Explicit Set Existence

3-16 July, 2009, Leeds Symposium on Proof Theory and Constructivism

.

Peter Aczel

petera@cs.man.ac.uk

Manchester University

Explicit Set Existence

I Inexplicitness of AC over ZF?

II Core Mathematics

III Fullness

IV The reals

V Explicit Fullness

VI Deterministic Inductive Definitions

I: Inexplicitness of AC over ZF?

- All the set existence axioms and schemes of ZF are explicit; e.g.
- Pairing: The class $\{a, b\} = \{x \mid x = a \vee x = b\}$ is a set, for all sets a, b.
- Replacement: For all sets $a, \ldots$,

$$\forall x \in a \, \exists! y \, \phi(x, y, \ldots) \;\Rightarrow\; \{y \mid \exists x \in a \, \phi(x, y, \ldots)\} \text{ is a set.}$$

- AC seems to be essentially inexplicit over ZF; i.e. every explicit theorem of ZFC seems to be 'equivalent' to an explicit theorem of ZF.
- Can this idea be made precise?
- $ZFC \vdash$ '$\{x \mid \neg AC\}$ is a set', but if ZF is consistent $ZF \nvdash$ '$\{x \mid \neg AC\}$ is a set'.

Some brands of mathematics

- Classical, with AC
- Classical, without any choice
- Topos
- Constructive, Brouwer style - Intuitionism
- Constructive, Markov style - Recursive
- Constructive, Bishop style
- Constructive, Richman style (= Bishop without any choice)

II: Core Mathematics

- All these, and others, are brands of mathematics.
- They are open conceptual frameworks.
- A lot of constructive mathematics can be derived in all these brands.
- Some mathematical principles are brand-essential.
  - Choice principles: AC, CC, DC, RDC, PA, ... etc
  - Logical: EM, REM, LPO, LLPO, MP, ... etc
  - Impredicative: Powerset, Full Separation, ...

Some criteria for Core Mathematics

- Extensional
- Adequate
- Compatible
- Local
- Explicit
- Some problems with $\mathbf{CZF}$ for a core system:
  - Strong Collection is inexplicit.
  - Subset Collection (Fullness) is inexplicit.
  - Set Induction is not local.
- Problems with $\mathbf{CZF}^{-}_{R,E}$:
  - Cannot show that $\mathbb{R}_d$ is a set.
  - Do not have apparatus to define the class of hereditarily countable sets, etc.

III: The Fullness Axiom, 1

- The Fullness axiom is an inexplicit set existence axiom that can be used instead of the Subset Collection Scheme in axiomatizing CZF.
- In CZF the axiom has been used to prove Myhill's Exponentiation axiom and also to prove that the class of Dedekind reals, $\mathbb{R}_d$, is a set and several other results.
- Some notation, for classes A, B, R:
  - $R : A \succ\!\!- B$ if $\forall x \in A\, \exists y \in B\ (x, y) \in R$.
  - $R : A \succ\!\!-\!\!\prec B$ if $R : A \succ\!\!- B\ \&\ R^{-1} : B \succ\!\!- A$.
  - $mv(A, B) = \{r \in Pow(A \times B) \mid r : A \succ\!\!- B\}$.
  - $B^A = \{f \in mv(A, B) \mid f$ is single valued$\}$.

The Fullness Axiom, 2

- Exponentiation Axiom: $Exp(A, B)$ for all sets A, B, where

$$Exp(A, B) \equiv B^A \text{ is a set.}$$

- Fullness Axiom: $Full(A, B)$ for all sets A, B, where

$$Full(A, B) \equiv mv(A, B) \text{ has a full subset,}$$

where, for a class $C \subseteq X$,

$$C \text{ is a full subclass of } X \text{ if } \forall r \in X\ \exists s \in C\ s \subseteq r.$$

- Strong Collection Scheme: For each class R and every set A, if $R : A \succ\!\!- V$ then $R : A \succ\!\!-\!\!\prec B$ for some set B.
- AC(A,B): B^A is a full subclass of $mv(A, B)$.

Fullness and Exponentiation

- The axiom system BCST has Extensionality, Pairing, Union, Δ_0-Separation and Replacement.
- Theorem: In BCST,
  1. $Full(A, B) \Rightarrow Exp(A, B)$,
  2. $AC(A, B) + Exp(A, B) \; \Rightarrow \; Full(A, B)$.

IV: The Dedekind Reals,1: Weak cuts

- $X \subseteq \mathbb{Q}$ is a weak left cut if

  1-l: $\exists r(r \in X)\ \&\ \exists s(s \notin X)$,

  2-l: $r \in X \iff \exists s\, r < s \in X$.

- $Y \subseteq \mathbb{Q}$ is a weak right cut if

  1-r: $\exists r(r \in Y)\ \&\ \exists s(s \notin Y)$,

  2-r: $r \in Y \iff \exists s\, r > s \in Y$.

- (X, Y) is a weak cut if
  - X is a weak left cut and Y is a weak right cut,
  - $X \cap Y = \emptyset$,
  - $r < s \Rightarrow (r \notin X \Rightarrow s \in Y)\ \&\ (s \notin Y \Rightarrow r \in X)$.
- (X, Y) is located if $r < s \Rightarrow (r \in X \vee s \in Y)$.
- X is located if $r < s \Rightarrow (r \in X \vee s \notin X)$.

The Dedekind Reals,2

- A (left) cut is a located weak (left) cut.
- Note: Classically every weak (left) cut is located.
- Proposition: The following are equivalent:
  - X is a left cut,
  - (X, Y) is a cut for some Y,
  - (X, Y) is a cut, where $Y = \{s \in \mathbb{Q} \mid \exists r < s\ r \notin X\}$.
- Definition: The class $\mathbb{R}_d$ of Dedekind reals is the class of all left cuts. Note: $\mathbb{R}_d$ is a Δ_0-class.
- Prop: A weak left cut X is located (and so in $\mathbb{R}_d$) iff

$$\forall \epsilon > 0\ \exists r \in X\ r + \epsilon \notin X; \quad \text{i.e. } R_X \in mv(\mathbb{Q}^{>0}, \mathbb{Q}),$$

where $R_X = \{(\epsilon, r) \in \mathbb{Q}^{>0} \times X \mid r + \epsilon \notin X\}$.

The Dedekind Reals,3

- Theorem: Assuming $Full(\mathbb{N}, \mathbb{N})$, the class of Dedekind reals is a set.
- Proof: Assuming $Full(\mathbb{N}, \mathbb{N})$, as $\mathbb{Q}^{>0} \sim \mathbb{N}$ and $\mathbb{Q} \sim \mathbb{N}$, we also have $Full(\mathbb{Q}^{>0}, \mathbb{Q})$.
- So we may choose a full subset $\mathcal{C}$ of $mv(\mathbb{Q}^{>0}, \mathbb{Q})$
- For $R \in mv(\mathbb{Q}^{>0})$ let

$$X_R = \{r \in \mathbb{Q} \mid r < s \text{ for some } (\epsilon, s) \in R\},$$

- Now let $\mathcal{C}_X = \{R \in \mathcal{C} \mid X_R \in \mathbb{R}_d\}$ and

$$\mathbb{R}' = \{X_R \mid R \in \mathcal{C} \;\&\; X_R \in \mathbb{R}_d\} = \{X_R \mid R \in \mathcal{C}_X.\}$$

- Then, by Δ_0-Separation and Replacement $\mathbb{R}'$ is a set.

The Dedekind Reals,4

- If $R \in \mathcal{C}$, $X_R = \{r \in \mathbb{Q} \mid r < s$ for some $(\epsilon, s) \in R\}$.
- $\mathbb{R}' = \{X_R \mid R \in \mathcal{C} \;\&\; X_R \in \mathbb{R}_d\}$ is a set.
- It suffices to show that $\mathbb{R}_d = \mathbb{R}'$.
- $\mathbb{R}_d \supseteq \mathbb{R}'$ trivially.
- If $X \in \mathbb{R}_d$,
  $R_X = \{(\epsilon, r) \in \mathbb{Q}^{>0} \times X \mid r + \epsilon \notin X\} \in mv(\mathbb{Q}^{>0}, \mathbb{Q})$.
- For $\mathbb{R}_d \subseteq \mathbb{R}'$ it suffices to prove
- Lemma (ECST) : *Let $X \in \mathbb{R}_d$ and $R \in \mathcal{C}$. Then*
  $R \subseteq R_X \;\Rightarrow\; X = X_R$.

$$
\begin{aligned}
X \in \mathbb{R}_d \quad &\Rightarrow R_X \in mv(\mathbb{Q}^{>0}, \mathbb{Q}), \text{ as } X \text{ is located} \\
&\Rightarrow R \subseteq R_X \text{ for some } R \in \mathcal{C}, \text{ as } \mathcal{C} \text{ is a full subset} \\
&\Rightarrow X = X_R, \text{ by the lemma .}
\end{aligned}
$$

V: Explicit Fullness; the scheme

- For classes F, X, A such that $F : X \to V$ and $A \subseteq X$, F is A-powerful if, for all $r \in A$ there is $r' \in A$ such that

$$(*)\ \ \forall s \in X[s \subseteq r' \ \Rightarrow\ s \in A \,\&\, Fs = Fr].$$

- The Explicit Fullness Scheme (EFS): If $F : mv(B, C) \to V$ is A-powerful, where B, C are sets and A is a Δ_0-subclass of $mv(B, C)$ then $FA = \{Fr \mid r \in A\}$ is a set.
- Note that EFS is an explicit set existence scheme.
- Theorem: In BCST, Fullness implies each instance of EFS. In fact Full(B,C) implies the above instances, EFull(B,C), of EFS.
- Lemma: If $F : X \to V$ is A-powerful, A is a Δ_0-subclass of X and X has a full subset D then FA is a set.

V: Explicit Fullness; applications

- Theorem(BCST+EFS): Exponentiation
- Proof: Given sets B, C, to show that C^B is a set, apply EFS with $A = C^B$ and $Fr = r$ for $r \in mv(B, C)$.
- Theorem(BCST+EFS): Let Q, A be sets such that $A \subseteq Q \times Q$. Then $\mathcal{R}$ is a set, where $\mathcal{R}$ is the class of subsets X of Q such that
  - X is open; i.e. $\forall x \in X \exists y \in X (x, y) \in A$, and
  - X is located; i.e. $\forall (x, y) \in A [x \in X \vee y \notin X]$.
- Note: The proof only uses EFS(A,2).
- Corollary(ECST+EFS$(\mathbb{N}, 2)$): $\mathbb{R}_d^e$ and $\mathbb{R}_d$ are sets.
- Here $\mathbb{R}_d^e$ is the class of open, located subsets of $\mathbb{Q}$, where $A = \{(r, s) \in \mathbb{Q} \times \mathbb{Q} \mid r < s\}$.
- Note that $\mathbb{R}_d \subseteq \mathbb{R}_d^e$.

VI: Deterministic Inductive Definitions, 1

- Let Φ be a class. A Φ-step, X/y, is a pair $(X, y) \in \Phi$. A class A is Φ-closed if

$$X \subseteq A \;\Rightarrow\; y \in A, \text{ for all } \Phi\text{-steps } X/y.$$

- Theorem (CZF-Subset Collection): For each class Φ there is a smallest Φ-closed class $I(\Phi)$.
- The proof makes essential use of Strong Collection and Set Induction.
- Φ is deterministic if

$$\text{If } X_1/y \text{ and } X_2/y \text{ are } \Phi\text{-steps then } X_1 = X_2.$$

- ECST is BCST+Strong Infinity.

Deterministic Inductive Definitions, 2

- Theorem(ECST+Set Induction): The smallest class $I(\Phi)$ exists for each deterministic class Φ.
- Examples:
  1. For each class A, $H(A) = I(\Phi_A)$, where Φ_A is the class of steps y/y such that y is an image of a set in A. So $H(\omega)$ is the class of hereditarily finite sets and $H(\omega \cup \{\omega\})$ is the class of hereditarily countable sets; i.e hereditarily finite or an image of ω. Here ω is the smallest inductive set, given by Strong Infinity.
  2. If A, R are classes, with $R \subseteq A \times A$ such that $R_y = \{x \in A \mid (x, y) \in R\}$ is a set for each $y \in A$, the class $WF(A, R) = I(\{R_y/y \mid y \in A\})$ is the well-founded part of R in A.
  3. Also the W-classes are given by deterministic inductive definitions.

CONCLUSION

- A possible useful axiom system for my core mathematics might be ECST+EFS+DIDS, where DIDS is a scheme in an extension of the language so as to obtain a class $I(\Phi)$ from a class Φ.
- The scheme should express that if Φ is deterministic then $I(\Phi)$ is the smallest Φ-closed class.
- The Replacement scheme and EFS need to be extended to the extended language.
- I conjecture that it has the same logical strength as CZF.