On Completeness of Logics Enriched with Transitive Closure Modality

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We give a sufficient condition for Kripke completeness of the extension of a modal logic with the transitive closure modality. More precisely, we show that if a logic is canonical and admits what we call definable filtration (ADF), then such an extension is complete (and again ADF).

The transitive closure of a binary relation R on a set W is denoted by R^+. Given a frame $F = (W, R)$, we write $F^{(+)} = (W, R, R^+)$, and for a class of frames $\mathcal{F}$, denote $\mathcal{F}^{(+)} = \{F^{(+)} \mid F \in \mathcal{F}\}$.

The extension of a normal modal logic $\mathbf{L}$ with the *transitive closure modality* $\boxplus$ is the minimal normal bimodal logic $\mathbf{L}^{\boxplus}$ that contains $\mathbf{L}$ and the axioms:

$$(\mathsf{A1}) \quad \boxplus p \to \Box p, \qquad (\mathsf{A2}) \quad \boxplus p \to \Box\boxplus p, \qquad (\mathsf{A3}) \quad \boxplus(p \to \Box p) \to (\Box p \to \boxplus p).$$

Fact 1 $(W, R, S) \models (\mathsf{A1}) \wedge (\mathsf{A2}) \wedge (\mathsf{A3})$ *iff* $S = R^+$.

Fact 2 *The class of* $\mathbf{L}^{\boxplus}$*-frames equals* $\mathcal{F}^{(+)}$*, where* $\mathcal{F}$ *is the class of* $\mathbf{L}$*-frames.*

A logic $\mathbf{L}$ is called *complete* if it is the logic of some class of frames. It is well known that $\mathbf{K}^{\boxplus}$ is complete (cf. [8]). To the best of our knowledge, no general conditions for completeness of $\mathbf{L}^{\boxplus}$ were known so far. Here we present one such condition, which in fact is rather strong, as it implies the finite model property (FMP) and hence, for finitely axiomatizable logics, decidability. On the other hand, many "standard" logics satisfy this condition.

Below we give definitions for unimodal logics (and 1-frames), while for bimodal logics (and 2-frames) they are introduced similarly.

Recall that the *canonical frame* for a logic $\mathbf{L}$ is defined as $F_{\mathbf{L}} = (W_{\mathbf{L}}, R_{\mathbf{L}})$, where $W_{\mathbf{L}}$ is the set of maximal $\mathbf{L}$-consistent sets of formulas, and for $x, y \in W_{\mathbf{L}}$, $x\, R_{\mathbf{L}}\, y$ iff $\{A \mid \Box A \in x\} \subseteq y$. A logic is called *canonical* if $F_{\mathbf{L}} \models \mathbf{L}$.

Fact 3 *Any canonical logic is complete.*

However, this way of proving completeness of $\mathbf{L}^{\boxplus}$ fails even in simple cases: although $\mathbf{K}$ is canonical, $\mathbf{K}^{\boxplus}$ is not, since $F_{\mathbf{K}^{\boxplus}} \not\models (\mathsf{A3})$. Below we give a condition on $\mathbf{L}$, which, together with canonicity of $\mathbf{L}$, implies completeness of $\mathbf{L}^{\boxplus}$.

Open problem 1: *Does the canonicity of* $\mathbf{L}$ *imply the completeness of* $\mathbf{L}^{\boxplus}$*?*

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Let Γ be a finite *sub-closed* (i.e., closed under subformulas) set of formulas. In any model M, the set Γ induces an equivalence relation on the worlds of M defined by: $x \sim_\Gamma y$ iff $\forall A \in \Gamma\ (M, x \models A \Leftrightarrow M, y \models A)$. Below, by an *$\mathcal{F}$-model* we mean any model based on some frame from the class $\mathcal{F}$.

Definition 1. By a *(modally) definable Γ-filtration* of a model $M = (W, R, V)$ we mean any model $\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{V})$ that satisisfies the following conditions:

- $\widehat{W} = W/{\sim}$, where $\sim\, = \,\sim_\Delta$, for some finite sub-closed set of formulas $\Delta \supseteq \Gamma$; the $\sim$-equivalence class of a world $x \in W$ will be denoted by $\widehat{x}$;
- the valuation $\widehat{V}$ satisfies: $x \models p \Leftrightarrow \widehat{x} \models p$, for all $x \in W$ and $p \in \mathsf{Var}(\Gamma)$;
- $R^{\mathsf{min}}_{\sim} \subseteq \widehat{R} \subseteq R^{\mathsf{max}}_{\Gamma,\Box}$, where the *minimal* and *maximal filtered* relations are:

$$\begin{array}{lcl} \widehat{x}\, R^{\mathsf{min}}_{\sim}\, \widehat{y} & \leftrightharpoons & \exists x' \sim x\ \exists y' \sim y\text{:}\quad x'\, R\, y, \\ \widehat{x}\, R^{\mathsf{max}}_{\Gamma,\Box}\, \widehat{y} & \leftrightharpoons & \text{for all formulas } \Box A \in \Gamma\ (\, M, x \models \Box A \ \Rightarrow\ M, y \models A\,). \end{array}$$

This is weaker than the notion of filtration *through* Γ [4, 7], in which $\Delta = \Gamma$.

Definition 2. We say that a class of frames $\mathcal{F}$ *admits definable filtration* (*ADF*) if, for every $\mathcal{F}$-model M and every finite sub-closed set of formulas Γ, there exists an $\mathcal{F}$-model $\widehat{M}$ that is a definable Γ-filtration of M.

We say that a logic *ADF* if the class of all its frames ADF.

Example 1 ([1, 2, 5, 6]). The logics **K**, **T**, **K4**, **S4**, **B**, **S5**, and their multi-modal versions admit definable filtration. Furthermore, the logics **S4**.**1**, $\mathbf{K}{+}\Box p \to \Box^m p$, $\mathbf{K} + \Box^m p \to \Box p$, for $m \geq 0$, admit definable filtration.

Fact 4 *If a logic is complete and ADF, then it has the FMP.*

The notion of *definable filtration* introduced above is slightly stronger than that of *filtration* we used in [3], where $\sim$ was an arbitrary equivalence relation of finite index that refines $\sim_\Gamma$. For instance, if a model M has two modally indistinguishable worlds x and y then any filtration $\widehat{M}$ that puts x and y into different $\sim$-equivalence classes is not a definable filtration of M. However, we do not know if every logic that admits filtation admits definable filtration. Note that, in contrast to [3], here we find it convenient to *not* include the completeness of a logic into the notion of ADF.

In [3, Th. 2.6] we proved that if $\mathcal{F}$ admits filtration, then so does $\mathcal{F}^{(+)}$. An inspection of the proof shows that the same holds for definable filtrations:

Fact 5 *If a logic* $\mathbf{L}$ *ADF, then* $\mathbf{L}^{\boxplus}$ *ADF, too.*

Theorem 1. *If a logic* $\mathbf{L}$ *is canonical and ADF, then* $\mathbf{L}^{\boxplus}$ *is complete and ADF.*

Open problem 2: *Can 'canonicity' of* $\mathbf{L}$ *be weakened to 'completeness' here?*

The proof of Theorem 1 combines the filtration construction from [3, Th. 2.6] with the following remarkable property of the induction axiom ($\mathsf{A3}$). Let us write $M \models A^*$ if all substitution instances of the formula A are true in the model M.

Fact 6 *Let M be a model based on a frame $F = (W, R, S)$. Let $\sim\, = \,\sim_\Delta$ for a finite set of formulas Δ. Denote the minimal filtered frame $F_\sim^{\mathsf{min}} = (\widehat{W}, R_\sim^{\mathsf{min}}, S_\sim^{\mathsf{min}})$. Then the implication holds:* *if* $M \models (\mathsf{A3})^*$ *then* $F_\sim^{\mathsf{min}} \models (\mathsf{A3})$.

Theorem 1 also holds for polymodal logics, where the new modality $\boxplus$ corresponds to the transitive closure of the union of a finite set of relations. Furthermore, results similar to Theorem 1, even with 'completeness' in place of 'canonicity', can be easily obtained for the modalities of the union and composition of relations, since these modalities are expressible in terms of the original ones. This makes our framework close to the propositional dynamic logic PDL. However, we cannot use Theorem 1 to iterate the operation of extending the language: the premise of the theorem for $\mathbf{L}$ is stronger than its conclusion for the resulting logic. Nevertheless, we believe that the following proposition holds.

Conjecture 1. Suppose that a set of formulas Φ axiomatizes a canonical logic that admits definable filtration. Then the logic $\mathsf{PDL}(\Phi)$, that is PDL extended by the axioms Φ that axiomatize atomic modalities, has the finite model property.

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