Local Constructive Set Theory

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Some settings for constructive mathematics

- Dependent Type Theory (DTT)
- Constructive Set Theory (CST)
- Local Constructive Set Theory (LCST)
- DTT is intensional and keeps the fundamental constructive notions explicit.
- CST is fully extensional and expressed in the single-sorted language of axiomatic set theory.
- LCST is also extensional, but many-sorted and is a predicative variation on higher order arithmetic.

My motivation: To have a setting for topics in constructive mathematics, such as point-free topology, that allows a rigorous presentation that can be simply translated into both the DTT and CST settings.

Pragmatic Constructivism

- LCST is a setting for pragmatic constructivism. It is a generalised predicative version of John Bell's local set theory for impredicative constructivism (topos mathematics)
- Pragmatic constructivism has its origins in Bishop style constructive mathematics, as further developed by Bridges, Richman et al and influenced by Martin-Lof's DTT, by CST and by topos theory.
- Roughly, it is generalised predicative mathematics with intuitionistic logic. But it uses no form of choice, so as to be compatible with topos mathematics and allow sheaf models.
- A lot of elementary mathematics can be carried out in LCST; e.g. the categorical axiomatisation of the constructive Dedekind reals.

Simple type structures over the set N.

- Impredicative: $N \quad \mathcal{P}N \quad \mathcal{P}\mathcal{P}N \quad \cdots$
  For each set A, $\mathcal{P}A$ is the set of all subsets of A.
- Predicative: $N \quad Pow(N) \quad Pow(Pow(N)) \quad \cdots$
  For each class A, $Pow(A)$ is the class of all subsets of A.
- N is a set, but the assertion that $Pow(N)$ is a set is taboo!.
- Given A, what is a set of elements of A?
- Some notions of set of:
  - logical
  - combinatorial
  - hybrid

Notions of set of

- Logical: Sets of elements of A are given as extensions $B = \{x : A \mid R(x)\}$ of propositional functions R on A. Then $a \in B \equiv R(a)$. But this is the notion of class on A.
- Combinatorial: Sets of elements of A are given as families $B = \{a_i\}_{i:I}$ of elements a_i of A, indexed by an index type I. Then $a \in B \equiv (\exists i : I)[a =_A a_i]$.
- Hybrid Sets of elements of A are given as $B = \{a_i \mid i : I \mid R(i)\}$, where $\{a_i\}_{i:I}$ is a family of elements a_i indexed by an index type I and R is a propositional function on I. Then $a \in B \equiv (\exists i : I)[R(i) \wedge a =_A a_i]$.
- The combinatorial notion works when the type theory uses propositions-as-types. The hybrid notion works more generally for type theories that use a suitable treatment of logic.

Interpreting CST in DTT

- The iterative notion of set, used to interprete CST in DTT, uses an inductive type V whose single introduction rule is

$$\frac{a \text{ is a set of elements of } V}{a : V}$$

- The combinatorial notion of set of is used, assuming propositions-as-types, or more generally the hybrid notion might be used. The index types of the families are the 'small' types; i.e. the types in some type universe.
- The interpretation of LCST in DTT does not need the inductively defined type V. The powertype of a type A is just the type of sets of elements of the type A.

Many-sorted predicate logic

- We assume given an infinite supply of variables, $x, y, \ldots$, and some sorts, $\alpha, \beta, \ldots$.
- A context Γ has the form $x_1 : \alpha_1, \ldots, x_n : \alpha_n$, where $\vec{x} = x_1, \ldots, x_n$ is a list of distinct variables.
- We assume that, for each context Γ, the Γ-terms of sort α are defined in the standard way using variables declared in Γ and sorted individual constants and function symbols.
- The formulae are generated from the atomic formulae in the usual way using the logical operations, the logical constants $\bot, \top$, the binary connectives $\wedge, \vee, \rightarrow$ and the quantifiers $(\forall x : \alpha), (\exists x : \alpha)$. Each formula being a Γ-formula for some context Γ that declare the variables that may occur free in the formula.

Sequents

- We use a sequent version of natural deduction to formulate the axioms and rules of inference for intuitionistic logic. Sequents have the form $(\Gamma)\ \Phi \Rightarrow \phi$ where Γ is a context, Φ is a finite set of Γ-formulae and ϕ is a Γ-formula. In writing sequents we will omit (Γ) when Γ is the empty set and omit $\Phi \Rightarrow$ when Φ is the empty set.
- We present the logical axioms and rules of inference schematically, suppressing the parametric variable declarations and parametric assumption formulae.

The logical rules of inference

$$\frac{\phi \Rightarrow \psi}{\phi \rightarrow \psi} \qquad \frac{\phi \quad \phi \rightarrow \psi}{\psi} \qquad \frac{\phi \quad \psi}{\phi \wedge \psi} \qquad \frac{\phi \wedge \psi}{\phi} \qquad \frac{\phi \wedge \psi}{\psi}$$

$$\frac{\phi}{\phi \vee \psi} \qquad \frac{\psi}{\phi \vee \psi} \qquad \frac{\phi \vee \psi \quad \phi \Rightarrow \theta \quad \psi \Rightarrow \theta}{\theta}$$

$$\frac{(\forall x : \alpha)\phi_0}{\phi_0[a/x]} \qquad \frac{(x : \alpha)\ \phi_0}{(\forall x : \alpha)\phi_0}$$

$$\frac{\phi_0[a/x]}{(\exists x : \alpha)\phi_0} \qquad \frac{(\exists x : \alpha)\phi_0 \quad (x : \alpha)\ \phi_0 \Rightarrow \theta}{\theta}$$

Structural rules

Weakening $\dfrac{(\Gamma)\ \Phi \Rightarrow \phi}{(\Gamma')\ \Phi' \Rightarrow \phi}$ if $\Gamma \subseteq \Gamma'$ and $\Phi \subseteq \Phi'$,

Cut $\dfrac{(\Gamma)\ \Phi \Rightarrow \phi \qquad (\Gamma)\ \Phi, \phi \Rightarrow \theta}{(\Gamma)\ \Phi \Rightarrow \theta}$

Substitution $\dfrac{(\Gamma)\ \Phi \Rightarrow \phi}{(\Delta)\ \Phi[\vec{b}/\vec{x}] \Rightarrow \phi[\vec{b}/\vec{x}]}$

where Δ is a context and if Γ is the context $x_1 : \alpha_1, \ldots, x_n : \alpha_n$ then $\vec{x}$ is $x_1, \ldots, x_n$ and $\vec{b}$ is $b_1, \ldots, b_n$, with b_i a Δ-term of sort α_i for $i = 1, \ldots, n$. Also $\phi[\vec{b}/\vec{x}]$ is the result of simultaneously substituting b_i for x_i in ϕ for $i = 1, \ldots, n$ and $\Phi[\vec{b}/\vec{x}]$ is the set $\{\psi[\vec{b}/\vec{x}] \mid \psi \in \Phi\}$.

Adding equality

For each sort α we allow the formation of atomic formulae $(a =_\alpha b)$ for terms a, b of sort α.

Reflexivity axiom $(a =_\alpha a)$

Equality rule $$\frac{(a =_\alpha b) \quad \phi_0[a/x]}{\phi_0[b/x]}$$

for terms a, b of sort α and $(x : \alpha)$-formula ϕ_0.

Adding classes

We now allow the formation of *classes* $\{x : \alpha \mid \phi_0\}$ *on sort* α whenever ϕ_0 is a $(x : \alpha)$-formula. We also allow atomic formulae $a \in A$ whenever a is a term of sort α and A is a class on sort α. We add the following axiom scheme for all terms a of sort α and all $(x : \alpha)$-formulae ϕ_0.

Comprehension: $a \in \{x : \alpha \mid \phi_0\} \quad \longleftrightarrow \quad \phi_0[a/x]$

Some abbreviations

In the following ϕ_0 is a $(x : \alpha)$-formula, A, B are classes on sort α and $a, b, a_1, \ldots, a_n$ are terms of sort α.

$(\forall x \in A)\ \phi_0$	$(\forall x : \alpha)\ (x \in A\ \rightarrow\ \phi_0)$
$(\exists x \in A)\ \phi_0$	$(\exists x : \alpha)\ (x \in A\ \wedge\ \phi_0)$
$\{x \in A \mid \phi_0\}$	$\{x : \alpha \mid x \in A\ \wedge\ \phi_0\}$
$A \subseteq B$	$(\forall x \in A)\ x \in B$
$A = B$	$A \subseteq B\ \wedge\ B \subseteq A$
$\{a_1, \ldots, a_n\}_\alpha$	$\{x : \alpha \mid x =_\alpha a_1 \vee \cdots \vee x =_\alpha a_n\}$
$\delta_\alpha(a, b)$	$\{x : \alpha \mid x =_\alpha a \wedge x =_\alpha b\}$
$A \cup B$	$\{x : \alpha \mid x \in A\ \vee\ x \in B\}$
$A \cap B$	$\{x : \alpha \mid x \in A\ \wedge\ x \in B\}$
$\neg A$	$\{x : \alpha \mid x \notin A\}$

Adding product sorts

- Given sorts $\alpha_1, \ldots \alpha_n$ for $n \geq 0$, form the product sort
$$\alpha_1 \times \cdots \times \alpha_n$$
written **1** when $n = 0$ and $\alpha_1$when $n = 1$.
- Given terms $a_1 : \alpha_1, \ldots a_n : \alpha_n$, form the term
$$(a_1, \ldots, a_n) : \alpha_1 \times \cdots \times \alpha_n$$
written $* : \mathbf{1}$ when $n = 0$ and just $a_1 : \alpha_1$ when $n = 1$.
- Given a term $c : \alpha_1 \times \cdots \times \alpha_n$, form terms $c_i : \alpha_i$ for $i = 1, \ldots, n$.
- Add the axioms
$$\begin{aligned} (a_1, \ldots, a_n)_i &=_{\alpha_i} a_i \qquad (i = 1, \ldots, n) \\ (c_1, \ldots, c_n) &=_{\alpha_1 \times \cdots \times \alpha_n} c \end{aligned}$$

Some abbreviations

In the following abbreviations $A_1, \ldots A_n$ are classes on sorts $\alpha_1, \ldots, \alpha_n$ respectively, with $n \geq 2$, A, B, R are classes on sorts $\alpha, \beta, \alpha \times \beta$ respectively and a is a term of sort α.

$A_1 \times \cdots \times A_n$	$\{x : \alpha_1 \times \cdots \times \alpha_n \mid x_1 \in A_1 \wedge \cdots \wedge x_n \in A_n\}$
R^{-1}	$\{x : \alpha \times \beta \mid (x_2, x_1) \in R\}$
R_a	$\{y : \beta \mid (a, y) \in R\}$
$\bigcup_{x \in A} R_x$	$\{y : \beta \mid (\exists x \in A)\ y \in R_x\}$
$\bigcap_{x \in A} R_x$	$\{y : \beta \mid (\forall x \in A)\ y \in R_x\}$
$R : A \succ\!\!- B$	$(\forall x \in A)(\exists y \in B)\ (x, y) \in R$
$R : A \succ\!\!-\!\!\prec B$	$R : A \succ\!\!- B \quad \wedge \quad R^{-1} : B \succ\!\!- A$
$R : A \to B$	$R \subseteq A \times B \quad \wedge \quad R : A \succ\!\!- B$
	$\wedge \quad (\forall x, y \in R)\ [x_1 =_\alpha y_1 \to x_2 =_\beta y_2]$

Adding a natural numbers sort

We add a sort N of *natural numbers*, with terms 0 and $s(a)$ for a a term of sort N together with the following axioms, where A is a class on sort N.

$$(\forall x : N)\ \neg[0 =_N s(x)]$$

$$(\forall x : N)(\forall y : N)\ [s(x) =_N s(y) \to x =_N y]$$

$$(0 \in A)\ \wedge\ (\forall x \in A)(s(x) \in A)\ \Rightarrow\ (\forall x : N)[x \in A]$$

Adding power sorts

- Given a sort α, form the sort $\mathcal{P}\alpha$ of sets on sort α.
- We require that every term of sort $\mathcal{P}\alpha$ is a class on sort α.
- Add the following axiom for terms a, b of sort $\mathcal{P}\alpha$.

Extensionality axiom: $(a = b) \Rightarrow (a =_{\mathcal{P}\alpha} b)$

- We need to have some set existence axioms. Local Set Theory assumes that every class on sort α is a term of sort $\mathcal{P}\alpha$.
- But local set theory is thoroughly impredicative. We obtain local constructive set theory by instead adding some predicative set existence axioms. But first we introduce some more abbreviations.

Some more abbreviations

In the following A, B are classes of sorts α, β respectively.

$Pow(A)$	$\{x : \mathcal{P}\alpha \mid x \subseteq A\}$
$\mathsf{S}A$	$(\exists y : \mathcal{P}\alpha)\ y = A$
B^A	$\{z \in Pow(A \times B) \mid z : A \to B\}$
$mv(B^A)$	$\{z \in Pow(A \times B) \mid z : A \succ\!\!\!- B\}$

Set Existence axioms

In the following axioms A, B, R are classes on sorts $\alpha, \beta, \alpha \times \beta$ respectively.

Finite sets $(\forall x_1, \ldots, x_n : \alpha)\ \mathsf{S}\{x_1, \ldots, x_n\}_\alpha\ (n \geq 0)$,

Equality sets $(\forall x, y : \alpha)\ \mathsf{S}\delta_\alpha(x, y)$,

Indexed Union $\mathsf{S}A \ \wedge\ (\forall x \in A)\mathsf{S}R_x \ \Rightarrow\ \mathsf{S}\bigcup_{x \in A} R_x$,

Infinity $\mathsf{S}N$,

Strong Collection

$$\mathsf{S}A \ \wedge\ (R : A \succ\!\!- B) \ \Rightarrow\ (\exists z \in Pow(B))\ R : A \succ\!\!-\!\!\prec z,$$

Subset Collection

$$\mathsf{S}A \ \wedge\ \mathsf{S}B \\ \Rightarrow\ (\exists z \in Pow(mv(B^A)))(\forall u \in mv(B^A))(\exists u' \in z)\ u' \subseteq u,$$