\mathbb{N}$ is a class Y satifying $(*)$.

We now show that, assuming that Y is a class satisfying $(*)$, $\mathbb{N} \subseteq Y$. We must show that if X is a transitive subset of Z then $X \subseteq Y$. So let X be a transitive subset of Z . We will use Set Induction on a to show that

$$a \in X \Rightarrow a \in Y.$$

So we may assume as induction hypothesis that

$$\forall x{\in}a[x \in X \to x \in Y].$$

Now let $a \in X$. Then $a \subseteq X$, as X is transitive, so that, by the induction hypothesis, $a \subseteq Y$. Also $a \in Z$ so that $a = 0$ or $a = b^+$ for some b. If $a = 0$ then $a \in Y$, as Y satisfies $(*)$. If $a = b^+$ then $b \in a$ so that $b \in Y$ and, as Y satisfies $(*)$, $a \in Y$.
■

It is easy to check that the class $\mathbb{N}$ with the object 0 and operator $(\ )^+$ satisfy the Dedekind-Peano axioms for the natural numbers. That mathematical induction holds is an immediate consequence of the inductive definition of $\mathbb{N}$. A basic result in axiomatic set theory is the principle of definition by primitive recursion.

Theorem: 3.3 *Let* A *be class,* $a_0 \in A$ *and* $F : \mathbb{N} \times A \to A$*. Then there is a unique function* $H : \mathbb{N} \to A$ *such that*

$$\begin{cases} H(0) & = a_0 \\ H(n^+) & = F(n, H(n)) \end{cases}$$

Proof: Call a set X **good** if, for some $m \in \mathbb{N}$, $X : m^+ \to A$ such that $X(0) = a_0$ and for all $n \in m$ $X(n^+) = F(n, X(n))$. Let

$$H = \bigcup\{X \mid X \text{ is good}\}.$$

Then standard arguments, using mathematical induction show that H is the required function. An easy mathematical induction show that any two good functions agree where they are both defined. That

$$\forall n{\in}\mathbb{N}\exists a{\in}A\ (n,a) \in H$$

is proved by mathematical induction on n. For the base case observe that

$$X_0 = \{(0, a_0)\} : 0^+ \to A$$

is good. For the induction step observe that if $X : n^+ \to A$ is good then so is

$$X^+ = X \cup \{(n^+, F(n, X(n)))\} : n^{++} \to A.$$

As good functions agree where they are both defined $H : \mathbb{N} \to A$ and H agrees with every good function. It follows that H satisfies the above equations and is the unique function to do so.

■

Using this result we can give the usual primitive recursive definitions of the standard arithmetical operations on $\mathbb{N}$ such as addition and multiplication.

3.4 Union-Replacement

Proposition: 3.4 *Given the Extensionality and Pairing axioms the Union-Replacement axiom scheme is equivalent to the combination of the Union axiom and the Replacement axiom scheme.*

Proof: Assume Union-Replacement and let $\forall x \in \exists! y\ \ \phi(x, y)$. Then, as singleton classes are sets,

$$\forall x{\in}a\mathbb{S}y\phi(x, y)$$

so that by Union-Replacement

$$\mathbb{S}y\exists x{\in}a\phi(x, y).$$

So we have proved replacement. The Union axiom follows from the instance of Union-replacement where $\phi(x, y)$ is $y \in x$.

Conversely, given the Union axiom and the Replacement scheme, suppose that
$\forall x{\in}a\mathbb{S}y\phi(x, y)$. Then

$$\forall x{\in}a\exists! z\forall y[y \in z\ \leftrightarrow\ \phi(x, y)].$$

So, by Replacement we may form the set

$$\{z \mid \exists x{\in}a \forall y[y \in z \;\leftrightarrow\; \phi(x,y)]\}.$$

By the Union axiom we may form the union set of this set, which is

$$\{y \mid \exists x{\in}a\phi(x,y)\}.$$

Thus we have proved the Union-Replacement axiom scheme.
■

We now consider a few consequences of Union-Replacement. If $(B_a)_{a\in A}$ is a family of classes and each B_a is actually a set then we have a **family of sets** and we may form the class

$$\{B_a \mid a \in A\} = \{z \mid \exists a{\in}A \; z = B_a\}.$$

If A is also a set then $\{B_a \mid a \in A\}$ is a set, by Replacement and $\bigcup_{a\in A} B_a$ is a set, by Union-Replacement.

The Union Axiom asserts that the class $\bigcup A$ is a set for each set A. So, using the Pairing axiom we get that the class $A \cup B$ is a set whenever A, B are sets and hence that $\{a_1, \ldots, a_n\}$ is a set whenever $a_1, \ldots, a_n$ are sets for $n > 0$.

Proposition: 3.5 *If A, B are sets then so is the class $A \times B$.*

Proof: Let A,B be sets. Then, as

$$\{a\} \times B = \{(a,b) \mid b \in B\}$$

is a set, by replacement, so is

$$A \times B = \bigcup_{a\in A} (\{a\} \times B)$$

by Union-Replacement.
■

3.4.1 Exponentiation of Sets

Note that if A is a set and $F : A \to B$ then, by Replacement,

$$F = \{(a, F(a)) \mid a \in A\}$$

is a set. If a is a set and B is a class then we may form the class ${}^aB$ of all functions from a to B, given by

$$ {}^aB = \{f \mid f : a \to B\}. $$

We may now state the following axiom.

Exponentiation $\forall a \forall b[{}^ab$ is a set.]

Proposition: 3.6 *The Exponentiation Axiom is a consequence of the Subset Collection Scheme.*

3.5 The class Ω of Truth values

Here we need to assume all of $BCST$. Let $\Omega = Pow(1)$ and, for $A \subseteq \Omega$. We think of the elements of Ω as truth values, with 0 representing falsity and 1 representing truth. In constructive mathematics we cannot assert that those are the only truth values. Moreover in constructive set theory we cannot even assert that the class of truth values form a set.

Let

- $\bigvee A = \{x \mid x \in 1 \ \wedge \ \exists y{\in}A \ x \in y\} = \bigcup A$,
- $\bigwedge A = \{x \mid x \in 1 \ \wedge \ \forall y{\in}A \ x \in y\}$.

For each set $a \in Pow(\Omega)$ the class $\bigvee a$ is a set in Ω by the Union axiom and $\bigwedge a$ is a set in Ω by the Infimum Axiom.

If θ is a formula and $c \in \Omega$ such that $[\theta \ \leftrightarrow \ 0 \in c]$ then, by Extensionality, c is unique and we call c the **truth value** of θ. For any formula θ we use $!\theta$ to abbreviate

$$ \exists c{\in}\Omega \ [\theta \ \leftrightarrow \ 0 \in c] $$

Proposition: 3.7

1. *If* $\forall x{\in}a \ !\phi(x)$ *then* $!\forall x{\in}a\phi(x)$ *and* $!\exists x{\in}a\phi(x)$.
2. *If* $!\phi_1 \wedge !\phi_2$ *then*
   (a) $![\phi_1 \wedge \phi_2]$
   (b) $![\phi_1 \vee \phi_2]$
   (c) $![\phi_1 \to \phi_2]$
3. $!\phi \ \to !\neg\phi$

4. $\forall a \forall b\ ![a = b]$.

5. $\forall a \forall b\ ![a \in b]$.

6. $!\phi$ *for each restricted formula* ϕ.

Proof:

1. By the assumption, using Union-Replacement we get
$$\mathbb{S}c[c \in \Omega\ \wedge\ \exists x{\in}a[\phi(x)\ \leftrightarrow\ 0 \in c]].$$
So we may form the set
$$b = \{c \in \Omega \mid \exists x{\in}a[\phi(x)\ \leftrightarrow\ 0 \in c]\}.$$
This is in $Pow(\Omega)$ so that $\bigvee b, \bigwedge b \in \Omega$ and
$$\forall x{\in}a\phi(x)\ \leftrightarrow\ 0 \in \bigwedge b$$
and
$$\exists x{\in}a\phi(x)\ \leftrightarrow\ 0 \in \bigvee b.$$

2. Let $c_1, c_2 \in \Omega$ such that
$$\phi_i\ \leftrightarrow\ 0 \in c_i$$
for $i = 1, 2$. Then $c_\wedge = \bigwedge\{c_1, c_2\} \in \Omega$ and
$$[\phi_1 \wedge \phi_2]\ \leftrightarrow\ 0 \in c_\wedge$$
giving (i). Similarily $c_\vee = \bigvee\{c_1, c_2\} \in \Omega$ and
$$[\phi_1 \vee \phi_2]\ \leftrightarrow\ 0 \in c_\vee$$
giving (ii). For (iii) let $c_\rightarrow = \bigwedge\{c_2 \mid x \in c_1\} \in \Omega$ and
$$[\phi_1 \rightarrow \phi_2]\ \leftrightarrow\ 0 \in c_\rightarrow.$$

3. As $0 \in \Omega$ and $0 = 1\ \leftrightarrow\ 0 \in 0$ and $\neg\phi\ \leftrightarrow\ [\phi \rightarrow 0 = 1]$.

4. This can be proved using 1,2, a double Set Induction on a, b and the equivalence
$$a = b\ \leftrightarrow\ \forall x{\in}a\exists x{\in}b[x = y]\ \wedge\ \forall y{\in}b\exists x{\in}a[x = y].$$

5. $a \in b\ \leftrightarrow\ \exists y{\in}b[a = y]$. Hence, by 1 and 4, $![a \in b]$.

6. By induction on the way a restricted formula is built up, using 4,5 for atomic ϕ, 2,3 for the connectives and 1 for the restricted quantifiers.

3.5.1 Restricted Separation

Lemma: 3.8 *For each formula $\phi(x)$*

$$\forall x{\in}a!\phi(x) \ \rightarrow \ \mathbb{S}x[x \in a \wedge \phi(x)].$$

Proof: Assume that $\forall x{\in}a!\phi(x)$. Then for each $x \in a$ we may form the set $b_y = \{y \mid y = 0 \wedge \phi(x)\}$ and hence, by Replacement, the set $\{(y, x) \mid y \in b_x\}$. Hence, by Replacement again we may form the set

$$f = \{(x, \{(y, x) \mid y \in b_x\}) \mid x \in a\}.$$

Note that f is a function with domain a such that for each $x \in a$ the value $f(x)$ is itself the constant function with value x defined on the set b_x. Hence, by Union-Replacement

$$\{x \in a \mid \phi(x)\} = \{x \mid x \in a \wedge \phi(x)\} = \bigcup_{x \in a} ran(f(x))$$

is a set.
■

Theorem: 3.9 *Every instance of restricted Separation can be proved in BCST.*

Proof: Let $\phi(x)$ be a restricted formula. Then by part 6 of Proposition 3.7 $\forall x{\in}a \ !\phi(x)$ and hence

$$\mathbb{S}x[x \in a \wedge \phi(x)].$$

■

3.5.2 Quotients

Let A be a set and let R be a subset of $A \times A$ that is an equivalence relation on A. Then for each $a \in A$ we may form its **equivalence class**

$$[a]_R = \{x \in A \mid xRy\}$$

which is actually a set, by Restricted Separation. We may use Replacement to form the quotient set of A with respect to R.

$$A/R = \{[a]_R \mid a \in A\}.$$

3.6 Transitive Closure

In the rest of this section, unless otherwise indicated we will only need to assume Extensionality, Set Induction, Pairing and Union-Replacement. We will show that for each set a there is a smallest set $TC(a)$ that is transitive and has a as a subset.

Let $TransClos(x, y)$ be the formula

$$x \subseteq y \ \wedge \ Trans(y) \ \wedge \ \forall z[x \subseteq z \ \wedge \ Trans(z) \rightarrow y \subseteq z].$$

Proposition: 3.10 *For all a*

$$(*) \qquad \exists! b \, TransClos(a, b).$$

Proof: This will be proved by Set Induction on a. So we may assume as induction hypothesis that

$$\forall x{\in}a \ \exists! y \, TransClos(x, y).$$

By Replacement there is a set b such that

$$\forall y[y \in b \ \leftrightarrow \ \exists x{\in}a \, TransClos(x, y).$$

Let $c = a \cup \bigcup b$. We show that $TransClos(a, c)$.

That $a \subseteq c$ is immediate from the definition of c. To see that c is transitive let $x \in c$. We must show that x is a subset of c. As $x \in c$, either $x \in a$ or $x \in y$ for some $y \in b$. Note that if $y \in b$ then $y \subseteq \bigcup b \subseteq c$. If $x \in a$ then $TransClos(x, y)$ for some $y \in b$ so that $x \subseteq y \subseteq c$. If $x \in y$ for some $y \in b$ then $x \subseteq y \subseteq c$, as $Trans(y)$.

It remains to show that $c \subseteq z$ for each set z such that $[a \subseteq z \wedge Trans(z)]$. By the definition of c, as $a \subseteq z$ it is only necessary to show that $\bigcup b \subseteq z$. So let $u \in \bigcup b$. Then $u \in y$ for some $y \in b$. So $TransClos(x, y)$ for some $x \in a$. Then $x \in z$ and hence $x \subseteq z$, as z is transitive. So, as $TransClos(x, y)$ and $Trans(z)$, $y \subseteq z$. As $u \in y$, $u \in z$.

We have now proved that $TransClos(a, c)$. The uniqueness of c is a consequence of the definition of $TransClos(x, y)$. If also $TransClos(a, c')$ then both $c \subseteq c'$ and $c' \subseteq c$ so that $c = c'$.
■

By this proposition for each a we can define $TC(a)$ to be the unique b such that $TransClos(a, b)$.

Proposition: 3.11

1. $[x \in a \vee x \subseteq a] \rightarrow TC(x) \subseteq TC(a)$.
2. $TC(a) = a \cup \bigcup_{x \in a} TC(x)$.

We can now prove a convenient variant of Set Induction.

Proposition: 3.12 (TC-Induction)

$$\forall a[\forall x{\in}TC(a)\theta(x) \rightarrow \theta(a)] \rightarrow \forall a\theta(a).$$

Proof: Let $\theta'(x)$ be the formula $\forall y{\in}TC(x)\theta(y)$. We will assume that

$$(*) \qquad \forall a[\theta'(a) \rightarrow \theta(a)]$$

and show that $\forall a\theta(a)$.

Claim: $\forall x{\in}a\ \theta'(x) \ \rightarrow \ \theta'(a)$.

Proof of Claim: Let $\forall x{\in}a\ \theta'(x)$. Then

$$(1) \qquad \forall x{\in}a\forall y{\in}TC(x)\ \theta(y)$$

and by $(*)$,

$$(2) \qquad \forall x{\in}a\ \theta(x).$$

Now let $z \in TC(a)$. Then, by 2 of the previous proposition, either $z \in a$ or $z \in TC(x)$ for some $x \in a$. In either case $\theta(z)$. Thus we have proved that $\theta'(a)$.
■

By the claim we may use Set Induction to get that $\forall a\theta'(a)$ and hence, by $(*)$, we get that $\forall a\theta(a)$.
■

3.7 Iterations of a monotone set operator

For any class Y of ordered pairs, for each set a let

$$Y^a = \{y \mid (a, y) \in Y\}$$

and

$$Y^{\in a} = \{y \mid \exists x{\in}a\ (x,y) \in Y\} = \bigcup_{x\in a} Y^x.$$

Also let

$$Y^{\infty} = \{y \mid \exists a\ (a,y) \in Y\} = \bigcup_{a\in V} Y^a.$$

Theorem: 3.13 *Let Γ be a monotone operation on sets; i.e. $\Gamma(X)$ is a set for each set X such that for all sets X, Y*

$$X \subseteq Y \to \Gamma(X) \subseteq \Gamma(Y).$$

Then there is a class J of ordered pairs having the following properties.

1. *For each set a*

   $(i)_a$ $J^a = \Gamma(J^{\in a})$,

   $(ii)_a$ J^a *is a set.*

2. *Let Y be a class such that for all sets X*

   $$X \subseteq Y \to \Gamma(X) \subseteq Y.$$

   Then

   (a) *$J^a \subseteq Y$ for all a and hence $J^{\in a} \subseteq Y$ for all a.*

   (b) *$J^{\infty} \subseteq Y$. Moreover, assuming Collection, J^{∞} is such a class Y and so is the smallest such class.*

Proof: We define $J = \{G \mid G \text{ is good}\}$ where we call a set G of ordered pairs **good** if, for some transitive set X

$$(*) \qquad G = \{(a,x) \mid a \in X \wedge x \in \Gamma(G^{\in a}).$$

1. The two parts $(i)_a$ and $(ii)_a$ will be proved simultaneously by TC-Induction. So, we assume as induction hypothesis that for all $b \in TC(a)$ both $(i)_b$ and $(ii)_b$ hold. So, as $a \subseteq TC(a)$, J^b is a set for each $b \in a$ so that by Union-Replacement $J^{\in a}$ is a set and hence so is $\Gamma(J^{\in a})$. Note that this shows that $(i)_a \to (ii)_a$. Also, if $b \in a$ then $x \in a$ and hence J^x is a set for each $x \in b$, so that $J^{\in b}$ is a set. It follows that G is a set where

   $$G = \bigcup_{b\in TC(a)\cup\{a\}} \{b\} \times \Gamma(J^{\in b}).$$

Note that $G^b = J^b$ for each $b \in TC(a)$ so that $G^{\in b} = J^{\in b}$ for all $b \in TC(a) \cup \{a\}$. So $(*)$ holds, with $X = TC(a) \cup \{a\}$ and so G is good. It follows that $\Gamma(J^{\in a}) = G^a \subseteq J^a$.

To see that $J^a \subseteq \Gamma(J^{\in a})$ let $x \in J^a$. Then $x \in G^a$ for some good G. So $x \in \Gamma(G^{\in a}) \subseteq \Gamma(J^{\in a})$.

So we have proved $(i)_a$ and so $(ii)_a$ as we have seen.

2. (a) An easy proof by Set Induction on a.
   (b) The first part is an immediate consequence of (a). For the second part let X be a subset of J^∞. Then $\forall x \in X \exists a [x \in J^a]$. So by Collection there is a set b such that $\forall x \in X \exists a \in b [x \in J^a]$. It follows that $X \subseteq J^{\in b}$ and hence $\Gamma(X) \subseteq \Gamma(J^{\in b}) = J^b$.

■

3.8 The set of Natural Numbers

Recall that we defined the class $\mathbb{N}$ and characterised it as the smallest class Y such that

1. $0 \in Y$,
2. $a \in Y \Rightarrow a^+ \in Y$.

We will now assume the Infinity Axiom to prove the following result.

Theorem: 3.14 $\mathbb{N}$ *is a set.*

Proof: Let Γ be the set operator such that for each set Y

$$\Gamma(Y) = \{0\} \cup \{y^+ \mid y \in Y\}.$$

Observe that, by the Successor axiom and Replacement, $\Gamma(Y)$ is a set. It follows that J^a is a set for each set a and hence that $J^{\in a}$ is also a set for each set a. Observe that for each set $X \subseteq \mathbb{N}$ we have $\Gamma(X) \subseteq \mathbb{N}$. Hence by part 2(a) of Theorem 3.13 $J^{\in a} \subseteq \mathbb{N}$ for all a. To show that $\mathbb{N}$ is a set it suffices to show the converse inclusion for some a, as then $\mathbb{N}$ is equal to the set $J^{\in a}$. For that it suffices to have $J^a \subseteq J^{\in a}$ for that a, as then $J^{\in a}$ is a set Y satisfying 1,2 above.

So it suffices to find such a set a. The Infinity Axiom will give such a set. Recall its formulation.

Infinity

$$\exists a\,[\exists x\; x \in a \;\wedge\; \forall x{\in}a\exists y{\in}a\; x \in y]$$

Now let a be a set assumed to exist by this axiom. We must show that $\Gamma(J^{\in a}) \subseteq J^{\in a}$. So let $b \in \Gamma(J^{\in a})$. Then either (i) $b = 0$ or (ii) $b = z^+$ for some $z \in J^{\in a}$.

If (i) then, as by the choice of a there is $x \in a$. So

$$b = 0 \in \Gamma(J^{\in x}) = J^x \subseteq J^{\in a}.$$

If (ii) then, $z \in J^x$ for some $x \in a$. Now, by the choice of a, there is $y \in a$ such that $x \in y$. It follows that

$$b = z^+ \in \Gamma(J^{\in y}) = J^y \subseteq J^{\in a}.$$

So, in either case $b \in J^{\in a}$, as desired.
■

3.9 The Real Numbers

We give a categorical axiomatisation of the pseudo-ordered field of real numbers. The axiom system is a relatively minor constructive refinement of the standard classical axiomatisation.

A **pseudo-ordering** $<$ of a class A is a relation on A that satisfies the following axioms for all $x, y \in A$

1. $\neg[x < y \wedge y < x]$
2. $x < y \to \forall z{\in}A[x < z \vee z < y]$
3. $\neg[x < y \vee y < x] \to x = y$

Given a pseudo-ordering it is usual to make the definitions

- $x \leq y \equiv \neg[y < x]$
- $x \sharp y \equiv [x < y \vee y < x]$

A pseudo-ordered class A is **Dedekind-complete** if every inhabited, located subset that is bounded above has a least upper bound. Here we have used the following notions. A set is **inhabited** if it has an element. For each subset X of A let

$$X^{<} = \{y \in A \mid \exists x \in X\; y < x\}$$

and let $X^{\geq} = A - X^{<}$. We say that X is **located** if for all $x, y \in A$

$$x < y \to [x \in X^{<} \vee y \in X^{\geq}]$$

and X is **bounded above** if $X^{\geq}$ is inhabited. Finally $a \in A$ is a **least upper bound** of X if $a \in X^{\geq}$ and $\forall y \in X^{\geq}[a \leq y]$.

In the following definitions we assume that the algebraic notion of a commutative ring with 1 has been defined in the standard way except that we allow the elements of the ring to form a class that need not be a set.

A commutative ring R with 1 is a **pseudo-ordered commutative ring with 1** if it comes with a pseudo-ordering satisfying the following axioms for all $x, y \in R$

1. $0 < 1$
2. $x < y \to \forall z{\in}R[x + z < y + z]$
3. $0 < x, y \to 0 < xy$

It is a **pseudo-ordered field** if also the following holds.

1. $\forall x{\in}R[0 < x \to \exists y{\in}R[xy = 1]]$

A pseudo-ordered field F is **Archimedean** if for all $x \in F$ there is $n \in \mathbb{N}$ such that

$$x < \overbrace{1 + \cdots + 1}^{n}.$$

Theorem: 3.15 *There is a unique up to isomorphism Dedekind-complete Archimedean pseudo-ordered field $\mathbb{R}$. If the class ${}^{\mathbb{N}}\mathbb{N}$ of all functions $\mathbb{N} \to \mathbb{N}$ forms a set*[4] *then $\mathbb{R}$ is a set.*

The construction of a class $\mathbb{R}$, to give the existence part of the result, may be carried out by a careful constructive refinement of any of the familiar classical constructions. We illustrate by describing the Dedekind cut approach. We assume that the set of $\mathbb{Q}$ of rational numbers has been constructed in a standard way to form an ordered field. There should be no problem with this. Then we can define $\mathbb{R}$ to be the class of **located left cuts** where a subset X of $\mathbb{Q}$ is a **left cut** if both X and $\mathbb{Q} - X$ are inhabited and $\forall x{\in}X \exists y{\in}X\ x < y$. The locatedness condition is constructively essential in order to derive the important property that every real number can be approximated arbitrarily closely by rational numbers. Another important fact is that a located left cut is a countably infinite set in the sense that it is in one-one correspondence with $\mathbb{N}$. This implies that $\mathbb{R}$ is a set if ${}^{\mathbb{N}}\mathbb{N}$ is a set.

[4]e.g. if Exponentiation holds

4 Inductive Definitions

We will think of an inductive definition as a generalised notion of axiom system. We may chacterise a (finitary) axiom system as follows. There are objects, which we will call the statements of the axiom system, and there are axioms and rules of inference. Each axiom is a statement and each rule of inference has instances that consist of finitely many premisses and a conclusion, both the premisses and conclusion being statements. So we may think of an instance of a rule of inference as an inference step $\frac{X}{a}$ where X is the finite set of premisses and a is the conclusion. It is also convenient to think of each axiom a as such a step where the set X of premisses is empty. The theorems of an axiom system may be characterised as the smallest set of statements that include all the axioms and are closed under the rules of inference. Here, a set of statements is closed under a rule if, for each instance of the rule, if the premisses are in the set then so is the conclusion. If we let Φ be the set of steps determined by the axioms and the instances of the rules then we may characterise the set of theorems as the smallest set of statements such that for every step in Φ, if the premisses are in the set then so is the conclusion. Our generalisation is to allow any objects to be statements and to start from an arbitrary class of steps, with each step having a set of premisses that need not be finite. So we are led to the following definitions.

4.1 Inductive Definitions of Classes

We define an **inductive definition** to be a class of ordered pairs. If Φ is an inductive definition and $(X, a) \in \Phi$ then we write

$$\frac{X}{a}\ \Phi$$

and call $\frac{X}{a}$ an **(inference) step** of Φ, with set X of **premisses** and **conclusion** a. For any class Y, if $\frac{X}{a}\ \Phi$ for some subset X of Y then we will write

$$\subset\!\frac{Y}{a}\ \Phi.$$

The class Y is **Φ-closed** if for all a

$$\subset\!\frac{Y}{a}\ \Phi \ \Rightarrow\ a \in Y.$$

We define the class **inductively defined by** Φ to be the smallest Φ-closed class. The main result of this section states that this class $I(\Phi)$ always exists.

Theorem: 4.1 (Class Inductive Definition Theorem) *For any inductive definition Φ there is a smallest Φ-closed class $I(\Phi)$.*

This result will be proved in the axiom system consisting of Extensionality, Set Induction, Pairing, Union-Replacement and Strong Collection. Note that Union can be used instead of Union-Replacement as Replacement is a consequence of Strong Collection.

4.1.1 The Proof

The proof involves the iteration of the following monotone operator on classes until it closes up at its least fixed point which turns out to be the required class $I(\Phi)$. For each class Y let

$$\Gamma(Y) = \{a \mid \subset\!\frac{Y}{a}\ \Phi\}.$$

Note that Γ is monotone; i.e. for classes Y_1, Y_2

$$Y_1 \subseteq Y_2 \Rightarrow \Gamma(Y_1) \subseteq \Gamma(Y_2).$$

As an inductive definition need not be finitary; i.e. it can have steps with infinitely many premisses, we will need transfinite iterations of Γ in general. In classical set theory it is customary to use ordinal numbers to index iterations. Here it is unnecessary to develop a theory of ordinal numbers and we simply use sets to index iterations. This is not a problem as we can carry out proofs by set induction. The following result gives us the iterations we want.

Lemma: 4.2 *There is a class J such that for each a,*

$$J^a = \Gamma(\bigcup_{x\in a} J^x).$$

Proof: Call a set G of ordered pairs **good** if

$$(*) \qquad (a, y) \in G \Rightarrow y \in \Gamma(G^{\in a}).$$

where

$$G^{\in a} = \{y' \mid \exists x{\in}a\ (x, y') \in G\}.$$

Let $J = \bigcup\{G \mid G \text{ is good}\}$. We must show that for each a

$$J^a = \Gamma(J^{\in a}).$$

First, let $y \in J^a$. Then $(a, y) \in G$ for some good set G and hence by $(*)$, above, $y \in \Gamma(G^{\in a})$. As $G^{\in a} \subseteq J^{\in a}$ it follows that $y \in \Gamma(J^{\in a})$. Thus $J^a \subseteq \Gamma(J^{\in a})$.

For the converse inclusion let $y \in \Gamma(J^{\in a})$. Then $\frac{Y}{y}\ \Phi$ for some set $Y \subseteq J^{\in a}$. It follows that $\forall y' {\in} Y \exists x {\in} a\ y' \in J^x$ so that

$$\forall y' {\in} Y \exists G\ [\ G \text{ is good and } y' \in G^{\in a}].$$

By Strong Collection there is a set Z of good sets such that

$$\forall y' {\in} Y\ \exists G {\in} Z\ y' \in G^{\in a}.$$

Let $G = \{(a, y)\} \cup \bigcup Z$. Then $\bigcup Z$ is good and, as $\frac{Y}{y}\ \Phi$ and $Y \subseteq G^{\in a}$, G is good. As $(a, y) \in G$ we get that $y \in J^a$. Thus $\Gamma(J^{\in a}) \subseteq J^a$.
■

We can now prove the theorem. It only remains to show that J^∞ is the smallest Φ-closed class. Our argument follows the lines of the proof of part 2 of Theorem 3.13.

To show that J^∞ is Φ-closed let $\frac{Y}{y}\ \Phi$ for some set $Y \subseteq J^\infty$. Then $\forall y' {\in} Y \exists x\ y' \in J^x$. So, by Collection, there is a set a such that

$$\forall y' {\in} Y \exists x {\in} a\ y' \in J^a;$$

i.e. $Y \subseteq J^{\in a}$. Hence $y \in \Gamma(J^{\in a}) = J^a \subseteq I$. Thus J^∞ is Φ-closed.

Now let I be a Φ-closed class. We show that $J^\infty \subseteq I$. It sufices to show that $J^a \subseteq I$ for all a. We do this by Set Induction on a. So we may assume, as induction hypothesis, that $J^x \subseteq I$ for all $x \in a$. It follows that $J^{\in a} \subseteq I$ and hence

$$J^a = \Gamma(J^{\in a}) \subseteq \Gamma(I) \subseteq I,$$

the inclusions holding because Γ is monotone and I is Φ-closed.
■

Call an inductive definition Φ **local** if $\Gamma(X)$ is a set for all sets X.

Proposition: 4.3 *If Φ is local then both J^a and $J^{\in a}$ are sets for each set a.*

Proof: We show that J^a is a set for each set a by Set Induction on a. The induction hypothesis is that J^x is a set for each $x \in a$. It follows that $J^{\in a}$ is a set by Union-Replacement. So, by the assumption, $J^a = \Gamma(J^{\in a})$ is a set.
■

Note that we could use the alternative construction of Theorem 3.13 to get this result, thereby avoiding the need for Lemma 4.2 which uses Strong Collection. The two constructions of classes J are easily seen to give (extensionally) the same class.

4.1.2 Examples

Let A be a class.

1. $H(A)$ is the smallest class X such that for each set a that is an image of a set in A
$$a \in Pow(X) \Rightarrow a \in X.$$
Note that $H(A) = I(\Phi)$ where Φ is the class of all pairs (a, a) such that a is an image of a set in A.

2. If R is a subclass of $A \times A$ such that $R_a = \{x \mid xRa\}$ is a set for each $a \in A$ then $Wf(A, R)$ is the smallest subclass X of A such that
$$\forall a \in A\, [R_a \subseteq X \Rightarrow a \in X].$$
Note that $Wf(A, R) = I(\Phi)$ where Φ is the class of all pairs (R_a, a) such that $a \in A$.

3. If B_a is a set for each $a \in A$ then $W_{a\in A}B_a$ is the smallest class X such that
$$a \in A \;\&\; f : B_a \to X \;\Rightarrow\; (a, f) \in X.$$
Note that $W_{x\in A} = I(\Phi)$ where Φ is the class of all pairs $(ran(f), (a, f))$ such that $a \in A$ and $f : B_a \to V$.

4.2 The Regular Extension Axiom

A class A is **transitive** if $\forall a{\in}A\ a \subseteq A$. A **regular set** is a transitive set A such that
$$\forall a{\in}A\ SubColl(a, A, A),$$
where $SubColl(a, b, c)$ abbreviates
$$\forall r\ [[r : a \succ\!\!- b] \;\to\; \exists b'{\in}c\ [r : a \succ\!\!-\!\!\prec b']].$$

The **Regular Extension Axiom (REA)** is as follows.

Every set is a subset of a regular set

Proposition: 4.4 *The Subset Collection Scheme is equivalent to the axiom*

$$\forall a \forall b \exists c \; SubColl(a, b, c).$$

Proposition: 4.5 *The Subset Collection Scheme is a consequence of REA.*

The following weakened notion may also be useful. We call a transitive set A **weakly regular** if

$$\forall a{\in}A \; \forall r \; [[r : a \succ\!\!- A] \;\rightarrow\; \exists b{\in}A \; r : a \succ\!\!- b].$$

The **weakly Regular Extension Axiom (wREA)** is as follows.

> Every set is a subset of a weakly regular set

4.3 Inductive definitions of Sets

We define a class B to be a **bound** for Φ if whenever $\frac{X}{a}\ \Phi$ then X is an image of a set $b \in B$; i.e. there is a function from b onto X. We define Φ to be **(regular, weakly regular) bounded** if

1. $\{y \mid \frac{X}{y}\ \Phi\}$ is a set for all sets X,
2. Φ has a bound that is a (regular, weakly regular) set.

Proposition: 4.6 *Assuming Exponentiation, every bounded inductive definition Φ is local; i.e. $\Gamma(X)$ is a set for each set X.*

Proof: Let B be a bound for Φ. If $\frac{Y}{y}\ \Phi$ then for some $b \in B$ there is a surjective $f : b \to Y$. So if X is a set then

$$\Gamma(X) = \bigcup_{f \in C} \{y \mid \frac{ran(f)}{y}\ \Phi\}$$

where $C = \bigcup_{b \in B} {}^{X}b$. By Exponentiation and Union-Replacement C is a set. As Φ is bounded $\{y \mid \frac{ran(f)}{y}\ \Phi\}$ is always a set, so that, by Union-Replacement $\Gamma(X)$ is a set.
■

The following result does not seem to need any form of Collection.

Theorem: 4.7 (Set Definition Theorem) *If Φ is a weakly regular bounded local inductive definition then there is a smallest Φ-closed class $I(\Phi)$ which is a set.*

Proof: Let A be a weakly regular bound for Φ. Then, as Φ is local, we may apply Proposition 4.3 to get that $J^{\in A}$ is a set. As $J^{\in A} \subseteq Y$ for any Φ-closed class Y it suffices to show that $J^{\in A}$ is Φ-closed.

So let $\frac{X}{x}\ \Phi$ with X a subset of $J^{\in A}$. Then, as A is a bound for Φ, there is $Z \in A$ and surjective $f : Z \to X$. So $\forall z \in Z\ f(z) \in J^{\in A}$ and hence $\forall z \in Z \exists a \in A\ f(z) \in J^a$. As A is a weakly regular set and $Z \in A$ there is $b \in A$ such that $\forall z \in Z \exists a \in b\ f(z) \in J^a$. Hence $X \subseteq \bigcup_{a \in b} J^a$ so that $x \in \Gamma(\bigcup_{a\in b} J^a) = J^b \subseteq J^{\in A}$.
■

Corollary: 4.8 *Assume the Exponentiation Axiom and wREA.*

1. *Every bounded Φ is weakly regular bounded.*
2. *If A is a set then*
   (a) *$H(A)$ is a set,*
   (b) *if $R \subseteq A \times A$ such that $R_a = \{x \mid xRa\}$ is a set for each $a \in A$ then $Wf(A, R)$ is a set.*
   (c) *if B_a is a set for each $a \in A$ then $W_{a\in A} B_a$ is a set.*

4.4 Tree Proofs

We may relativise the notion of theorem for an axiom system to a set, X, of assumptions treated as additional axioms. The set of theorems relative to X is then the smallest set of statements of the axiom system that include the axioms, are closed under the rules of inference and also include the assumptions from X. We generalise this idea to inductive definitions. Let $I(\Phi, X)$ be the smallest Φ-closed class that has X as a subclass. This exists as it can be defined as $I(\Phi_X)$ where

$$\Phi_X = \Phi \cup (\{\emptyset\} \times X).$$

We will give another characterisation of $I(\Phi, X)$ in terms of a suitable notion of proof. These will be well-founded trees having two kinds of nodes, **assumption nodes** that will always be leaves and **step nodes**.

We define the class $\mathbb{P}$ of **proto-proofs** to be the smallest class Y such that for any a

1. $(0, a) \in Y$,
2. If Z is a subset of Y then $(1, a, Z) \in Y$.

For each proto-proof $p \in \mathbb{P}$ we can define its **conclusion** $concl(p)$ as follows.

$$\begin{cases} concl((0,a)) & = a \\ concl((1,a,Z)) & = a \end{cases}$$

For each set $Z \subseteq \mathbb{P}$ of proto-proofs let

$$\tau(Z) = \{concl(p) \mid p \in Z\}.$$

By recursion on $p \in \mathbb{P}$ we can define the sets $ass(p)$ of **assumptions** of p and $steps(p)$ of **steps** of p using the following equations.

$$\begin{cases} ass((0,a)) & = \{a\} \\ ass((1,a,Z)) & = \bigcup_{p \in Z} ass(p) \end{cases}$$

$$\begin{cases} steps((0,a)) & = \emptyset \\ steps((1,a,Z)) & = \{(\tau(Z),a)\} \cup \bigcup_{p \in Z} steps(p) \end{cases}$$

Theorem: 4.9 *For any class Φ of ordered pairs and any class X*

$$I(\Phi, X) = I'(\Phi, X),$$

where

$$I'(\Phi, X) = \{concl(p) \mid p \in \mathbb{P} \ \&\ ass(p) \subseteq X \ \&\ steps(p) \subseteq \Phi\}.$$

Proof: Let

$$\mathbb{P}' = \{p \in \mathbb{P} \mid concl(p) \in I(steps(p), ass(p))\}.$$

Claim 1: $p \in \mathbb{P} \Rightarrow p \in \mathbb{P}'$

We will show this by $\mathbb{P}$-induction. So we must show that for all a

1. $(0, a) \in \mathbb{P}'$,
2. if $Z \in Pow(\mathbb{P}')$ then $(1, a, Z) \in \mathbb{P}'$.

For 1: Let $p = (0, a)$. Then

$$\begin{cases} concl(p) & = a \\ steps(p) & = \emptyset \\ ass(p) & = \{a\} \end{cases}$$

So $p \in \mathbb{P}'$ as $a \in I(\emptyset, \{a\})$.

For 2: Let $p = (1, a, Z)$ with $Z \in Pow(\mathbb{P}')$. Then

$$\begin{cases} concl(p) & = a \\ steps(p) & = \{(\tau(Z), a)\} \cup \bigcup_{p' \in Z} steps(p') \\ ass(p) & = \bigcup_{p' \in Z} ass(p') \end{cases}$$

As $Z \in Pow(\mathbb{P}')$

$$\forall p' {\in} Z\ [concl(p') \in I(steps(p'), ass(p'))];$$

But

$$\forall p' {\in} Z\ [\ steps(p') \subseteq steps(p) \text{ and } ass(p') \subseteq ass(p)\ ].$$

It follows that $\tau(Z) \subseteq I(steps(p), ass(p))$. As $(\tau(Z), a) \in steps(p)$ we get that $concl(p) = a \in I(steps(p), ass(p))$; i.e. $p \in \mathbb{P}'$ as required.

Claim 2: $I'(\Phi, X) \subseteq I(\Phi, X)$.
Let $a \in I'(\Phi, X)$. Then $a = concl(p)$ for some $p \in \mathbb{P}$ such that $ass(p) \subseteq X$ and $steps(p) \subseteq \Phi$. By claim 1 $p \in \mathbb{P}'$ so that

$$concl(p) \in I(steps(p), ass(p)).$$

It follows that $a = concl(p) \in I(\Phi, X)$, as required.

Claim 3: $X \subseteq I'(\Phi, X)$ and $I'(\Phi, X)$ is Φ-closed.
If $a \in X$ then $a = concl(p)$, where $p = (0, a) \in \mathbb{P}$ with $ass(p) = \{a\} \subseteq X$ and $steps(p) = \emptyset \subseteq \Phi$, so that $a \in I'(\Phi, X)$.

To see that $I'(\Phi, X)$ is Φ-closed let $\frac{Y}{y}\ \Phi$ with $Y \subseteq I'(\Phi, X)$. Let R be the class of all pairs (a, p) such that $p \in \mathbb{P}$, $concl(p) = a$, $ass(p) \subseteq X$ and $steps(p) \subseteq \Phi$. Then

$$\forall a {\in} Y \exists p\ aRp.$$

By Strong Collection there is a set Z such that

$$\forall a {\in} Y \exists p {\in} Z\ aRp \quad \wedge \quad \forall p {\in} Z \exists a {\in} Y\ aRp.$$

It follows that $Z \in Pow(\mathbb{P})$ and $\tau(Z) = Y$ so that $p = (1, y, Z) \in \mathbb{P}$ so that $y = concl(p) \in I'(\Phi, X)$.

Claim 4: Let I' be a class such that $X \subseteq I'$ and I' is Φ-closed. Then $I'(\Phi, X) \subseteq I'$.
We must show that

$$\forall p{\in}\mathbb{P}\ \forall a\ [aRp \Rightarrow a \in I'].$$

We will do this by induction on $p \in \mathbb{P}$. So we must show that

1. if $p = (0, a)$ then $aRp \Rightarrow a \in I'$,
2. if $p = (1, a, Z)$, with $Z \in Pow(\mathbb{P})$ such that

$$\forall p'{\in}Z\ \forall a'\ [a'Rp' \Rightarrow a' \in I'],$$

then $aRp \Rightarrow a \in I'$.

For 1: if $p = (0, a)$ and aRp then $a \in ass(p) \subseteq X \subseteq I'$ so that $a \in I'$.

For 2: let $p = (1, a, Z)$, with $Z \in Pow(\mathbb{P})$ such that

$$\forall p'{\in}Z\ \forall a'\ [a'Rp' \Rightarrow a' \in I'],$$

and let aRp. Then $\frac{\tau(Z)}{a}\ \Phi$ and $\tau(Z) \subseteq I'$ so that $a \in I'$.

By claims 3 and 4 we get that $I(\Phi, X) \subseteq I'(\Phi, X)$, so that with claim 2 we are done.
■

Theorem: 4.10 *Let A be a regular set such that $2 \in A$. For any class Φ of ordered pairs such that $\Phi \subseteq A$ and any class X*

$$I(\Phi, X) = I_A(\Phi, X),$$

where

$$I_A(\Phi, X) = \{concl(p) \mid p \in (\mathbb{P} \cap A)\ \&\ ass(p) \subseteq X\ \&\ steps(p) \subseteq \Phi\}.$$

Moreover $\mathbb{P} \cap A = \mathbb{P}_A$ is a set, where $\mathbb{P}_A$ is the smallest class Y satisfying the conditions 1,2, used in defining $\mathbb{P}$, for all $a, Z \in A$.

Proof: Trivially $I_A(\Phi, X) \subseteq I'(\Phi, X) \subseteq I(\Phi, X)$. To show that $I(\Phi, X) \subseteq I_A(\Phi, X)$ it suffices to show that $X \subseteq I_A(\Phi, X)$ and $I_A(\Phi, X)$ is Φ-closed. This can be done along the lines of the proof of claim 3 in the previous proof, using the assumption that A is a regular set instead of Strong Collection.

For the last part, that $\mathbb{P}_A \subseteq (\mathbb{P}\cap A)$ follows from the fact that $Y = (\mathbb{P}\cap A)$ satisfies the conditions 1,2, used in defining $\mathbb{P}$, for all $a, Z \in A$. To show that $(\mathbb{P} \cap A) \subseteq \mathbb{P}_A$ it is enough to show that $Y = \{p \mid p \in A \to p \in \mathbb{P}_A\}$ satisfies the conditions 1,2, used in defining $\mathbb{P}$. Finally observe that $\mathbb{P}_A = I(\Psi)$ where Ψ is a weakly regular bounded local inductive definition, so that by Theorem 4.7 $\mathbb{P}_A$ is a set.
■

4.5 The Set Compactness Theorem

Theorem: 4.11 (In $CZF + REA$) *Let A be a set and let Φ be an inductive definition that is a subset of $Pow(A) \times A$. Then there is a set B of subsets of A such that for all classes X and all a*

$$a \in I(\Phi, X) \;\Rightarrow\; \exists Y{\in}B\,[Y \subseteq X \;\&\; a \in I(\Phi, Y)].$$

Proof: By REA let A' be a regular set such that $\{2\} \cup A \cup \Phi \subseteq A'$. Let $B = \{ass(p) \mid p \in \mathbb{P}_{A'} \;\&\; steps(p) \subseteq \Phi\}$. By Theorem 4.10 and Replacement, B is a set and if $a \in I(\Phi, X)$ then $a \in I_{A'}(\Phi, X)$ so that $a = concl(p)$ for some $p \in \mathbb{P}_{A'}$ such that $ass(p) \subseteq X$ and $steps(p) \subseteq \Phi$. Now let $Y = ass(p)$. Then $Y \in B$, $Y \subseteq X$ and $a \in I(\Phi, Y)$, as desired.
■

4.6 Closure Operations on a po-class

Given a class A a **partial ordering** of A is a subclass $\leq$ of $A \times A$ satisfying the standard axioms for a partial ordering; i.e.

1. $a \leq a$ for all $a \in A$,
2. $[a \leq b \wedge b \leq c] \rightarrow a \leq c$,
3. $[a \leq b \wedge b \leq a] \rightarrow a = b$,

A **po-class** is a class A with a partial ordering $\leq$.
Let A be a po-class. Then $f : A \rightarrow A$ is **monotone** if

$$x \leq y \rightarrow f(x) \leq f(y).$$

We define $c : A \rightarrow A$ to be a **closure operation** on A if it is monotone and for all $a \in A$

$$a \leq c(c(a)) \leq c(a).$$

Note that, for a closure operation c on A, if $a \in A$ then

$$c(a) \leq a \;\leftrightarrow\; c(a) = a \;\leftrightarrow\; \exists y{\in}A[a = c(y)].$$

We call a subclass C of A a **closure class** on A if for each $a \in A$ there is $\overline{a} \in C$ such that

1. $a \leq \overline{a}$,
2. $a \leq y \rightarrow \overline{a} \leq y$ for all $y \in C$.

Proposition: 4.12 *There is a one-one correspondence between closure operations and closure classes on a po-class A. To each closure operation $c : A \rightarrow A$ there corresponds the closure class $C = \{a \mid c(a) = a\}$ of fixed points of c. Conversely to each closure class C there corresponds the closure operation c which associates with each $a \in A$ the unique $\overline{a} \in C$ satisfying 1,2 above. These correspondences are inverses of each other.*

Example: Let A be a set. Then $Pow(A)$ is a class that is a po-class, when partially ordered by the subset relation on $Pow(A)$.

Let Φ be an inductive definition that is a subset of $Pow(A) \times A$. We call Φ an **inductive definition on** A. Let

$$C_\Phi = \{X \in Pow(A) \mid X \text{ is } \Phi\text{-closed}\}.$$

Then C_Φ is a closure class on $Pow(A)$ whose associated closure operation $c_\Phi : Pow(A) \to Pow(A)$ can be given by

$$c_\Phi(X) = I(\Phi, X)$$

for all sets $X \subseteq A$.

Which closure operations arise in this way? Call a monotone operation $f : Pow(A) \to Pow(A)$ **set-based** if there is a subset B of $Pow(A)$ such that whenever $a \in f(X)$, with $X \in Pow(A)$, then there is $Y \in B$ such that $Y \subseteq X$ and $a \in f(Y)$. We call B a **baseset** for f.

Theorem: 4.13 *Let $c : Pow(A) \to Pow(A)$, where A is a set. Then $c = c_\Phi$ for some inductive definition Φ on A if and only if c is a set-based closure operation on $Pow(A)$.*

Proof: Let $c = c_\Phi$, where Φ is an inductive definition on the set A. That c is a closure operator is an easy consequence of its definition. That it is set-based is the content of Theorem 4.11. For the converse, let c be a set based closure operator on $Pow(A)$, with baseset B and associated closure class C. Let Φ be the set of all pairs (Y, a) such that $Y \in B$ and $a \in c(Y)$. This is a set by Union-Replacement, as $B = \bigcup_{Y \in B}(\{Y\} \times c(Y))$. It is clearly an inductive definition on A. It is easy to check that for any set $X \subseteq A$ X is Φ-closed if and only if $X \in C$, which will give us the desired result that $c = c_\Phi$.
■

4.7 Set-generated $\bigvee$-Semilattices

Let S be a po-class. If $X \subseteq S$ and $a \in S$ then a is a **supremum** of X if for all $x \in S$

$$\forall y{\in}X[y \leq x] \;\leftrightarrow\; a \leq x.$$

Note that a supremum is unique if it exists. The supremum of a subclass X of S will be written $\bigvee X$. A po-class is a **$\bigvee$-semilattice** if every subset has a supremum.

Let S be a $\bigvee$-semilattice . A subset G is a **generating set for** S if for every $a \in S$

$$G_a = \{x \in G \mid x \leq a\}$$

is a set and $a = \bigvee G_a$. An $\bigvee$-semilattice is **set-generated** if it has a generating set.

Example: For each set A the po-class $Pow(A)$ is a set-generated $\bigvee$-semilattice with set $G = \{\{a\} \mid a \in A\}$ of generators.

Theorem: 4.14 *Let C be a closure class on an $\bigvee$-semilattice S. Then C is a $\bigvee$-semilattice, when given the partial ordering induced from S. If S is set-generated then so is C. Moreover every set-generated $\bigvee$-semilattice arises in this way from a closure class C on a $\bigvee$-semilattice $Pow(A)$ for some set A.*

Proof: Let c be the closure operator associated with the closure class C on the $\bigvee$-semilattice S. It is easy to check that C has the supremum operation $\bigvee^C$ given by $\bigvee^C X = c(\bigvee X)$ for each subset X of C. Now assume that S has a set G of generators. Let

$$G^C = \{c(x) \mid x \in G\}.$$

We show that G^C is a set of generators for C. For each $a \in C$ let

$$G^C_a = \{y \in G^C \mid y \leq a\}.$$

We must show that G^C_a is a set and $a = \bigvee^C G^C_a$. Observe that

$$\begin{aligned} G^C_a &= \{c(x) \mid x \in G \wedge c(x) \leq a\} \\ &= \{c(x) \mid x \in G \wedge x \leq a\} \\ &= \{c(x) \mid x \in G_a\} \end{aligned}$$

so that G^C_a is a set. Also observe that $\bigvee^C G^C_a = c(\bigvee\{c(x) \mid x \in G_a\})$. It follows first that $a = \bigvee\{x \mid x \in G_a\} \leq \bigvee\{c(x) \mid x \in G_a\} \leq \bigvee^C G^C_a$ and second that $\bigvee^C G^C_a = \bigvee\{c(x) \mid x \in G_a\} \leq a$, as if $x \in G_a$ then $x \leq a$ so that $c(x) \leq a$. So we get that $a = \bigvee^C G^C_a$.

Finally suppose that S is a set-generated $\bigvee$-semilattice , with set G of generators. Let $c : Pow(G) \to Pow(G)$ be given by

$$c(X) = G_{\bigvee X}$$

for all $X \in Pow(G)$. Then it is easy to observe that c is a closure operation on $Pow(G)$. If C is the associated closure class then the function $C \to S$ that maps each $X \in C$ to $\bigvee X \in S$ is an isomorphism between C and S with inverse the function that maps each $a \in S$ to $G_a \in C$.

■

4.8 Set Presentable $\bigvee$-Semilattices

Given a generating set G for S a subset R of $G \times Pow(G)$ is a **relation set over** G **for** S if for all $(a, X) \in G \times Pow(G)$

$$a \leq \bigvee X \ \leftrightarrow\ \exists Y \subseteq X\ [\ (a, Y) \in R\ ].$$

A **set presentation of** S is a pair (G, R) consisting of a generating set G for S and a relation set R over G for S.

Definition: 4.15 *A* **set presentable $\bigvee$-semilattice** *is a $\bigvee$-semilattice that has a set presentation.*

Example: For each set A the po-class $Pow(A)$ is a set presentable $\bigvee$-semilattice with set $G = \{\{a\} \mid a \in A\}$ of generators and relation set

$$R = \{(\{a\}, \{\{a\}\}) \mid a \in A\}.$$

Theorem: 4.16 *If $S = Pow(A)$, for some set A and C is a closure class then C is set-presentable if and only if the closure operation associated with C is set-based.*

Proof: Assume that $S = Pow(A)$, for some set A, and that c is the closure operation on S associated with C. Also assume that $B \subseteq S$ is a baseset for c. Then for all $X \subseteq A$ and all $a \in A$

$$(*) \qquad a \in c(X)\ \leftrightarrow\ \exists Y {\in} B\ [Y \subseteq X \wedge a \in c(Y)].$$

Now let A' be a regular set such that $B \cup G \subseteq A'$ and let

$$R = \{(Q, Z) \mid Q \in G \wedge Z \in A' \wedge Q \subseteq c(\cup Z) \wedge Z \subseteq G\}.$$

Claim 1: R is a set.

Proof: First observe that $T = \{Z \in A' \mid Z \subseteq G\}$ is a set. Also, for each $Z \in T$ we may form the set $\cup Z$ so that $c(\cup Z)$ is also a set and hence $S_Z = \{Q \in G \mid Q \subseteq c(\cup Z)\}$ is a set. Hence, by Union-Replacement $R = \bigcup_{Z \in T}(S_z \times \{Z\})$ is a set.
■

Now let $X \in Pow(G)$ and $Q \in C$.

Claim 2: $Z \subseteq X \wedge QRZ \;\rightarrow\; Q \subseteq \bigvee X$.

Proof: Let $Z \subseteq X \wedge QRZ$. Then $Q \subseteq c(\cup Z) \subseteq c(\cup X)$ and hence $Q \subseteq \bigvee X$.

■

Claim 3: $Q \subseteq \bigvee X \;\rightarrow\; \exists Z[Z \subseteq X \wedge QRZ]$.

Proof: Let $Q \subseteq \bigvee X$. Then by $(*)$ there is $Y \in B$ such that

$$Y \subseteq \cup X \wedge Q \subseteq c(Y).$$

As $Y \subseteq \cup X$

$$\forall y{\in}Y \exists Q'{\in}X \; y \in Q'.$$

As A' is regular, $Y \in A'$ and $X \subseteq A'$ there is $Z \in A'$ such that

$$\mathbb{B}(y{\in}Y, Q'{\in}Z)[\; y \in Q' \wedge Q' \in X \;].$$

So $Y \subseteq \cup Z$ and $Z \subseteq X$ so that $Q \subseteq c(\cup Z)$ and $Z \subseteq X \subseteq G$ and hence also QRZ.

■

It follows from these claims that (G, R) is a set presentation of C.

Now let (G, R) be a set presentation of a $\bigvee$-semilattice S. We show that S is isomorphic to a set presentable $\bigvee$-semilattice obtained from an inductive definition as above. Let Φ be the converse relation to R; i.e. it is the set of all pairs (X, a) such that aRX. Then Φ is an inductive definition that is a subset of $Pow(G) \times G$. Observe that there is a one-one correspondence between the class C of subsets X of G that are Φ-closed and the elements of S given by the function $C \rightarrow S$ mapping $X \mapsto \bigvee X$ and its inverse function $S \rightarrow C$ mapping $a \mapsto G_a = \{x \in G \mid x \leq a\}$. This is easily seen to be an isomorphism of the po-classes.

■

4.9 $\bigvee$-congruences on a $\bigvee$-semilattice

Let S be a $\bigvee$-semilattice . We define an equivalence relation $\approx$ on S to be a **$\bigvee$-congruence** on S if, for each set I, if $x_i, y_i \in S$ such that $x_i \approx y_i$ for all $i \in I$ then

$$\bigvee_{i\in I} x_i \approx \bigvee_{i\in I} y_i.$$

A preorder $\preceq$ on S is a **$\bigvee$-congruence pre-order** on S if for each subset X of S and each $a \in S$

$$\bigvee X \preceq a \;\leftrightarrow\; \forall x{\in}X\ [x \preceq a].$$

Proposition: 4.17 *There is a one-one correspondence between $\bigvee$-congruences and $\bigvee$-congruence pre-orders on S. To each $\bigvee$-congruence $\approx$ there corresponds the $\bigvee$-congruence pre-order $\preceq$ where*

$$x \preceq y \;\leftrightarrow\; \bigvee\{x, y\} \approx y.$$

Conversely to each $\bigvee$-congruence pre-order $\preceq$ corresponds the $\bigvee$-congruence $\approx$ where

$$x \approx y \;\leftrightarrow\; [x \preceq y \wedge y \preceq x].$$

These correspondences are inverses of each other.

Proposition: 4.18 *If $c : S \to S$ is a closure operation on S and we define $\approx$ by*

$$x \approx y \;\leftrightarrow\; c(x) = c(y)$$

for all $x, y \in S$ then $\approx$ is a $\bigvee$-congruence on S.

Proof: The relation $\approx$ is obviously an equivalence relation on S. Now suppose that $x_i \approx y_i$ for all $i \in I$, where I is a set. So $c(x_i) = c(y_i)$ for all $i \in I$. Let $x = \bigvee_{i\in I} x_i$ and $y = \bigvee_{i\in I} y_i$. Note that, as $y_i \leq c(y_i) = c(x_i)$ for all $i \in I$,

$$y = \bigvee_{i\in I} y_i \leq \bigvee_{i\in I} c(y_i) = \bigvee_{i\in I} c(x_i).$$

As $x_i \leq x$ for each $i \in I$ and c is monotone, $y \leq \bigvee_{i\in I} c(x_i) \leq c(x)$ and hence $c(y) \leq c(x)$. Similarily $c(x) \leq c(y)$ so that $c(x) = c(y)$. Thus we have shown that $\approx$ is a $\bigvee$-congruence on S.
■

Proposition: 4.19 *Let $\preceq$ be a $\bigvee$-congruence preorder on $S = Pow(A)$, where A is a set. Then the associated $\bigvee$-congruence $\approx$ comes from a closure operation c, as in the previous theorem, provided that for every $X \in S$ the class $\{a \in A \mid \{a\} \preceq X\}$ is a set. Then we can define $c(X)$ to be that set.*

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