Aspects of General Topology in Constructive Set Theory

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1 Introduction

The subject of formal topology was originally developed in [19] and developed further in [21, 8, 20, 9] and other papers, as a constructive version of the point-free approach to general topology, as carried out in the setting of Martin-Löf's dependent type theory, [15]. The ideas for point-free topology have their origins in Brouwer's intuitionistic conception of the continuum and were developed more recently using the notion of a locale, particularly in connection with topos theory. Topos mathematics, the mathematics that can be generally carried out in a topos, is fully impredicative. So the point-free topology that can be developed in topos mathematics needs to be adapted to the setting of the predicative dependent type theory of Per Martin-Löf and, for that reason the notion of a formal topology has become a focus of attention as the appropriate substitute for the notion of a locale.

As in classical mathematics the mainstream approach to the constructive study of topological notions has not been point-free. The main focus in connection with constructive analysis has been on the notion of a separable metric space of points and not on the general notion of a topological space of points, see [7]. A possible exception is Bishop's, [7], notion of a neighborhood space, which corresponds to the notion, in Definition 2 of this paper, of a set-indexed base on a set. But Bishop's notion does not play a significant role in [7]. Moreover the notion of a real number has been developed using Cauchy sequences of rationals assuming countable and dependent choices. By contrast the main notion of real number in topos mathematics has been in terms of Dedekind cuts and the axioms of countable and dependent choices are not assumed.

There is a need to reconcile the different constructive approaches to the treatment of the continuum and topological notions in general, both point-set and point-free. The aim of this paper is to initiate such a reconciliation by developing a constructive set-theoretical approach to some notions of general topology that includes a treatment of both point-set and point-free notions and some relationships between them.

Constructive set theory is the set theoretical approach to a certain minimalist brand of constructive mathematics. It is minimalist in the sense that it makes fewer foundational assumptions than the other brands on the market and so can be viewed as a central core of constructive mathematics, the other brands being obtained by allowing additional principles to be used.

Let us review some of the standard approaches to constructive mathematics. The historically first serious approach is the intuitionistic mathematics of Brouwer. Brouwer was keen to accept principles that were inconsistent with classical mathematics, for example the continuity principle that all totally defined functions on the reals are continuous. Another significant approach to constructive mathematics has been the recursive constructivism of Markov and his school. This approach is also inconsistent with classical mathematics, having Church's Thesis in the form that all totally defined functions on the natural numbers are recursive. It also has Markov's principle, which expresses that every computation that does not continue for ever must terminate. A contrast to these two approaches has been the more recently developed approach of Bishop and his school, which only assumes principles that are common to classical mathematics and each of the two other approaches to constructivism we have mentioned. Brouwer, Markov and Bishop all accepted the principles of intuitionistic logic and did not accept the classical law of excluded middle. Intuitionistic logic has also played a central role in categorical logic and in particular is the internal logic holding in all toposes. For this reason many category theorists have considered that the mathematics that generally holds in a topos is also a brand of constructive mathematics. This brand of constructivism is also compatible with classical mathematics. But it differs from the Bishop brand in at least two respects. First it is explicitly impredicative while Bishop's constructive mathematics is essentially predicative in that it does not seem to use any fully impredicative principles. The second respect in which topos mathematics, as I will here call the category theorists brand of constructive mathematics, differs from Bishop mathematics is that Bishop, along with Brouwer and Markov, accepted the principles of countable choice and dependent choices[1]. These choice principles are definitely not accepted in topos mathematics as they generally do not hold in a topos, although of course they do hold in some

[1]Note that the countable choice principle is a consequence of the dependent choices principle.

toposes.

From the above review it should now be clear that the minimalist brand of constructivism to be considered here will be obtainable from Bishop constructivism by simply dropping from its principles the countable choice and dependent choices principles.

Constructive set theory was initiated by Myhill in his paper [16] as a set theoretical foundation for Bishop constructivism. Another foundational approach to Bishop constructivism has been Martin-Löf's Type Theory, [15]. Both Myhill and Martin-Löf accept the countable and dependent choice principles. Aczel, in [1, 2, 3], introduced the system **CZF** and $\mathbf{CZF}^+ = \mathbf{CZF} + \mathbf{REA}$ which we take to be the main formal systems for constructive set theory. When dependent choices is added to **CZF** we get a system that is close to Myhill's original formal system. The natural subsystem of **ZF** for topos mathematics is the system **IZF** which has the same logical strength as **ZF**. By contrast the system **CZF** is a logically much weaker subsystem of **IZF**. We refer the reader to [5] for a more recent presentation of the ideas of constructive set theory and the axiom systems **CZF** and $\mathbf{CZF}^+$.

The paper is organised as follows. In section 2 we discuss the classical Galois adjunction between topological spaces and locales and some of the problems involved in getting a version of the adjunction to hold in constructive set theory. In the section we state the Galois Adjunction Theorem, Theorem 1, which is eventually proved at the end of section 5. This is our first main result, giving a resolution of the problems in **CZF**. Section 3 is devoted to first setting up the general notion of a topological superspace which has a class of points and a collection of open classes of points. More specialised is the more familiar notion of a topological space with a set of points and a class of open subsets. But more important for us will be the notions of ct-space and standard ct-space , where the points still need not form a set, but where there is a set-indexed family of basic open classes that form a base for the collection of open classes. When a ct-space is small, i.e. the points form a set, then the space is standard and we have the notion of a topological space with a base of open sets that forms a set.

In section 4 we turn to the point-free approach to topology. We start with the general algebraic notion of a locale which is a partially ordered class satisfying suitable properties. Although the locales of interest will generally not be known to have their elements form a set, there will usually be a set of generators, so that the main notion of interest will be that of a set-generated locale. This leads to the notion of a formal topology, which is a notion essentially equivalent to that of a set-generated locale. Each standard ct-space determines a formal topology, this construction being our constructive version of the classical operation that associates with each topological space its locale of open subsets. Conversely each formal topology determines a ct-space of its formal points. The formal topology is defined to be standard if the resulting ct-space is standard and finally these operations give the Galois adjunction between the superlarge categories of standard ct-space and standard formal topologies that we were seeking.

In general the formal points of a formal topology need not form a set. But often they do. The main aim of the rest of the paper is to obtain some results about this. In section 6 we focus on the notion of a set-presentable formal topology, which corresponds to the notion of a set-presentable locale but we prefer to limit our attention to formal topologies. This notion is closely related to the notion of a covering system and the use of a covering system to inductively generate a formal topology. In fact we show that in $\mathbf{CZF}^+$ the inductively generated formal topologies are exactly the set-presentable ones. We show that in **CZF** every locally compact formal topology is set-presentable. We end the section by giving a set theoretical variant of a construction of Erik Palmgren. We use it to show that in an extension of $\mathbf{CZF}^+$ if the formal points of a set-presentable formal topology form a T_1 ct-space then the points form a set. The main construction can be put in an abstract form which is proved in Appendix 1.

Section 7 is devoted to the notion of a regular formal topology and a set theoretical generalisation of a type theoretic result of Giovanni Curi. He showed that in Martin-Löf type theory the formal points of any regular locally compact formal topology form a set. We show in **CZF** that 'locally compact' can be weakened to 'set-presentable'. The main result used in the proof is left to Appendix 2.

2 The Galois Adjunction Theorem

At the heart of the classical point-free approach to topology is the Galois adjunction[2] between the category Top of topological spaces and the category Loc of locales. This adjunction still works over any topos and works in the Intuitionistic set theory **IZF**. The aim here is to present the adjunction in the context of the much weaker subtheory **CZF** of **IZF**. The theory **CZF** avoids the impredicativity of **IZF** by not assuming the Powerset Axiom, using Subset Collection in its place, and only assumes the Restricted Separation Scheme, rather than the full Separation Scheme. To compensate **CZF** has

[2]It is the fact that the adjunction is Galois that gives rise to the equivalence between the full subcategories of sober spaces and spatial locales. We will explain when an adjunction is Galois below.

Strong Collection, which in **IZF** follows from the ordinary Collection Scheme using full Separation. So **IZF** may be viewed as being obtained from **CZF** by adding the Powerset Axiom and the full Separation Scheme. Obviously we want to construct in **CZF** an adjunction between two categories which will give us the standard adjunction when taken in **IZF**.

It is a feature of our work in **CZF** that we have to work with superlarge entities of the size of the category of classes, whose objects and maps are class-sized. So we are working in a metatheory which has not only classes of sets as well as sets but also collections of classes. We presume that this metatheory is conservative over **CZF**, but do not attempt to spell out the details here, which we expect to be routine but tedious. An alternative approach might be to understand the notion of a collection of classes in the following way. As usual each class can be taken to be given as $\{x \mid \phi(x,\ldots)\}$, for some formula $\phi(x,\ldots)$. A collection of such classes can be understood to be given by a formula $\theta(X,\ldots)$ in the language of **CZF** augmented with a unary predicate symbol X. The formula is taken to define the collection of those classes $\{x \mid \phi(x,\ldots)\}$ such that θ' holds where θ' is the result of substituting $\phi(y)$ for each atomic formula $X(y)$ in $\theta(X,\ldots)$.

Galois Adjunctions

The data for an adjunction between two categories can be given as a pair of adjoint functors with the appropriate natural transformations. Here we will use the following standard alternative. A *right adjoint* is a functor $G : D \to C$ and assignments of an object A_X of D and a map $\eta_X : X \to GA_X$ of C to each object X of C such that for each object A' of C and map $\eta' : X \to GA'$ of D there is a unique map $h : A_X \to A'$ of C such that $\eta' = (Gh) \circ \eta_X$.

This data uniquely determines a left adjoint of G; i.e. a functor $F : C \to D$ given by the following. $FX = A_X$ for each object X of C and if $g : X \to X'$ in C then Fg is the unique $h : A_X \to A_{X'}$ such that

$$\eta_{X'} \circ g = (Gh) \circ \eta_X.$$

It follows that the map $\eta_X : X \to G(FX)$ is natural in X. Note that a right adjoint G is an *equivalence* if the functor G is full and faithfull and each map η_X is an isomorphism $X \cong G(FX)$. More generally the right adjoint is a *Galois right adjoint* if η_X is an isomorphism whenever X has the form GY for some object Y of D. When G is a Galois right adjoint its restriction to the full subcategory $\hat{D}$ of D consisting of those objects Y of D such that $Y \cong F(GY)$ is an equivalence between that category and the full subcategory $\hat{C}$ of C consisting of those objects X in C such that

$X \cong G(FX)$. An adjunction is a *Galois adjunction* if its right adjoint is Galois.

There does not seem to be much of a literature on the notion of a Galois adjunction. The source for the notion used here is Exercise 19D on page 299 of the book [6].

The theorem

In the case of the classical Galois adjunction between the category Top of topological spaces and the category Loc of locales there are functors $\Omega : Top \to Loc$ and $Pt : Loc \to Top$ with Pt a right adjoint of Ω. The functor Ω associates with each topological space the locale of open sets of the space and conversely the functor Pt associates with each locale the topological space of its formal points; e.g. represented as its completely prime filters. The natural transformation $\eta : Id_{Top} \to Pt \circ \Omega$ associates to each topological space X the map $\eta_X : X \to Pt(\Omega X)$ where, for each point $x \in X$, $\eta_X(x)$ is the completely prime filter of open neighborhoods of x.

Our aim is to prove the following result.

Theorem: 1 (The Galois Adjunction Theorem) *In* **CZF** *there are (superlarge) categories Top' and Loc' and functors $\Omega' : Top' \to Loc'$ and $Pt' : Loc' \to Top'$ with Pt' a Galois right adjoint of Ω', such that, in* **IZF**, *Top' is equivalent to Top and Loc' is equivalent to Loc and the adjoint functors correspond to the classical ones.*

Note that Top' and Loc' are superlarge like the category of classes; i.e. the objects of the category are themselves possibly large objects as are the maps.

The first barrier to a simple proof of this result is that the open sets of a topological space cannot generally be shown to form a set in **CZF** so that we need a class sized notion of locale. This gives rise to the problem that the formal points of a class sized locale, when taken to be the completely prime filters, also become class sized and so cannot be the elements of a topological space. This problem can be overcome by restricting attention to set-generated class sized locales so that the completely prime filters can be represented by suitable sets of generators. But using these to be the formal points still leaves the problem that the formal points cannot generally be shown to form a set and so cannot form a (set-sized) topological space. This problem can be overcome by formulating a class sized notion of topological space having a set-indexed family of open classes. There are further problems to overcome. But in the end we are able to prove the theorem in section 5.

3 Spaces

3.1 Topological superspaces

First we review the notion of a topological space as it must be in **CZF**. As usual we take a topological space to be a set equipped with a topology on it. We must take a topology on a set to be a class of open subsets satisfying the usual closure properties; i.e. the empty set and the whole set must be open, the intersection of two open sets must be open and the union of any set of open sets must be open. We cannot expect that the open sets will form a set in general. This is because if the open sets of an inhabited topological space were to form a set then in **CZF** we could deduce the Powerset Axiom, something we do not want to assume. But we can often expect that there will be a set basis of basic open sets for the topology. More specifically we will use the following notion which is essentially the same as Bishop's, [7], notion of a neighborhood space.

Definition: 2 *A* set-indexed base *on a set X is a family $\{B_a\}_{a\in S}$ of subsets B_a of X indexed by a set S such that the following conditions hold.*

1. $X = \bigcup_{a\in S} B_a$,
2. $B_{a_1} \cap B_{a_2} = \bigcup_{a\in a_1\downarrow a_2} B_a$ *for* $a_1, a_2 \in S$, *where*

$$a_1 \downarrow a_2 = \{a \in S \mid B_a \subseteq B_{a_1} \cap B_{a_2}\}.$$

Each set-indexed base on a set X determines the topology on X where a set is open if it is a union of basic open sets B_a. Note that in **IZF** each topological space X has a canonical set-indexed base $\{Y\}_{Y\in\mathcal{O}}$, where $\mathcal{O}$ is the set of open subsets of X.

In the following we will represent topological spaces that are equipped with a set-indexed base as special examples of the following notion.

Definition: 3 *A* topological superspace *consists of a class X of its points, a class S of its neighborhood indices and a class relation, $\Vdash$, from the class X to the class S; i.e. $\Vdash$ is a subclass of $X \times S$. The conditions, 1,2 of Definition 2 must hold, where, for $a \in S$, B_a is the class $\{x \in X \mid x \Vdash a\}$.*

Note that each topological space X can be viewed as a topological superspace where the class, S, of neighborhood indices is just the topology of its open subsets and the relation $\Vdash$ is given by

$$x \Vdash a \iff x \in a$$

for $(x, a) \in X \times S$. A subclass Y of X is defined to be *open* if it is a union of basic open classes B_a. Note that the open classes of a topological superspace X form a superlarge 'topology' on the class X; i.e.

1. $\emptyset, X$ are open classes,

2. if Y_1, Y_2 are open classes then so is $Y_1 \cap Y_2$,

3. if $\{Y_i\}_{i \in I}$ is a family of open classes Y_i, indexed by the class I; i.e. for some subclass Y of $I \times X$ each class $Y_i = \{x \in X \mid (i, x) \in Y\}$ is open.

Moreover the classes B_a form a class base for the 'topology'. For each subclass U of S let

$$B_U = \bigcup_{a \in U} B_a$$

and let

$$\mathcal{A}U = \{a \in S \mid B_a \subseteq B_U\}.$$

Note that B_U is always an open class and, for subclasses U, V of S,

1. $U \subseteq \mathcal{A}U$,

2. $U \subseteq \mathcal{A}V \Rightarrow \mathcal{A}U \subseteq \mathcal{A}V$,

3. $\mathcal{A}U \cap \mathcal{A}V \subseteq \mathcal{A}(U \downarrow V)$,

   where $U \downarrow V = \{a \in S \mid (\exists u \in U)(\exists v \in V)\ B_a \subseteq (B_u \cap B_v)$.

Call a subclass U of S *saturated* if $\mathcal{A}U = U$ or, equivalently if $U = \mathcal{A}V$ for some subclass V of S. For each open class Y the class

$$\{a \in S \mid B_a \subseteq Y\}$$

is the unique saturated class U such that $B_U = Y$. Thus there is a one-one correspondence between the open classes of points and the saturated classes of neighborhood indices.

A map from a topological superspace X to a topological superspace X' is defined to be a continuous function $g : X \to X'$; i.e. a class function such that $g^{-1}B'_{a'}$ is an open class for each $a' \in S'$. It follows that $g^{-1}Y'$ is an open subclass of X for each open subclass Y' of X'. With this definition of map it is routine to check that the superspaces form a superlarge category Top_0 and we can define the category Top to be the full subcategory of topological spaces equipped with their canonical set-indexed bases. For each subclass U' of S' let

$$g^*U' = \{a \in S \mid B_a \subseteq g^{-1}B'_{U'}\}.$$

Observe that, for all $U' \in Pow(S')$,

$$g^{-1}B'_{U'} = B_{g^*U'}.$$

3.2 Constructive and Standard topological spaces

Constructive topological spaces

The category Top_0 is too large for our purposes while the full subcategory Top is too small. We next focus on a full subcategory Top_1 of Top_0 which is equivalent, in **IZF** to the category Top.

Definition: 4 *We call a topological superspace X a* constructive topological space, abbreviated ct-space *if*

1. *S is a set,*
2. *$\alpha_x = \{a \in S \mid x \Vdash a\}$ is a set for each $x \in X$,*
3. *$\{y \in X \mid \alpha_y = \alpha_x\}$ is a set for each $x \in X$.*

We define Top_1 to be the full subcategory of Top_0 consisting of the ct-spaces.

Standard topological spaces

In order to associate a 'locale of opens' to each space the notion of ct-space seems to be a little too general. We define a suitable subcategory Top_2 of Top_1.

Definition: 5 *A* standard space *is a ct-space X such that $\mathcal{A}U$ is a set for each subset U of S. A continuous function $g : X \to X'$, where X, X' are ct-spaces, is defined to be* standard continuous *if X is standard and $g^*a' = g^*\{a'\}$ is a set for each $a' \in S'$. It follows that g^*U' is a set for each subset U' of S'.*

Proposition: 6

1. *If X is a standard space then $id_X : X \to X$ is standard continuous.*
2. *If $g : X \to X'$ and $g' : X' \to X''$ are standard continuous functions then so is $g' \circ g : X \to X''$.*

Corollary: 7 *The standard spaces and standard continuous functions between them form a subcategory Top_2 of Top_1.*

The T_n Separation Properties

Here we only consider weak constructive versions of the T_n separation properties. These and other versions are discussed in more detail in [4]. Let X be a ct-space with set S of neighborhood indices. For subsets α, β of S let

$$\alpha \sim \beta \quad \text{if } \forall a \in \alpha \, \forall b \in \beta \, \exists z \in X \ (a, b \in \alpha_z).$$

For $n = 0, 1, 2$ we have the following separation properties T_n.

$$\begin{array}{ll} X \text{ is } T_0 & \text{if } (\forall x, y \in X)(\alpha_x = \alpha_y \Rightarrow x = y) \\ X \text{ is } T_1 & \text{if } (\forall x, y \in X)(\alpha_x \subseteq \alpha_y \Rightarrow x = y) \\ X \text{ is } T_2 & \text{if } (\forall x, y \in X)(\alpha_x \sim \alpha_y \Rightarrow x = y) \end{array}$$

Regular spaces

In order to define the notion of a regular ct-space , if Y, Y' are open subclasses of act-space X let $Y' \prec Y$ iff there is a a set U of neighborhood indices such that, if $Y'' = B_U$ then $Y'' \cap Y' = \emptyset$ and $Y'' \cup Y = X$. The idea here is that, classically, $Y' \prec Y$ expresses that the *closure* of Y' is a subclass of Y, without mentioning *closure*.

Definition: 8 *The* ct*-space X is defined to be* regular *if, for each $x \in X$*

$$\forall a \in \alpha_x \, \exists b \in \alpha_x \ B_b \prec B_a.$$

A T_3 space is a regular T_0 space.

Sober spaces

This property can be considered as another separation property. But because of its significance for point free topology we consider it separately. First let us call a set $\alpha \subseteq S$ of neighborhood indices of a ct-space X an *ideal point* of X if the following conditions hold.

1. $(\exists b)[b \in \alpha]$,
2. $(\forall b, c \in \alpha)(\exists a \in \alpha)[B_a \subseteq B_b \cap B_c]$,
3. $B_a \subseteq B_U \Rightarrow (\exists b \in \alpha)[b \in U]$, for all $a \in \alpha$ and $U \in Pow(S)$.

Note that α_x is an ideal point of X for every point x of X. We call the space X a *sober space* if for every ideal point α of X there is a unique point x of X

such that $\alpha = \alpha_x$. Note that every sober space is T_0. The ideal points form a ct-space $sob(X)$ having the same set S of neighborhood indices as X, with

$$\alpha \Vdash a \iff a \in \alpha$$

for $a \in S$ and $\alpha \in sob(X)$. Note that if

$$B'_a = \{\alpha \in sob(X) \mid \alpha \Vdash a\}$$

is the basic open set of $sob(X)$ with index $a \in S$ then

$$B_a \subseteq B_U \;\Leftrightarrow\; B'_a \subseteq B'_U$$

for $a \in S$ and $U \in Pow(S)$, where

$$B'_U = \bigcup_{b \in U} B'_b.$$

It follows that the ideal points of $sob(X)$ are exactly the points of $sob(X)$ so that $sob(X)$ is sober and $sob(sob(X)) = X$. Also $sob(X)$ is standard iff X is standard.

Let $\eta_X : X \to sob(X)$ be given by

$$\eta_X(x) = \alpha_x$$

for $x \in X$. Then η_X is continuous and is injective iff X is T_0 and is bijective iff X is sober. We may call $sob(X)$ the soberification of X.

Which spaces are sober? Classically all Hausdorff spaces are sober. But the proof is highly non-constructive and it seems unlikely that there is any interesting constructive version of that result. Nevertheless we do have the result that all complete metric spaces are sober. But the **CZF** theory of metric spaces and their completions will not be presented here but left for another occasion. Because we do not assume countable choice we need to use the Dedekind reals when defining the notion of metric space and replace the notion of a Cauchy sequence in a metric space by a Dedekind-cut-like notion.

Small and quasi-small spaces

A ct-space X is defined to be *small* if the class X of its points form a set. The small ct-spaces are essentially just the topological spaces equipped with a set indexed family $\{B_a\}_{a \in S}$ of open sets that form a base. Note that all small ct-spaces are standard. A weaker notion will also be useful. Call a subclass Y of a ct-space X *full* if, for every $x \in X$ and $a \in \alpha_x$ there is $y \in Y$ such that $a \in \alpha_y$ and $\alpha_y \subseteq \alpha_x$. We call a ct-space *quasi-small* if it has a full subset. Obviously every small space is quasi-small. Moreover observe that every quasi-small space is standard.

The constructive topological spaces in IZF

Our next result shows that in **IZF** some of the distinctions we have been making collapse.

Theorem: 9 (IZF) *Every* ct*-space is small. So the notions of* ct*-space, standard space and topological space coincide and all continuous functions are standard. Thus the categories* Top, Top_1, Top_2 *coincide.*

Proof: Let X be a ct-space . First observe that $P = \{\alpha_x \mid x \in X\}$ is a set in **IZF**. This is because, in **IZF** $Pow(S)$ is a set by the Powerset Axiom and, as $P = \{\alpha \in Pow(S) \mid \exists x \in X\ \alpha = \alpha_x\}$, by the Full Separation Scheme of **IZF** we get that P is a set.

By condition 3 in Definition 4, for each $\alpha \in P$ the class $X_\alpha = \{y \in X \mid \alpha_y = \alpha\}$ is a set. As $X = \bigcup_{\alpha \in P} X_\alpha$ it follows that X is a set.
■

4 Locales and formal topologies

4.1 Frames and Locales

Definition: 10 *A* class-frame *is a partially ordered class* A *having a top* $\top$*, binary meets,* $a_1 \wedge a_2$*, for* $a_1, a_2 \in A$ *and set sups* $\bigvee Y$*, for sets* $Y \subseteq A$*, such that meets distribute over sups; i.e.* $a \wedge \bigvee Y = \bigvee\{a \wedge y \mid y \in Y\}$ *for all* $a \in A$ *and all subsets* Y *of* A*. If* A *and* A' *are class-frames then a frame map* $A \to A'$ *is a class function* $G : A \to A'$ *that preserves the top, binary meets and set sups.*

This gives us the superlarge category of class-frames. The superlarge category of *class-locales* is the opposite category; i.e. the objects are the same, but the direction of the maps is reversed. So a class-locale is just a class-frame, but a locale map $A \to A'$ is a frame map $A' \to A$. In **CZF** class-frames and class locales will generally not be known to be small; i.e. to have a set of elements. So we drop the *class* prefix and just talk about frames and locales.

Examples of locales

The class of open sets of any topological space X form a locale when partially ordered by the subset relation. It has the set X as top, binary meets are binary intersections and set sups are set unions. In particular, for any set S the discrete topological space in which all subsets are open forms a locale.

More generally if S is a poset then the topological space $Pow_{\downarrow}(S)$, in which the opens are the downward closed subsets, forms a locale. A subset U of S is *downward closed* if $U\!\downarrow\, \subseteq U$, where $U\!\downarrow\, = \{a \in S \mid (\exists u \in U)\ a \leq u\}$.

We now show how to associate a locale $loc(X)$ to each standard space X. Recall that with each standard space X is a set S and operation assigning $\mathcal{A}U \in Pow(S)$ to each $U \in Pow(S)$, and that $U \in Pow(S)$ is saturated if $\mathcal{A}U = U$. Then the class of all saturated subsets of S forms a locale when partially ordered by the subset relation. Note that if $\mathcal{U}$ is a subset of this locale then its supremum, $\bigvee \mathcal{U}$, is $\mathcal{A}(\bigcup \mathcal{U})$.

The following construction will be useful. A class-function $j : A \to A$ on a locale A is a *nucleus* on A if, for all $x, y \in A$,

1. $x \leq jx$,
2. $jjx \leq jx$,
3. $x \leq y \Rightarrow jx \leq jy$,
4. $jx \wedge jy \leq j(x \wedge y)$.

If j is a nucleus on a locale A then the subpoclass

$$A_j = \{x \in A \mid jx \leq x\}$$

forms a locale with the same top as A, the binary meet operation being the restriction of the binary meet operation on A and the sup operation $\bigvee_j$ given by $\bigvee_j X = j(\bigvee X)$ for all subsets X of A_j. An example of this construction is the previous construction of the locale of saturated subsets of S associated with a standard space. This is just A_j, where $A = Pow_{\downarrow}(S)$ and $jU = \mathcal{A}U$ for each $U \in A$.

4.2 Set-generated locales

Next we define the superlarge category Loc_1 of set-generated locales.

Definition: 11 *A* set-generated locale *is a locale A equipped with a set-indexed family $\{\gamma(a)\}_{a \in S}$ of generators of the locale. By this we mean that S is a set such that for each $x \in A$ the class $S_x = \{a \in S \mid \gamma(a) \leq x\}$ is a set such that $a = \bigvee\{\gamma(x) \mid x \in S_a\}$.*

Note that the locale $loc(X)$ of saturated subsets of S associated with a standard space X becomes set-generated when equipped with $\{\gamma(a)\}_{a \in S}$, where S is the set of neighborhood indices of X and $\gamma(a) = \mathcal{A}\{a\}$ for each $a \in S$.

We define Loc_1 to be the category of set-generated locales with locale maps between the underlying locales.

If j is a nucleus on the locale $Pow_{\downarrow}(S)$, where S is a poset, then $Pow_{\downarrow}(S)_j$ has the set-indexed family of generators

$$\{j(\downarrow\{a\})\}_{a\in S}.$$

Moreover we have the following result.

Theorem: 12 *Every locale A with set-indexed family of generators $\{\gamma(a)\}_{a\in S}$ is isomorphic to the locale $Pow_{\downarrow}(S)_j$, where S is preordered by $\leq_S$, given by*

$$a \leq_S b \iff \gamma(a) \leq \gamma(b)$$

for $a, b \in S$, and the nucleus j is given by

$$jU = \{a \in S \mid \gamma(a) \leq \bigvee\{\gamma(b) \mid b \in U\}\}$$

for $U \in Pow_{\downarrow}(S)$. The isomorphism maps $x \in A$ to $S_x = \{a \in S \mid \gamma(a) \leq x\} \in Pow_{\downarrow}(S)_j$.

A locale A is a *small locale* if both A and $\{(x,y) \in A \times A \mid x \leq y\}$ are sets. The small locales form a full subcategory *Loc* of the category of locales. In **IZF** this is essentially the same notion as that of a set-generated locale.

Proposition: 13 (IZF) *There is an equivalence between the category Loc_1 of set-generated locales and the category Loc of small locales.*

Proof: The equivalence is given by the forgetfull functor $Loc_1 \to Loc$ which associates with each set-generated locale the small locale obtained by forgetting the family of generators. A set-generated locale A is small in **IZF** because $A = \{\bigvee\{\gamma(a) \mid a \in U\} \mid U \in Pow(S)\}$ is a set, using the Powerset axiom and Replacement. In the other direction there is the functor $Loc \to Loc_1$ that associates with each small locale A the same locale with set-indexed family of generators $\{a\}_{a\in A}$.
■

4.3 Formal topologies

At this point it will be convenient to introduce the notion of a formal topology. Formal topologies will be made to form a category $FTop_1$ that is equivalent to Loc_1.

Definition: 14 *A* formal topology *is a poset S equipped with a class cover relation $\triangleleft$ between S and its powerclass $Pow(S)$ such that $\mathcal{A}U = \{a \in S \mid a \triangleleft U\}$ is a set for each subset U of S and, for all sets $U, V \subseteq S$, if $U{\downarrow} = \{a \in S \mid (\exists u \in U)\ a \leq u\}$ and $U \downarrow V = U{\downarrow} \cap V{\downarrow}$,*

1. $\downarrow U \subseteq \mathcal{A}U$,
2. $U \subseteq \mathcal{A}V \Rightarrow \mathcal{A}U \subseteq \mathcal{A}V$,
3. $\mathcal{A}U \cap \mathcal{A}V \subseteq \mathcal{A}(U \downarrow V)$.

Note that for any formal topology S the restriction of $\mathcal{A}$ to the downward closed subsets of S is a nucleus j on $Pow_{\downarrow}(S)$ and

$$\mathcal{A}U = j(U{\downarrow})$$

for all subsets U of S. We let $sat(S)$ be the set-generated locale $Pow_{\downarrow}(S)_j$. Every formal topology arises in this way from a nucleus j on a poset S.

Note that on a formal topology

$$a \leq b \Rightarrow a \triangleleft \{b\}$$

for all $a, b \in S$. If we are not interested in the preorder $\leq$ we can take it to be defined by

$$a \leq b \iff a \triangleleft \{b\}$$

as the axioms for a formal topology still hold with this change.

The category of formal topologies

Given formal topologies S, S' a *formal topology map* $S \to S'$ is defined to be a subset r of $S \times S'$ such that the following conditions hold where, for each subset U' of S',

$$rU' = \{a \in S \mid \exists a' \in U'\ ara'\}.$$

FTM1 $S \subseteq \mathcal{A}(rS')$,

FTM2 $r\{b'\} \cap r\{c'\} \subseteq \mathcal{A}(r(\{b'\} \downarrow \{c'\}))$,

FTM3 $r(\mathcal{A}'U') \subseteq \mathcal{A}(rU')$,

FTM4 $\mathcal{A}(r\{a'\}) \subseteq r\{a'\}$.

for all $a', b', c' \in S'$ and all $U' \in Pow(S')$. Note the following consequence of **FTM3** and **FTM4**.

$$\mathcal{A}(r\{b'\}) = r\{b'\} = r(\mathcal{A}'\{b'\}) = \mathcal{A}(r(\mathcal{A}'\{b'\}))$$

It is now easy to see that formal topologies and formal topology maps form a category $FTop_1$, where

$$\{(a_1, a_2) \in S \times S \mid a_1 \in \mathcal{A}\{a_2\}\}$$

is the identity map on S for each formal topology S and the composition $r' \circ r$ of formal topology maps $r : S \to S'$ and $r' : S' \to S''$ is defined to be

$$\{(a, a'') \in S \times S'' \mid a \in \mathcal{A}(r(r'\{a''\}))\}.$$

Theorem: 15 *The category $FTop_1$ is equivalent to the category Loc_1.*

Proof: The equivalence is obtained using the adjoint functors $sat : FTop_1 \to Loc_1$ and $gen : Loc_1 \to FTop_1$ defined as follows.

We have already defined, for each formal topology S, the set-generated locale $sat(S) = Pow_{\downarrow}(S)_j$, where $jU = \mathcal{A}U = \{a \in S \mid a \lhd U\}$ for each $U \in Pow_{\downarrow}(S)$. If $r : S \to S'$ is a formal topology map we define $sat(r)$ to be the locale map $G_r : sat(S) \to sat(S')$ where, for sets $U' \in sat(S')$,

$$G_r U' = \mathcal{A}(rU').$$

We leave as a routine exercise the proof that this does indeed give a functor.

Conversely, given a set-generated locale A, with set-indexed family $\gamma(a)\}_{a\in S}$ of generators, let $gen(A)$ be the formal topology having the cover relation on S given by

$$a \lhd U \iff \gamma(a) \leq \bigvee\{\gamma(b) \mid b \in U\}$$

for $(a, U) \in S \times Pow(S)$. If $G : A \to A'$ is a locale map between the set-generated locales A, A', with set indexed families $\{\gamma(a)\}_{a\in S}$, $\{\gamma'(a')\}_{a'\in S'}$ of generators, then we define $gen(G)$ to be the formal topology map $r_G : gen(A) \to gen(A')$ given by

$$r_G = \{(a, a') \in S \times S' \mid \gamma(a) \leq G(\gamma'(a'))\}.$$

Again we leave as a routine exercise the proof that this is a functor that is, moreover, full and faithfull.

To complete the proof of the theorem we need to prove that the two functors, sat and gen, form an equivalence. This follows from the following result whose proof is left as an exercise.

Lemma: 16 *For each formal topology S there is an isomorphism $\eta_S : S \to gen(sat(S))$ of $FTop_1$ such that for each formal topology map $r : S \to gen(A')$, where A' is a set-generated locale, there is a unique locale map $G : sat(S) \to A'$ such that $r = gen(G) \circ \eta_S$.*

■

5 Proof of the Galois Adjunction Theorem

5.1 Formal points

Classically there are several equivalent approaches to defining the notion of 'point' of a locale. The most suitable from our point of view is to use the notion of a completely prime filter. A subclass F of a class-locale A is a *completely prime filter* if it is a filter; i.e. it has an element, is both upwards closed and meet closed and is such that for all subsets X of A

$$\bigvee X \in F \Rightarrow (\exists x \in X)\, x \in F.$$

If A is a set-generated locale, with set-indexed family of generators $\{\gamma(a)\}_{a \in S}$, a completely prime filter F is *set-generated* if the class $\alpha_F = \{a \in S \mid \gamma(a) \in F\}$ is a set.

Definition: 17 *A* formal point *of a formal topology S is a subset α of S such that*

> FP1: $\exists a(a \in \alpha)$,
>
> FP2: $a, b \in \alpha \Rightarrow \exists c \in \alpha(c \in \{a\}{\downarrow}\{b\})$,
>
> FP3: $a \in \alpha \;\Rightarrow\; (\forall U \in Pow(S))[a \triangleleft U \;\Rightarrow\; (\exists c \in \alpha)(c \in U)]$.

We justify the terminology 'formal point' for this notion by the following relationship with the notion of a set-generated completely prime filter.

Proposition: 18 *Let A be a set-generated locale with associated formal topology S. Then there is a one-one correspondence between the set-generated completely prime filters on A and the formal points of S. With each set-generated completely prime filter F on A is associated the formal point $\alpha_F = \{a \in S \mid \gamma(a) \in F\}$ and with each formal point α of S is associated the set-generated completely prime filter $F_\alpha = \{x \in A \mid \exists a \in \alpha\; a \in S_x\}$.*

5.2 The Galois adjunction

We now define a functor $Pt_1 : FTop_1 \to Top_1$.

Definition: 19 *If S is a formal topology then let $Pt_1(S)$ be the* ct*-space of formal points of S having the set S as its set of neighborhood indices and having the relation $\Vdash$ given by*

$$\alpha \Vdash a \iff a \in \alpha$$

for each $a \in S$ and each point α.

If $r : S \to S'$ is a map of $FTop_1$ then let

$$Pt_1(r)(\alpha) = \{a' \in S' \mid \exists a \in \alpha \; a \; r \; a'\}$$

for every formal point α of S.

Proposition: 20 *Pt_1 is a functor $Pt_1 : FTop_1 \to Top_1$.*

Proof: This is a routine exercise. We only point out the main steps. First it must be shown that if $r : S \to S'$ is a map of $FTop_1$ then $Pt_1(r) : Pt_1(S) \to Pt_1(S')$ is a map of Top_1; i.e. for each formal point α of S the set $Pt_1(r)(\alpha)$ is a formal point of S' and the function $Pt_1(r)$ is a continuous function $Pt_1(S) \to Pt_1(S')$.

Next it must be shown that if $id_S = \{(a, a') \mid a \lhd \{a'\}\}$ is the identity formal topology map on S then $Pt_1(id_S)$ is the identity function on $Pt_1(S)$.

Finally it must be shown that if $r : S \to S'$ and $r' : S' \to S''$ are formal topology maps then $Pt_1(r' \circ r) = Pt_1(r') \circ Pt_1(r)$.

$\blacksquare$

Note that each standard space X, with set S of neighborhood indices, has the associated formal topology $ft(X)$ with preorder on S given by

$$a \leq b \;\Leftrightarrow\; B_a \subseteq B_b$$

for $a, b \in S$ and cover relation on S given by

$$a \lhd U \;\Leftrightarrow\; B_a \subseteq B_U$$

for $a \in S, U \in Pow(S)$.

Recall from section 3 the notion of a sober ct-space space and the operation that associates with each ct-space X the sober ct-space $sob(X)$ and the continuous function $\eta_X : X \to sob(X)$, where $\eta_X(x) = \alpha_x$ for each $x \in X$. Also recall that the ct-space X is standard iff $sob(X)$ is standard.

Theorem: 21 *Let X be a standard* ct*-space with associated formal topology $ft(X)$.*

1. *$Pt_1(ft(X)) = sob(X)$ and the function $\eta_X : X \to Pt_1(ft(X))$ is standard continuous.*

2. *For each formal topology S', if $g : X \to Pt_1(S')$ is a standard continuous function then*

$$r_g = \{(a, a') \in S \times S' \mid a \in g^* a'\}$$

is the unique formal topology map $r : ft(X) \to S'$ such that

$$g = Pt_1(r) \circ \eta_X.$$

3. *If $X = Pt_1(S)$ for some formal topology S then X is sober so that $Pt_1(ft(X)) = X$ and η_X is the identity function on X.*

Proof: Let X be a standard formal topology.

1. Observe that the definition of $sob(X)$ was designed so that when X is standard $Pt_1(ft(X)) = sob(X)$. It remains to check that the continuous function $\eta_X : X \to sob(X)$ is standard continuous; i.e. that for all $a' \in S$ the class $\eta_X^* a'$ is a set. For that it is sufficient to observe that this class is the class $\{a \in S \mid B_a \subseteq B_{a'}\}$, which is a set because X is standard.

2. This is a routine exercise. It is necessary to show that if S' is a formal topology and $g : X \to Pt_1(S')$ then

   (a) r_g is a formal topology map $ft(X) \to S'$,

   (b) $g = Pt_1(r_g) \circ \eta_X$,

   (c) if $r : ft(X) \to S'$ is a formal topology map such that $g = Pt_1(r) \circ \eta_X$ then $r = r_g$.

3. Assume that S is a formal topology and $X = Pt_1(S)$. To show that X is sober let α be an ideal point of X. We want to show that there is a unique point x of X such that $\alpha = \alpha_x$. Note that each point x of X is a formal point of S and so is a subset of S and $\alpha_x = x$. So it suffices to check that the ideal point α is a formal point of S; i.e. that α satisfies the conditions $FP1, FP2, FP3$ of Definition 17.

   FP1: This is condition 1 for an ideal point.

FP2: Let $b, c \in \alpha$. Then, by condition 2 for an ideal point, there is $a' \in \alpha$ such that $B_{a'} \subseteq B_a \cap B_c$. Let $U = (b \downarrow c)$. We show that $B_{a'} \subseteq B_U$. It will then follow from condition 3 for an ideal point that there is $a \in \alpha$ such that $a \in (b \downarrow c)$.

So let $\alpha' \in B_{a'}$. Then $\alpha' \in B_a \cap B_c$ so that $b, c \in \alpha'$. Hence, by $FP2$ for α', there is $a'' \in \alpha'$ such that $a'' \in (b \downarrow c)$. It follows that $\alpha' \in B_{a''} \subseteq B_U$. Thus $B_{a'} \subseteq B_U$, as desired.

FP3: Let $a \in \alpha$ and $U \in Pow(S)$ such that $a \lhd U$. Then $B_a \subseteq B_U$ so that, by condition 3 for an ideal point, there is $c \in \alpha$ such that $c \in U$.

As X is sober, $X = sob(X) = Pt_1(ft(X))$. We have already seen that if $x \in X$ then $\eta_X(x) = \alpha_x = x$ so that η_X is the identity function on X.

■

This almost gives us a Galois adjunction between standard spaces and formal topologies. But not quite. The problem is that if S is a formal topology then we do not seem able to show that the ct-space $Pt_1(S)$ is standard. But we can bypass the problem using the following definition.

Definition: 22 *Call a formal topology S a* standard formal topology *if $Pt_1(S)$ is a standard* ct*-space and, if S is standard, call a formal topology map $r : S \to S'$ a* standard formal topology map *if $Pt_1(r) : Pt_1(S) \to Pt_1(S')$ is a standard continuous function.*

Now observe that the standard formal topologies and standard formal topology maps between them form a subcategory $FTop_2$ of $FTop_1$. Also observe that if X is a standard ct-space then $Pt_1(ft(X)) = sob(X)$ is also a standard ct-space so that $ft(X)$ is a standard formal topology. Moreover if also $g : X \to Pt_1(S')$ is standard continuous then the formal topology map $r_g : ft(X) \to S'$, defined in Theorem 21, is a standard formal topology map. It follows that we now do have a Galois adjunction between the category Top_2 and the category $FTop_2$.

In **IZF** every formal topology is standard so that the categories $FTop_1$, $FTop_2$, Loc_1, Loc are all equivalent. So, if we define Top' to be Top_2, Loc' to be $FTop_2$ and Pt' to be the restriction of Pt_1 to $FTop_2$ and finally let $\Omega' : Top' \to FTop'$ be the left adjoint of Pt' that assigns to each standard space its associated standard formal topology then we have completed the proof of Theorem 1.

6 Set-presentable formal topologies

6.1 Covering systems

We consider a fundamental way to inductively generate formal topologies. We exploit the following general result concerning inductive definitions in constructive set theory. For each class Φ a class A is defined to be Φ-*closed* if

$$X \subseteq A \Rightarrow a \in A$$

whenever the pair (X, a) is in Φ.

Theorem: 23 *Let Φ be a subclass of $Pow(S) \times S$, where S is a class.*

1. *For each subclass U of S there is a smallest Φ-closed class that includes U. We write $I(\Phi, U)$ for this class.*

2. *Assuming* **REA**, *if S and Φ are sets then*

   (a) *$I(\Phi, U)$ is a set for each set $U \subseteq S$,*

   (b) *there is a set B of subsets of $Pow(S)$ such that for each set $U \subseteq S$ and each $a \in I(\Phi, U)$ there is a set $V \in B$ such that $V \subseteq U$ and $a \in I(\Phi, V)$.*

The *Regular Extension Axiom*, **REA**, is discussed in [5] and states that every set is a subset of a regular set, where a set A is defined to be *regular* if it is (i) a transitive set, i.e. every element is a subset, and (ii) is such that whenever $a \in A$ and $R \subseteq a \times A$ such that for every $x \in a$ there is $y \in A$ such that $(x, y) \in R$ then there is $b \in A$ such that

$$\forall x \in a\, \exists y \in b\, (x, y) \in R \ \ \&\ \ \forall y \in b\, \exists x \in a\, (x, y) \in R.$$

It is a natural axiom of constructive set theory to add to **CZF**. The system **CZF** + **REA** is abbreviated **CZF**$^+$. The theorem is a consequence of results in section 5 of [5]. More specifically, part 1 of the theorem is a consequence of Theorem 5.1 of [5], part 2(i) is a consequence of Theorem 5.7 and part 2(ii) is a consequence of the Set Compactness Theorem, Theorem 5.11.

The notion of a covering system has been discussed under various names by Dragalin, by Maclane and Moerdijk, by Peter Johnstone, by Coquand and by Coquand et al., see [8, 9, 12, 13, 14].

Definition: 24 *A* covering system *consists of a poset S equipped with a function $C : S \to Pow(Pow(S))$ such that for all $a \in S$ and $X \in C(a)$*

1. $X \subseteq \downarrow\{a\}$,

2. if $b \leq a$ then there is $Y \in C(b)$ such that $Y \subseteq \downarrow X$,

Proposition: 25 *Given a covering system $C : S \to Pow(Pow(S))$ on a poset S, for each subclass U of S there is a smallest subclass $\mathcal{A}U$ of S such that $\downarrow U \subseteq \mathcal{A}U$ and for all $a \in S$ and $X \in C(a)$*

$$X \subseteq \mathcal{A}U \Rightarrow a \in \mathcal{A}U.$$

Moreover $\downarrow(\mathcal{A}U) \subseteq \mathcal{A}U$ and the operator $\mathcal{A}$ also has the following properties.

1. $U \subseteq \mathcal{A}V \Rightarrow \mathcal{A}U \subseteq \mathcal{A}V$,

2. $\mathcal{A}U \cap \mathcal{A}V \subseteq \mathcal{A}(U{\downarrow}V)$,

By part 1 of Theorem 23, for each subclass U of the set S the class $\mathcal{A}U$ is the class $I(\Phi, \downarrow U)$ where

$$\Phi = \{(X, a) \mid a \in S \;\&\; X \in C(a)\}.$$

Call a covering system *standard* if $\mathcal{A}U$ is a set for all sets $U \subseteq S$. Note that Φ is a set so that, by part 2(ii) Theorem 23, assuming **REA**, the covering system is standard.

Proposition: 26 *Every standard covering system on a poset S determines a formal topology with covering relation $\lhd$ on S given by*

$$a \lhd U \iff a \in \mathcal{A}U$$

for all $(a, U) \in S \times Pow(S)$.

If a formal topology can be obtained by this construction we call it an *inductively generated formal topology.*

6.2 Set-presentations

A *set-presentation* of a formal topology S is a function $C : S \to Pow(Pow(S))$ such that

$$a \lhd U \iff \exists V \in C(a)\; V \subseteq U.$$

A *set-presentable formal topology* is a formal topology that has a set-presentation.

Proposition: 27 *Every set-presentable formal topology is inductively generated.*

Proof: To see this let $C_0 : S \to Pow(Pow(S))$ be a set-presentation of a formal topology S and observe that we get a covering system on S that inductively generates the formal topology by defining for $a, b \in S$

$$a \leq b \iff a \lhd \{b\}$$

and for $a \in S$

$$C(a) = \{X{\downarrow}\{a\} \mid (\exists b \geq a)\ X \in C_0(b)\}.$$

■

Theorem: 28 (CZF$^+$) *Every covering system is standard and inductively generates a set-presentable formal topology.*

Proof: Let C be a covering system on a poset S and let $\mathcal{A}$ be as in Proposition 25. As already explained after Proposition 25, because we are assuming **REA** , C is standard and so it inductively generates a formal topology on S with

$$a \lhd U \iff a \in \mathcal{A}U.$$

By part 2(ii) of Theorem 23 there is a subset B of $Pow(S)$ such that

$$a \lhd U \iff \exists V \in B\ (V \subseteq U\ \&\ a \lhd V).$$

So $C' : S \to Pow(Pow(S))$ is a set-presentation of the formal topology where

$$C'(a) = \{V \in B \mid a \in \mathcal{A}V\},$$

for all $a \in S$.

■

Corollary: 29 (CZF$^+$) *A formal topology is inductively generated iff it is set-presentable.*

Proposition: 30 *If X is a quasi-small (standard) space then its associated formal topology is set-presentable.*

Proof: We obtain a set-presentation using Subset Collection to first obtain a set G of subsets of S such that whenever $a \in S$ and $R \in \mathbf{mv}(S^a)$ then there is $Z \in G$ such that $R \in \mathbf{mv}(Z^a)$ and $\breve{R} \in \mathbf{mv}(a^Z)$, where $\breve{R} = \{(b, x) \mid (x, b) \in R\}$.

For $a \in S$ let $C(a) = \{\cup Z \mid Z \in G \ \&\ a \subseteq \cup Z\}$. Trivially $V \in C(a) \Rightarrow a \lhd V$. Now let $a \lhd U$. Then $R \in \mathbf{mv}(S^a)$, where $R = \{(x,b) \mid x \in b \ \&\ b \in U\}$. It follows that there is $Z \in G$ such that $R \in \mathbf{mv}(Z^a)$ and $\breve{R} \in \mathbf{mv}(a^Z)$. So if $V = \cup Z$ then $Z \in C(a)$ and $Z \subseteq U$. Thus $a \lhd U \Rightarrow (\exists Z \in C(a))(Z \subseteq U)$ and we have shown that C is a set-presentation of the formal topology.
■

6.3 Locally compact formal topologies

The notion of a locally compact formal topology was first introduced by Inger Sigstam in [22] in an impredicative form and a predicative version was then given by Sara Negri in [17]. Since then Giovanni Curi formulated a slight improvement in [11]. The notion is a constructive version, for formal topologies, of the standard notion of a locally compact locale. See, for example [13]. Here we show that every locally compact formal topology is set-presentable. Local compactness involves the notion of a finite cover and we need to be clear about which constructive notion of finite is appropriate here. In fact it is the following notion. A set Y is defined to be *finitely enumerable* if, for some natural number $n \in \mathbb{N}$ there is a surjection $\{i \in \mathbb{N} \mid i < n\} \to Y$.

Definition: 31 *A formal topology S is defined to be* locally compact *if for every $a \in S$ there is a subset U of S such that*

$$(*) \qquad a \lhd U \quad \& \quad \forall b \in U \ b \prec\!\!\prec a$$

where, for $a, b \in S$, $b \prec\!\!\prec a$ if, for every subset V of S

$$a \lhd V \Rightarrow b \lhd V_0 \textit{ for some finitely enumerable } V_0 \subseteq V.$$

Using Strong Collection it is straightforward to prove that if S is locally compact then there is a function $i : S \to Pow(S)$ such that for each $a \in S$ the set $U = i(a)$ satisfies $(*)$. This gives us the definition of local compactness to be found in [10].

Theorem: 32 *Every locally compact formal topology is set-presentable.*

Proof: Let S be a locally compact formal topology via $i : S \to Pow(S)$. So for all $a \in S$ we have $a \lhd i(a)$ and if $a \lhd U$ then

$$(\forall b \in i(a))(\exists V \in \mathcal{F})[V \subseteq U \ \&\ b \lhd V],$$

where $\mathcal{F}$ is the set of finitely enumerable subsets of S. By Subset Collection there is a set G of subsets of $\mathcal{F}$ such that for all $a \in S$ and all $U \in Pow(S)$, if $a \lhd U$ then, for some $F \in G$,

$$(i) \qquad (\forall b \in i(a))(\exists V \in F)[V \subseteq U \ \& \ b \lhd V]$$

$$(ii) \qquad (\forall V \in F)(\exists b \in i(a))[V \subseteq U \ \& \ b \lhd V]$$

So, given $a \lhd U$ let $F \in G$ such that (i) and (ii) and let $Z = \cup F$. $Z \subseteq U$ and also $a \lhd Z$, as $(\forall b \in i(a)(b \lhd Z)$ and $a \lhd i(a)$. For $a \in S$ let

$$C(a) = \{\cup F \mid F \in G \ \& \ a \lhd \cup F\}.$$

Then C gives a set-presentation of the formal topology.
■

6.4 A variant of a result of Palmgren

This section was inspired by a draft paper, [18], of Erik Palmgren, where it was shown in a version of constructive type theory that if the formal points of a set generated formal topology are always maximal formal points then the formal points form a set. Here we prove this result in constructive set theory. We will make use of a more abstract result, Theorem 38 below, that will be proved in Appendix 1 in the formal system $\mathbf{CZF} + \mathbf{uREA} + \mathbf{DC}$. So our results in this subsection only hold in this system which is an extension of $\mathbf{CZF}^+$. The axiom **uREA** is a strengthening of the axiom **REA** which is explained in Appendix 1.

It will be convenient to use some terminology from domain theory. Call a partially ordered class a *directed complete partial order (dcpo)* if every directed subset has a sup, where a subset is *directed* if it is inhabited any pair of elements of the subset have an upper in the subset. A dcpo $\mathcal{X}$ is *set-generated* if there is a subset X such that, for every $a \in \mathcal{X}$, $\{x \in X \mid x \leq a\}$ is a directed set whose sup is a. It is easy to observe that the class of formal points of any formal topology, when ordered by the subset relation, form a dcpo. Our main result is the following.

Theorem: 33 ($\mathbf{CZF} + \mathbf{uREA} + \mathbf{DC}$) *The dcpo of formal points of a set-presented formal topology is a set-generated dcpo.*

Corollary: 34 ($\mathbf{CZF} + \mathbf{uREA} + \mathbf{DC}$) *The* ct*-space of formal points of a set-presented formal topology is quasi-small.*

Corollary: 35 (CZF + uREA + DC) *The Galois adjunction between the superlarge categories of standard topological spaces and standard formal topologies restricts to give a Galois adjunction between the full subcategories of quasi-small* ct*-spaces and the set-presentable standard formal topologies*

Note that this restricted Galois adjunction could have been used in proving the statement of the Galois Adjunction Theorem except for the fact that the proof here is not in **CZF** but in the extension **CZF + uREA + DC**.

Call a partially ordered class *flat* if $x \leq y \Rightarrow x = y$. Note that the assumption on a formal topology that the formal points are always maximal formal points can be rephrased as the assumption that the poclass of formal points is flat, or equivalently as the assumption that the associated ct-space is T_1. So the statement of Palmgren's result, expressed in constructive set theory, becomes the following.

Corollary: 36 (CZF + uREA + DC) *If the* ct*-space of formal points of a set-presented formal topology is* T_1 *then it is small; i.e. is a topological space with set-base.*

To prove this from the theorem it suffices to observe the following result.

Lemma: 37 *The elements of any flat set-generated dcpo form a set.*

Proof: If X is a set of generators for the dcpo then for any element a there must be $x \in X$ such that $x \leq a$, as X_a is directed. As the dcpo is flat $a = x \in X$. Thus the set X is the class of all the elements of the dcpo.
■

We will obtain the theorem from a more abstract result. To state the abstract result we need some definitions. Let S, S' be sets and let $\Gamma : Pow(S) \to Pow(S')$. The operator Γ is *monotone* if, for all sets $Y, Y' \in Pow(S)$,

$$Y \subseteq Y' \Rightarrow \Gamma(Y) \subseteq \Gamma(Y'),$$

and is *finitary* if, for every set $Y \in Pow(S)$ and every $a \in \Gamma(Y)$ there is a finitely enumerable set $Y_0 \subseteq Y$ such that $a \in \Gamma(Y_0)$.

Let $R : S' \to Pow(S)$. We define $\alpha \in Pow(S)$ to be Γ, R*-closed* if

$$(\forall x \in \Gamma(\alpha))(\exists y \in \alpha)\; y \in R_x.$$

It is easy to see that the poclass of Γ, R-closed subsets of S, when ordered by the subset relation, form a dcpo, when Γ is monotone and finitary. We have the following abstract result, which is proved in Appendix 1.

Theorem: 38 (CZF + uREA + DC) *If* Γ *is monotone and finitary then the dcpo of* Γ, R*-closed sets is a set-generated dcpo.*

To apply this to get Theorem 33 it suffices, given a formal topology S with set presentation $C : S \to Pow(Pow(S))$, to define a set S', a monotone, finitary $\Gamma : Pow(S) \to Pow(S')$ and $R : S' \to Pow(S)$ so that a subset of S is a formal point iff it is Γ, R-closed. We now do this. For each $\alpha \in Pow(S)$ let

$$\Gamma(\alpha) = \{0\} + (\alpha \times \alpha) + \sum_{a \in \alpha} C(a)$$

and let $S' = \Gamma(S)$. Then $\Gamma : Pow(S) \to Pow(S')$ is monotone and finitary. Let $R_b \in Pow(S)$ for $b \in S'$ be given by

$$\begin{cases} R_{(1,0)} & = S, \\ R_{(2,(b_1,b_2))} & = \{b_1\} \downarrow \{b_2\} \quad \text{for } (b_1, b_2) \in S \times S, \\ R_{(3,(b,V))} & = V \quad \text{for } (b, V) \in \sum_{a \in S} C(a). \end{cases}$$

It is now easy to see that the three conditions $1, 2, 3'$ for a formal point, can be combined into one using Γ and R to give us the following result.

Lemma: 39 *A subset* α *of* S *is a formal point iff* α *is* Γ, R*-closed.*

7 Regular Formal Topologies

Definition: 40 *Let* S *be a formal topology. For* $a \in S$ *let*

$$wc(a) = \{b \in S \mid (\forall d \in S)(d \triangleleft \{a\} \cup b^*)\}$$

where, for $b \in S$,

$$b^* = \{c \in S \mid \forall a \in (b \downarrow c) \;\; a \triangleleft \emptyset\}.$$

The formal topology is defined to be regular *if* $a \triangleleft wc(a)$ *for all* $a \in S$.

Proposition: 41 *A standard space* X*, is a regular* ct*-space iff the associated formal topology is a regular formal topology.*

Theorem: 42 *If* S *is a regular formal topology then the* ct*-space of its points is a* T_3 ct*-space .*

Proof: Let S be a regular formal topology with the ct-space X of its formal points. Recall that $B_b = \{\alpha \in X \mid b \in \alpha\}$ for $b \in S$. We will use the following lemma. Only part 3 requires regularity.

Lemma: 43 *Let $\alpha \in X$. Then*

1. $b, c \in \alpha \Rightarrow c \notin b^*$,
2. $(\exists c \in \alpha)(c \in b^*) \;\Rightarrow\; \alpha \in \neg B_b$,
3. *For each $a \in \alpha$ there is $b \in \alpha$ such that for any formal point β*
$$a \in \beta \text{ or } (\exists c \in \beta)(c \in b^*).$$

Proof:

1. Let $b, c \in \alpha$. Then, by condition FP2 of Definition 17, there is $a \in \alpha$ such that $a \in \{b\} \downarrow \{c\}$ and hence if $c \in b^*$ then $a \lhd \emptyset$. But, by condition FP3 of Definition 17, as $a \in \alpha$, $\neg(a \lhd \emptyset)$ and hence $c \notin b^*$.
2. Assume that $(\exists c \in \alpha)(c \in b^*)$. Then $\alpha \in B_c$ and
$$\begin{aligned} \gamma \in B_c \cap B_b \;&\Rightarrow b, c \in \gamma \\ &\Rightarrow c \notin b^*, \text{ by part 1,} \end{aligned}$$
contradicting $c \in b^*$. So $B_c \cap B_b = \emptyset$. Thus $\alpha \in \neg B_b$.
3. If $a \in \alpha$ then, as $a \lhd wc(a)$, by condition FP3 of Definition 17, there is $b \in \alpha$ such that $b \in wc(a)$. Now, for any formal point β choose $d \in \beta$ by condition FP1 of Definition 17. Then, as $b \in wc(a)$, we have $d \lhd \{a\} \cup b^*$ so that, by condition 3 of Definition 17 there is $c \in \beta$ such that $c \in \{a\} \cup b^*$; i.e. either $c = a$ or $c \in b^*$ so that $a \in \beta$ or $(\exists c \in \beta)(c \in b^*)$.

■

To complete the proof of the theorem we next show that X is T_1; i.e. we show that when α, β are formal points of S with $\beta \subseteq \alpha$ then $\alpha \subseteq \beta$. So let $a \in \alpha$. We show that $a \in \beta$. By part 3 of the lemma there is $b \in \alpha$ such that either $a \in \beta$ or $c \in b^*$ for some $c \in \beta$. In the latter case, as $\beta \subseteq \alpha$, we have $b, c \in \alpha$, so that, by part 1 of the lemma, $c \notin b^*$, contradicting $c \in b^*$. So we get $a \in \beta$, as desired.

It remains to show that X is a regular space; i.e. given $a \in S$ and $\alpha \in B_a$ we must show that $\alpha \in B_b$ for some $b \in S$ such that $X \subseteq B_a \cup \neg B_b$. By part 3 of the lemma there is $b \in \alpha$ such that $X \subseteq \{\beta \mid \beta \in B_a \text{ or } (\exists c \in \beta)(c \in b^*)\}$, so that, by part 2 of the lemma we are done.

■

7.1 A generalisation of a result of Giovanni Curi

In [10] Giovanni Curi has shown that in constructive type theory the points of any locally compact regular formal topology form a set. Here we show in constructive set theory that the formal points of any set-presentable regular formal topology form a set. By Theorem 32 this generalises Curi's result. It will be useful to relativise the notions of regular formal topology and formal point to a subset P of the formal topology.

Definition: 44 *Let S be a formal topology and let P be a subset of S.*

1. *For $a \in S$ let*

$$wc_P(a) = \{b \in S \mid (\forall d \in S)(d \lhd \{a\} \cup b^*_P)\},$$

where, for $b \in S$, $b^*_P = \{c \in S \mid P \cap (b \downarrow c) = \emptyset\}$. The formal topology is P-regular *if $a \lhd wc_P(a)$.*

2. *A* P-point *of S is a formal point of S that is a subset of P.*

Lemma: 45 *Let S be a formal topology and let $P_0 = \{a \in S \mid a \lhd \emptyset\}$.*

1. *If S is regular then S is P_0-regular.*

2. *Every formal point of S is a P_0-point.*

Theorem: 46 *If S is a P-regular formal topology, where P is a subset of S, then the P-points of S form a subclass of a set.*

Proof: Let S be a P-regular formal topology, where P is a subset of S. Define

$$\begin{array}{lcl} a \prec b & \Leftrightarrow & a \in wc_P(b), \\ b \asymp c & \Leftrightarrow & (\exists a \in P)\ a \in (b \downarrow c) \end{array}$$

Note that $b^*_P = \{c \in S \mid b \not\asymp c\}$. By the following result it will be enough to show that the class of subsets α of S that satisfy $A1, A2, A3$, below, form a set. We leave that as the Main Lemma of Appendix 2.

Lemma: 47 *If α is a P-point of S then*

A1: $b, c \in \alpha \Rightarrow b \asymp c$,

A2: $a \in \alpha \Rightarrow (\exists b \in \alpha)(b \prec a)$.

A3: $b \prec a \Rightarrow (\exists c \in \alpha)(b \asymp c \Rightarrow c = a)$.

Proof: Let α be a P-point of S. We must show that $A1, A2, A3$ hold.

A1 Let $b, c \in \alpha$. Then, by $FP2$, there is $a \in \alpha$ such that $a \in (b \downarrow c)$. As α is a P-point $a \in P$. Thus $b \asymp c$.

A2 Let $a \in \alpha$. As $a \lhd wc_P(a)$ we may apply $FP3$ to get that $b \in \alpha$ for some $b \in wc_P(a)$; i.e.

$$(\exists b \in \alpha)(b \prec a).$$

A3 Let $b \prec a$; i.e. $b \in wc_P(a)$, so that for all $d \in S$

$$d \lhd \{a\} \cup b^*_P.$$

By $FP1$ we can choose $d \in \alpha$ so that, by FP3,

$$(\exists c \in \alpha)(c \in b^*_P \vee c = a).$$

It follows that, because $b \asymp c \Rightarrow c \notin b^*_P$,

$$(\exists c \in \alpha)((b \asymp c) \Rightarrow (c = a)).$$

■

Recall the observation after Definition 17 that the points of a set-presentable formal topology form a predicative class.

Corollary: 48 *The P-points of a set-presentable P-regular formal topology form a set. In particular the points of a set-presentable regular topology form a set.*

Corollary: 49 *The formal points of a locally compact regular formal topology form a set.*

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8 Appendix 1: Proof of Theorem 38

The proof of the theorem seems to make essential use of both of the axioms **uREA** and **DC**. Recall that the Regular Extension Axiom, **REA**, states that every set is a subset of a regular set. The axiom **uREA** states that every set is a subset of a union-closed regular set; i.e. a regular set A such that for every set a in A its union, $\bigcup a$, is also in A. This would seem to be a natural minor strengthening of **REA**. The axiom **DC** of Dependent Choices is formulated as usual; i.e if the set R is a subset of $A \times A$, where A is a set, if $a_0 \in A$ and , for every $x \in A$ there is a $y \in A$ such that $(x, y) \in R$ then there is a function $f : \mathbb{N} \to A$ such that $f(0) = a_0$ and $(f(n), f(n+1)) \in R$ for all $n \in \mathbb{N}$.

Let S, S', Γ, R be as in the statement of the theorem. Let $Fin(S)$ be the set of all finitely enumerable subsets of S. By **uREA** we may choose a union-closed regular set A so that $S \cup \{\mathbb{N}\} \cup \{\Gamma(\alpha) \mid \alpha \in Fin(S)\}$ is a subset of A.

Lemma: 50 *For all sets $\alpha \subseteq S$*

1. $\alpha \in A \;\Rightarrow\; Fin(\alpha) \in A$,
2. $\alpha \in A \;\Rightarrow\; \Gamma(\alpha) \in A$.

Proof: Let α be a subset of S in A.

1. $Fin(\alpha) = \bigcup_{n \in \mathbb{N}} \{ran(f) \mid f \in ({}^{\{1,\ldots,n\}}\alpha)\}$. As ${}^{\{1,\ldots,n\}}\alpha \in A$ can be proved by induction on $n \in \mathbb{N}$ we get that $Fin(\alpha) \in A$.
2. Observe that, as Γ is finitary, $\Gamma(\alpha) = \bigcup\{\Gamma(\alpha_0) \mid \alpha_0 \in Fin(\alpha)\}$ and apply part 1 and the assumption that A is a union-closed regular set.

■

Now let γ be a Γ, R-closed subset of S. We must show that the set A_γ of Γ, R-closed subsets of γ that are in A is directed and has union γ. The class $P = A \cap Pow(\gamma)$ is a set, by Restricted Separation. Let

$$T = \{(\alpha, \beta) \in P \times P \mid \alpha \subseteq \beta \;\&\; (\forall x \in \Gamma(\alpha))(\exists y \in \beta)\; y \in R_x\}.$$

Lemma: 51 $(\forall \alpha \in P)(\exists \beta \in P)\; (\alpha, \beta) \in T$.

Proof: Let $\alpha \in P$. So $\alpha \in A$ and $\alpha \subseteq \gamma$. If $x \in \Gamma(\alpha)$ then $x \in \Gamma(\gamma)$ so that $y \in R_x$ for some $y \in \gamma$, as γ is Γ, R-closed. Thus, as $\gamma \subseteq S \subseteq A$,

$$(\forall x \in \Gamma(\alpha))(\exists y \in A)[y \in R_x \cap \gamma].$$

As A is regular and, by part 2 of Lemma 50, $\Gamma(\alpha) \in A$, there is $\beta_0 \in A$ such that

$$(\forall x \in \Gamma(\alpha))(\exists y \in \beta_0)[y \in R_x \cap \gamma]$$

and

$$(\forall y \in \beta_0)(\exists x \in \Gamma(\alpha))[y \in R_x \cap \gamma].$$

Let $\beta = \alpha \cup \beta_0$. Then $\beta \subset \gamma$ and $\beta \in A$, as A is closed under unions. So $\beta \in P$ and also

$$\alpha \subseteq \beta \ \& \ (\forall x \in \Gamma(\alpha))(\exists y \in \beta)[y \in R_x].$$

Thus $(\alpha, \beta) \in T$.
■

Corollary: 52 *If $\alpha_0 \in P$ then there is $\alpha \in A_\gamma$ such that $\alpha_0 \subseteq \alpha$.*

Proof: Let $\alpha_0 \in P$. Then, by **DC**, there is an infinite sequence $\alpha_0, \alpha_1, \ldots$ of elements of P such that $(\alpha_n, \alpha_{n+1}) \in T$ for all $n \in \mathbb{N}$. It follows that

$$\alpha_0 \subseteq \alpha_1 \subseteq \cdots \subseteq \gamma$$

and each $\alpha_n \in A$. As $\mathbb{N} \in A$ and A is a union-closed regular set, $\alpha = \bigcup_{n \in \mathbb{N}} \alpha_n$ is in A and $\alpha_0 \subseteq \alpha \subseteq \gamma$. It remains to show that α is Γ, R-closed. We must show that

$$(\forall x \in \Gamma(\alpha))(\exists y \in \alpha)\ y \in R_x.$$

So let $x \in \Gamma(\alpha)$. As Γ is finitary $x \in \Gamma(\alpha_n)$ for large enough n and then $y \in R_x$ for some $y \in \alpha_{n+1} \subseteq \alpha$, giving what we want.
■

The proof of the theorem is completed with the following result.

Corollary: 53

1. *A_γ has an element.*
2. *If $\alpha_1, \alpha_2 \in A_\gamma$ then there is $\alpha \in A_\gamma$ such that $\alpha_1 \cup \alpha_2 \subseteq \alpha$.*
3. *If $x \in \gamma$ then there is $\alpha \in A_\gamma$ such that $x \in \alpha$.*

Proof: Apply the previous corollary with $\alpha_0 = \emptyset$ for part 1, $\alpha_0 = \alpha_1 \cup \alpha_2$ for part 2 and $\alpha_0 = \{x\}$ for part 3.
■

9 Appendix 2: An application of Subset Collection

Let A, B be sets. A class relation $R \subseteq A \times B$ is *total from A to B* if

$$(\forall x \in A)(\exists y \in B)[(x, y) \in R].$$

We write $\mathbf{mv}(B^A)$ for the class of all such total relations from A to B that are sets. The Subset Collection Scheme is equivalent to the following axiom.

> For all sets A, B there is a subset C of the class $\mathbf{mv}(B^A)$ such that every set in $\mathbf{mv}(B^A)$ has a subset in C. We write $\mathbf{subcoll}(A, B)$ for the class of all such sets C.

Recall that a class is *predicative* if it can be defined by a restricted formula, possibly having set parameters. Note that, by Restricted Separation, the intersection of any predicative class with a set is a set. It follows that any predicative subclass of a set is a set.

Lemma: 54 *Let A, B be sets and let $\mathcal{D}, \mathcal{R}$ be classes, with $\mathcal{D}$ a predicative subclass of $\mathbf{mv}(B^A)$ such that there are class functions mapping $R : \mathcal{D} \mapsto \alpha_R : \mathcal{R}$ and $\alpha : \mathcal{R} \mapsto R_\alpha : \mathcal{D}$ such that if $\alpha \in \mathcal{R}$ and $R \in \mathbf{mv}(B^A)$ is a subset of R_α then $R \in \mathcal{D}$ and $\alpha_R = \alpha$. Then $\mathcal{R}$ is a set.*

Proof: By the above formulation of Subset Collection choose $C \in \mathbf{subcoll}(A, B)$ and let $D = C \cap \mathcal{D}$. As $\mathcal{D}$ is a predicative class D is a set. It is now easy to see that under our assumptions

$$\mathcal{R} = \{\alpha_R \mid R \in D\}$$

so that using the Replacement Scheme we get that $\mathcal{R}$ is a set.
■

The Main Lemma

We assume given $\mathcal{S} = (S, \prec, \asymp)$, where $\prec$ and $\asymp$ are set relations on the set S.

Definition: 55 *Call a subset α of S an* adequate set (for $\mathcal{S}$) *if*

> A1: $b, c \in \alpha \Rightarrow b \asymp c$,
>
> A2: $a \in \alpha \Rightarrow (\exists b \in \alpha)(b \prec a)$.

It is strongly adequate (for $\mathcal{S}$) *if also*

A3: $b \prec a \Rightarrow (\exists c \in \alpha)(b \asymp c \Rightarrow c = a)$.

Note the following observation.

Proposition: 56 *If α satisfies A3 and β is adequate then*

$$\alpha \subseteq \beta \Rightarrow \beta \subseteq \alpha.$$

Proof: Assume that $\alpha \subseteq \beta$ and $a \in \beta$. Then, by $A2$ for β,

$$b \prec a \text{ for some } b \in \beta.$$

By $A3$ for α,

$$b \asymp c \Rightarrow c = a \text{ for some } c \in \alpha.$$

As $\alpha \subseteq \beta$, $b, c \in \beta$ so that, by $A1$ for β, $b \asymp c$ and hence $c = a$, so that $a \in \alpha$.
■

An application of this observation is that every strongly adequate set is a maximally adequate set; i.e. it is maximal among the adequate sets.

The Main Lemma: *If $\prec$ and $\asymp$ are set relations on a set S then the strongly adequate sets for $(S, \prec, \asymp)$ form a set.*
Proof: Let $W = \{(a, b) \in S \times S \mid b \prec a\}$ and let $\mathcal{R}$ be the class of strongly adequate sets for $\mathcal{S}$. For $\alpha \in \mathcal{R}$ let

$$R_\alpha = \{((a, b), c) \in W \times S \mid c \in \alpha \;\&\; (b \asymp c \Rightarrow c = a)\}.$$

Then, by $A3$, $R_\alpha \in \mathbf{mv}(S^W)$. For $R \in \mathbf{mv}(S^W)$ let

$$\alpha_R = \{c \in S \mid (\exists w \in W)(w, c) \in R\}.$$

> **Lemma: 57** *Let $\alpha \in \mathcal{R}$, $R \in \mathbf{mv}(S^W)$ and $R \subseteq R_\alpha$. Then $\alpha_R = \alpha$.*
>
> **Proof:** To show that $\alpha_R \subseteq \alpha$ let $a \in \alpha_R$. Then $(w, a) \in R$ for some $w \in W$ so that $(w, a) \in R_\alpha$, as $R \subseteq R_\alpha$. It follows that $a \in \alpha$.
>
> To show that $\alpha \subseteq \alpha_R$ let $a \in \alpha$. Then, by $A2$, there is $b \in \alpha$ such that $b \prec a$. As $(a, b) \in W$ and $R \in \mathbf{mv}(S^W)$ there is c such that $((a, b), c)$ is in R and so in R_α, so that $c \in \alpha$ and
>
> $$b \asymp c \Rightarrow c = a.$$

As $b, c \in \alpha$, by $A1$, $b \asymp c$ and so $c = a$ so that $((a, b), a) \in R$ and hence $a \in \alpha_R$.

■

Now let $\mathcal{D} = \{R \in \mathbf{mv}(S^W) \mid \alpha_R \in \mathcal{R}\}$. Then $\mathcal{D}$ is a predicative class and trivially $R \in \mathcal{D} \Rightarrow \alpha_R \in \mathcal{R}$. By Lemma 57 $\alpha \in \mathcal{R} \Rightarrow R_\alpha \in \mathcal{D}$. So, by Lemma 54 and Lemma 57 again we get that $\mathcal{R}$ is set.

■