Local Goldblatt–Thomason Theorem

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Abstract

The celebrated theorem proved by Goldblatt and Thomason in 1974 gives necessary and sufficient conditions for an elementary (i.e., first-order definable) class of Kripke frames to be modally definable. Here we obtain a local analogue of this result, where in place of frames we consider n*-frames*, which are frames with n distinguished worlds. For talking about n-frames, we generalize customary modal formulas to what we call *modal expressions*. Unlike a modal formula, which is evaluated at a single world of a Kripke model, a modal expression with n individual variables is evaluated at an n-tuple of worlds, just as a first-order formula with n free variables. We introduce operations on n-frames that preserve validity of modal expressions, and show that closure under these operations is a necessary and sufficient condition for an elementary class of n-frames to be modally definable. We also discuss the relationship between modal expressions and hybrid logic and leave open questions.

Keywords: modal logic, modal definability, elementary class, hybrid logic

1 Introduction

In modal logic, local and global notions and results often go hand in hand. One of the most prominent pair of notions comes from Modal Correspondence Theory, which studies the relationship between modal and first-order formulas on Kripke frames. A modal formula φ is said to *globally correspond* to a closed first-order formula A (in the signature $\{R, =\}$) if, for every frame $F = (W, R)$, the formula φ is valid on F iff the formula A is true in F. At the same time, φ is said to *locally correspond* to a first-order formula with one free variable $B(x)$ if, for every frame F and every world a in it, φ is valid at the world a in F iff $B(a)$ is true in F. It is easily seen that if φ locally corresponds to $B(x)$ then φ globally corresponds to $\forall x\, B(x)$; the converse does not hold in general.

It is the global notions that are used in the famous Goldblatt–Thomason Theorem (GTT) proved in 1974 [7]. A class of frames $\mathbb{K}$ is called *modally definable* (resp., *elementary*) if there is a (possibly infinite) set of modal (resp., closed first-order) formulas such that $\mathbb{K}$ is exactly the class of frames on which all these formulas are valid (resp., true). The GTT claims that an elementary class of frames $\mathbb{K}$ is modally definable iff it is closed under taking *disjoint unions* of frames ($\biguplus$), *generated subframes* ($\hookrightarrow$), *p-morphic images* of frames ($\twoheadrightarrow$), and reflects *ultrafilter extensions* of frames ($\mathfrak{ue}$); the latter means that the complement of $\mathbb{K}$ is closed under the operation $\mathfrak{ue}$.

It is natural to ask what this result will look like under the local viewpoint. The notions of an elementary and a modally definable class of pointed Kripke frames (F, a) are defined in an obvious way. Moreover, one can easily give natural local analogues of the above operations $\hookrightarrow$, $\twoheadrightarrow$, and $\mathfrak{ue}$. But then two questions are raised: what is the local analogue of $\biguplus$, and are the resulting operations enough to obtain the "unary" version of GTT? The answer to the first one is rather simple: instead of the closure of a class of frames under $\biguplus$, one needs the following closure condition on a class $\mathbb{K}$ of pointed frames: whenever a pointed frame (F, a) is in $\mathbb{K}$ and G is an arbitrary frame (not necessarily related to $\mathbb{K}$) disjoint with F, then $(F \cup G, a)$ is in $\mathbb{K}$. We call this operation the *disjoint extension* of a pointed frame (F, a) with the frame G. After this, the second question is answered affirmatively, as we show in this paper.

We go even further and consider *n-frames*, which are frames with n distinguished worlds $(F, \vec{a})$, where $F = (W, R)$ is a frame and $\vec{a} = (a_1, \dots, a_n)$, $a_i \in W$, $n \geqslant 1$. The operations on 1-frames ($\biguplus$, $\hookrightarrow$, $\twoheadrightarrow$, $\mathfrak{ue}$) generalize easily to the n-ary case. Furthermore, the notion of an elementary class $\mathbb{K}$ of n-frames comes naturally: there is a set $\Gamma(\vec{x})$ of first-order formulas with free variables among $\vec{x} = (x_1, \dots, x_n)$ such that $\mathbb{K}$ is exactly the class of n-frames $(F, \vec{a})$ that satisfy $F \models \Gamma(\vec{a})$. However, this time we encounter an obstacle with generalizing the notion of a modally definable class of n-frames, because so far we had no syntactic entities in the modal language that could be evaluated at an n-tuple of worlds.

To resolve this issue, we use the so-called modal expressions, which we first introduced in [10]. *Modal expressions* are built up from modal formulas (φ, etc.) and individual variables $x_0, x_1, \dots$ (the same as in the first-order language) as follows: $x\colon \varphi$ is a modal expression, for any individual variable x and any modal formula φ; if Φ and Ψ are modal expressions, then so are $\Phi \wedge \Psi$, $\Phi \vee \Psi$, and $\neg\Phi$. The truth of a modal expression $\Phi(\vec{x})$ at an n-tuple of worlds $\vec{a}$ in a model M is denoted by $(M, \vec{a}) \models \Phi(\vec{x})$ or shorter by $M \models \Phi(\vec{a})$ and defined inductively: $M \models a\colon \varphi$ if $M, a \models \varphi$; the clauses for $\wedge, \vee, \neg$ are obvious. This resembles the semantics of first-order formulas with free variables. Finally, the notion of validity of a modal expression $\Phi(\vec{x})$ on an n-frame $(F, \vec{a})$ comes naturally: $F \Vdash \Phi(\vec{a})$ if $M \models \Phi(\vec{a})$, for all model M based on the frame F.

Now we can say that a class $\mathbb{K}$ of n-frames is *modally definable* if there is a set $\Delta(\vec{x})$ of modal expressions with individual variables among $\vec{x}$ such that $\mathbb{K}$ is exactly the class of n-frames on which all modal expressions from $\Delta(\vec{x})$ are valid. The main result of our paper is the following local version of GTT: *For any $n \geqslant 1$, an elementary class of n-frames is modally definable iff it is closed under taking disjoint extensions of n-frames, generated n-subframes, p-morphic images of n-frames, and reflects ultrafilter extensions of n-frames.* The uniformity of definitions and proofs given below for all $n \geqslant 1$ justifies that modal expressions are the "right" generalization of modal formulas.

The original proof of the GTT given in [7] was based on the Birkhoff HSP theorem from algebra. The latter says that a class of algebras forms a variety (i.e., is defined by a set of identities) iff it is closed under taking of homomorphic images (H), subalgebras (S), and direct products (P). Later, van Benthem [21] gave a different proof of the GTT based on results from the first-order model theory; this proof can also be found in [1, Sect. 3.8]. Our proof is an adaptation of van Benthem's argument, in which we "localize" intermediate notions and lemmas. We follow the presentation of the proof given in [1] with further simplifications. In particular, the proof of the Local GTT itself is made short, while auxiliary propositions (not related directly to the main construction and possibly interesting *per se*) are given separately.

Remark 1.1
Although modal expressions as an n-ary generalization of modal formulas were introduced in [10], the idea traces back to earlier works of Kracht and van Benthem.

In his developments on modal definability of first-order formulas with several free variables [11, Sect. 5.4], Kracht was already considering modal expressions, although implicitly and in a restricted form. Specifically, he investigated n-tuples of modal formulas $(\varphi_1, \ldots, \varphi_n)$, which, in our notation, correspond to modal expressions of the form $x_1 : \varphi_1 \vee \ldots \vee x_n : \varphi_n$; in [10] we called them *Kracht disjunctions*.

In fact, the definability of a class of n-frames by a set of arbitrary modal expressions is equivalent to the definability by a set of Kracht disjunctions. Indeed, in any modal expression, one can push negations inwards, so that they occur only in the form $\neg x : \varphi$, which is equivalent to $x : \neg\varphi$. Furthermore, any modal expression in which negation occurs only inside modal formulas is equivalent to a conjunction of Kracht disjunctions (by distributivity of $\wedge$ over $\vee$ and vice versa). Finally, a set of conjunctions of Kracht disjunctions can be equivalently represented as a set of Kracht disjunctions. Thus, any class of n-frames definable by a set of modal expressions is also definable by a set of Kracht disjunctions. However, in the proof of Local GTT, we will need at some point to take the negation of a modal expression, which is not so convenient if we only deal with Kracht disjunctions.

In [20, Ch. 3], van Benthem considered modal expressions in the form of their standard translations. To be more precise, he investigated what he calls *m-formulas*, which are exactly the translations of modal expressions into the first-order language in the signature $\{R\} \cup \{P_0, P_1, \ldots\}$, where P_i is a unary predicate symbol corresponding to the propositional variable p_i; see our Section 2 below for the definition of this translation. In our terminology, van Benthem's result [20, Th. 3.9] can be stated as follows: *A first-order formula $A(\vec{x})$ in the above signature is equivalent (on the class of all Kripke models) to the standard translation of some modal expression $\Phi(\vec{x})$ iff $A(\vec{x})$ is invariant for total bisimulations and generated submodels.*

Remark 1.2

Modal expressions are closely related to formulas of the hybrid logic [2, Ch. 14].s Indeed, an expression $x : \varphi$ has the same semantics[1] as the hybrid formula $@_x\varphi$. Thus, modal expressions can be regarded as formulas in the language $\mathsf{Fm}(\Box, @)$, i.e., the usual modal language extended with the @-operator (but unlike customary hybrid formulas, here nominals cannot occur in the position of propositional variables). Moreover, in fact, every formula in $\mathsf{Fm}(\Box, @)$ is equivalent to some modal expression, since @-operators can always be pushed upwards, due to equivalences like $\Box @_x\varphi \leftrightarrow @_x\varphi$ and $@_y @_x\varphi \leftrightarrow @_x\varphi$, see e.g. [18, Th. 3.3.2]. See Section 7 for further discussion.

2 Preliminaries

We assume the reader to be familiar with syntax and semantics of modal logic [1]; let us briefly recall some notions and fix notation. *Modal formulas* are built up from a countable[2] set of *propositional variables* $\mathbb{P} = \{p_0, p_1, \ldots\}$ according to the syntax:

$$\varphi, \psi \quad ::= \quad \bot \mid p_i \mid \varphi \to \psi \mid \Box\varphi.$$

We use other connectives as standard abbreviations, e.g. $\Diamond\varphi = \neg\Box\neg\varphi$.

A Kripke *frame* $F = (W, R)$ consists of a nonempty set W of *worlds* and a binary *accessibility relation* $R \subseteq W \times W$. A Kripke *model* based on F is a pair $M = (F, \theta)$,

[1] With a negligible difference that the variable x in $x : \varphi$ is interpreted as an element of the set of worlds, while the nominal x in $@_x\varphi$ is interpreted as a one-element set.

[2] Later, we will also need sets of propositional variables of larger cardinality.

where F is a frame and V a *valuation* on F which assigns to each variable $p \in \mathbb{P}$ a subset $\theta(p) \subseteq W$. The *truth* of a formula at a world in a model, denoted by $M, a \models \varphi$, is defined as usual; in particular,

$$M, a \models \Box\varphi \quad \leftrightharpoons \quad \text{for all } b \in W, \text{ if } a\,R\,b \text{ then } M, b \models \varphi.$$

We denote by $\theta(\varphi) = \{a \in W \mid M, a \models \varphi\}$ the set of worlds where φ is true. A formula φ is called *true in a model* M if it is true at all worlds in M; *valid on a frame* F if it is true in all models based on F; notation: $M \models \varphi$ and $F \Vdash \varphi$. Furthermore, a formula is called *satisfiable in a model* M if it is true at some world in M; *satisfiable in a frame* F if it is true at some world of some model based on F. For a subset $X \subseteq W$, we denote $R(X) = \{b \mid \exists a \in X\colon a\,R\,b\}$ and $R^{-1}(X) = \{a \mid \exists b \in X\colon a\,R\,b\}$. We also use notation $\Diamond X := R^{-1}(X)$, cf. [1, Def. 2.55]. It is easily seen that $\theta(\Diamond\varphi) = \Diamond\theta(\varphi)$.

Next, we fix a countable set of *individual variables* $\mathbb{X} = \{x_0, x_1, \ldots\}$ and consider the first-order (FO) language with a binary relation symbol R and unary relation symbols P_i for each $p_i \in \mathbb{P}$. Then a Kripke model $M = (W, R, \theta)$ can serve as a FO interpretation for this language, where P_i is interpreted as $\theta(p_i)$. Similarly, a frame $F = (W, R)$ serves as an interpretation of the FO language with the symbol R (and equality). The *standard translation* of a modal formula φ is a FO formula $\varphi^*(x)$ with one free variable defined inductively (where y is a fresh individual variable):

$$\begin{array}{rclcrcl} \bot^* & = & \bot, & \qquad & (\varphi \to \psi)^*(x) & = & \varphi^*(x) \to \psi^*(x), \\ p_i^*(x) & = & P_i(x), & & (\Box\varphi)^*(x) & = & \forall y\,(x\mathsf{R}y \to \varphi^*(y)). \end{array}$$

Clearly, the definition of φ^* mimics the semantics of φ, and so $M, a \models \varphi$ iff $M \models \varphi^*(a)$.

2.1 Operations on frames

Definition 2.1 (Disjoint union)
The *disjoint union* of frames $\{F_i = (W_i, R_i)\}_{i \in I}$ is $F = (W, R) = \biguplus_{i \in I} F_i$, where
– $W = \biguplus_{i \in I} W_i := \{\langle i, a\rangle \mid i \in I,\, a \in W_i\}$,
– $\langle i, a\rangle\, R\, \langle j, b\rangle \;\leftrightharpoons\; (i = j \text{ and } a\, R_i\, b)$.

Definition 2.2 (Generated subframe)
A frame $F' = (W', R')$ is called a *generated subframe* of a frame $F = (W, R)$, notation: $F' \hookrightarrow F$, if $W' \subseteq W$, $R' = R{\restriction}W' := R \cap (W' {\times} W')$, and $R(W') \subseteq W'$.

If, additionally, $W' = R^*(X) = X \cup R(X) \cup R^2(X) \cup \ldots$ for some $X \subseteq W$, then F' is called the subframe of F *generated by* X and denoted by $F' = F_X$. Equivalently, F_X is the smallest generated subframe of F containing X. If $F = F_X$, then F is called a frame generated by X; if $X = \{a\}$ here, then F is called *rooted* with the root a.

Generated and rooted (sub)models are defined in the obvious way.

Definition 2.3 (p-morphic image)
A frame $F' = (W', R')$ is called a *p-morphic image* of a frame $F = (W, R)$, notation: $F \twoheadrightarrow F'$, if there is a surjective *p-morphism* from F onto F', i.e., a function $h\colon W \to W'$ satisfying, for all $a, b \in W$ and $c \in W'$:
(sur) h is surjective, i.e., $h(W) = W'$;
(zig) $a\,R\,b \implies h(a)\,R'\,h(b)$;
(zag) $h(a)\,R'\,c \implies \exists b \in W\colon a\,R\,b$ and $h(b) = c$.

A family of sets $\alpha \subseteq 2^W$ is called an *ultrafilter* over W if, for all $X, Y \subseteq W$,

(1) if $X, Y \in \alpha$ then $X \cap Y \in \alpha$,
(2) if $X \in \alpha$ and $X \subseteq Y$ then $Y \in \alpha$,
(3) $X \notin \alpha$ iff $\overline{X} \in \alpha$, where $\overline{X} = W \setminus X$.

This definition implies that $\varnothing \notin \alpha$ and $W \in \alpha$, for any ultrafilter α over W.

Definition 2.4 (Ultrafilter extension)
The *ultrafilter extension* of a frame $F = (W, R)$ is the frame $F^{\mathfrak{ue}} = (W^{\mathfrak{ue}}, R^{\mathfrak{ue}})$, where $W^{\mathfrak{ue}}$ is the set of all ultrafilters over W, and $\alpha\, R^{\mathfrak{ue}}\, \beta$ holds iff $X \in \beta$ implies $\Diamond X \in \alpha$. The *ultrafilter extension* of a model $M = (F, \theta)$ is the model $M^{\mathfrak{ue}} = (F^{\mathfrak{ue}}, \theta^{\mathfrak{ue}})$, where the valuation $\theta^{\mathfrak{ue}}$ is defined so that $\alpha \models p$ iff $\theta(p) \in \alpha$, for every variable $p \in \mathbb{P}$.

In fact, the latter equivalence holds for all modal formulas: $M^{\mathfrak{ue}}, \alpha \models \varphi$ iff $\theta(\varphi) \in \alpha$ [1, Prop. 2.59]. For every point a in M, the model $M^{\mathfrak{ue}}$ has a "counterpart", namely, the *principal ultrafilter* $a^{\mathfrak{ue}} := \{X \subseteq W \mid a \in X\}$. It immediately follows that the points a and $a^{\mathfrak{ue}}$ are indistinguishable by modal formulas: $M, a \models \varphi$ iff $M^{\mathfrak{ue}}, a^{\mathfrak{ue}} \models \varphi$, for any $a \in W$ and any modal formula φ. Therefore, $F^{\mathfrak{ue}} \Vdash \varphi$ always implies $F \Vdash \varphi$.

Lemma 2.5 (See, e.g., [1, Theorem 3.14, Corollary 3.16])
For all frames mentioned below and any modal formula φ, we have:

– Let $F = \biguplus_{i \in I} F_i$. If $F_i \Vdash \varphi$ for all $i \in I$, then $F \Vdash \varphi$.
– Let $F' \hookrightarrow F$. If $F \Vdash \varphi$ then $F' \Vdash \varphi$.
– Let $F \twoheadrightarrow F'$. If $F \Vdash \varphi$ then $F' \Vdash \varphi$.
– Let $F' = F^{\mathfrak{ue}}$. If $F' \Vdash \varphi$ then $F \Vdash \varphi$.

3 Modal expressions and n-frames

Definition 3.1 (Modal expressions)
Modal expressions are built up from the modal formulas (φ, etc.) defined above and the individual variables in $\mathbb{X} = \{x_0, x_1, \ldots\}$ according to the following syntax:

$$\Phi, \Psi \ ::= \ x\colon \varphi \ \mid \ \neg\Phi \ \mid \ \Phi \wedge \Psi \ \mid \ \Phi \vee \Psi.$$

Other Boolean connectives can be treated as shortcuts. For example, $x\colon \Box p \to y\colon p$ is a modal expression. If Φ is a modal expression with individual variables among $\vec{x} = (x_1, \ldots, x_n)$, $n \geqslant 1$, then we write it as $\Phi(\vec{x})$ and say that Φ is a *(modal) n-expression*. Naturally, semantics of n-expressions is given in terms of n-models.

Definition 3.2 (n-frames and n-models)
An *n-frame* is a pair $(F, \vec{a})$, where $F = (W, R)$ is a frame and $\vec{a} = (a_1, \ldots, a_n)$ is an n-tuple of worlds: $a_i \in W$, $1 \leqslant i \leqslant n$. An *$n$-model* is a pair $(M, \vec{a})$ with M a model.

We define the notion "a modal n-expression $\Phi(\vec{x})$ is *true* in an n-model $(M, \vec{a})$", written as $(M, \vec{a}) \models \Phi(\vec{x})$ or shorter as $M \models \Phi(\vec{a})$, by induction on Φ as follows:

$$\begin{array}{lcl}
M \models a\colon \varphi & \leftrightharpoons & M, a \models \varphi, \\
M \models \neg\Phi(\vec{a}) & \leftrightharpoons & M \not\models \Phi(\vec{a}), \\
M \models \Phi(\vec{a}) \wedge \Psi(\vec{a}) & \leftrightharpoons & M \models \Phi(\vec{a}) \text{ and } M \models \Psi(\vec{a}), \\
M \models \Phi(\vec{a}) \vee \Psi(\vec{a}) & \leftrightharpoons & M \models \Phi(\vec{a}) \text{ or } M \models \Psi(\vec{a}).
\end{array}$$

A modal expression $\Phi(\vec{x})$ is *valid* on an n-frame $(F,\vec{a})$, written as $(F,\vec{a}) \Vdash \Phi(\vec{x})$ or $F \Vdash \Phi(\vec{a})$, if $M \models \Phi(\vec{a})$ for all models M based on F. The notions of truth and validity extend naturally to classes of n-frames and n-models and to sets of n-expressions.

Definition 3.3 (First-order and modal definability of classes of n-frames)
A class of n-frames $\mathbb{K}$ is called *modally definable* if, for some (possibly infinite) set $\Delta(\vec{x})$ of modal n-expressions with individual variables among $\vec{x} = (x_1, \dots, x_n)$,

$$\mathbb{K} \;=\; \{\, (F,\vec{a}) \mid F \Vdash \Delta(\vec{a}) \,\}.$$

A class of n-frames $\mathbb{K}$ is called *elementary* (or *first-order definable*) if, for some (possibly infinite) set $\Gamma(\vec{x})$ of FO formulas in the signature $\{\mathsf{R}, =\}$, we have

$$\mathbb{K} \;=\; \{\, (F,\vec{a}) \mid F \models \Gamma(\vec{a}) \,\}.$$

We say that a FO formula $A(\vec{x})$ *corresponds* to a modal n-expression $\Phi(\vec{x})$ if, for every n-frame $(F,\vec{a})$, we have $F \models A(\vec{a})$ iff $F \Vdash \Phi(\vec{a})$. Finally, we call a FO formula $A(\vec{x})$ *modally definable*[3] if it corresponds to some modal n-expression.

Example 3.4
The reader may easily check that
– the FO formula $x\mathsf{R}y$ corresponds to $\Phi(x,y) = x\colon \Box p \to y\colon p$,
– the FO formula $x\mathsf{R}y \wedge y\mathsf{R}x$ corresponds to $\Phi(x,y) = x\colon (\Box p \wedge q) \to y\colon (p \wedge \Diamond q)$,
– the FO formula $\exists z\,(x\mathsf{R}z \wedge y\mathsf{R}z)$ corresponds to $\Phi(x,y) = x\colon \Box p \to y\colon \Diamond p$.

All three FO formulas are examples of the so called *existential conjunctive* formulas (also known as *conjunctive queries*), as they have the form $\exists \vec{y} B(\vec{x},\vec{y})$, where B is a conjunction of atomic formulas of the form $u\mathsf{R}v$. A complete criterion for their modal definability was obtained in [10]; moreover, it is effective in the sense that there is an algorithm that, given an existential conjunctive formula, determines whether it is modally definable, and if it is, produces the corresponding modal expression. Note that for arbitrary FO formulas, the problem of recognizing their modal definability is undecidable already in the case of one free variable, due to Chagrova's results [4].

3.1 Operations on n-frames

All operations on n-frames, except possibly for the disjoint extension, are natural analogues of the operations on frames defined in Section 2.1.

Definition 3.5 (Operations on n-frames)

- The *disjoint extension* of an n-frame $(F,\vec{a})$ with a frame G is the frame

$$(F,\vec{a}) \uplus G \;:=\; (F \uplus G, \vec{a}\,').$$

  Here $F \uplus G = (\{0\} \times F) \cup (\{1\} \times G)$ is the disjoint union of the frames F and G as per Definition 2.1 and $\vec{a}\,' = \{0\} \times \vec{a}$.
- $(F',\vec{a})$ is a *generated n-subframe* of $(F,\vec{a})$ if $F' \hookrightarrow F$. Notation: $(F',\vec{a}) \hookrightarrow (F,\vec{a})$. If $F = F_{\vec{a}}$ then the n-frame $(F,\vec{a})$ is called *rooted*; similarly for n-models.
- $(F',\vec{a}\,')$ is a *p-morphic image* of $(F,\vec{a})$ if $F \twoheadrightarrow F'$ via a p-morphism $h\colon W \to W'$ such that $h(a_i) = a_i'$, $1 \leqslant i \leqslant n$. Notation: $(F,\vec{a}) \twoheadrightarrow (F',\vec{a}\,')$.

[3] In [10], such a FO formula was called *modally expressible*, while the term *modally definable* was used for the special case of correspondence with some Kracht disjunction.

- The *ultrafilter* extension of an n-frame $(F, \vec{a})$ is the n-frame $(F^{\mathfrak{ue}}, \vec{a}^{\mathfrak{ue}})$.

Below, we use notation $\mathcal{F} = (F, \vec{a})$ for n-frames. Then the notation $\mathcal{F}_X$, $\mathcal{F}_{\vec{a}}$, $\mathcal{F}^{\mathfrak{ue}}$, etc. are self-explanatory. The following is an immediate consequence of Lemma 2.5.

Lemma 3.6 (Preservation and antipreservation)
For all n-frames $\mathcal{F}, \mathcal{F}'$, any frame G, and any modal n-expression $\Phi(\vec{x})$, we have:

(a) Let $\mathcal{F}' = \mathcal{F} \uplus G$. If $\mathcal{F} \Vdash \Phi(\vec{x})$ then $\mathcal{F}' \Vdash \Phi(\vec{x})$.
(b) Let $\mathcal{F}' \hookrightarrow \mathcal{F}$. If $\mathcal{F} \Vdash \Phi(\vec{x})$ then $\mathcal{F}' \Vdash \Phi(\vec{x})$.
(c) Let $\mathcal{F} \twoheadrightarrow \mathcal{F}'$. If $\mathcal{F} \Vdash \Phi(\vec{x})$ then $\mathcal{F}' \Vdash \Phi(\vec{x})$.
(d) Let $\mathcal{F}' = \mathcal{F}^{\mathfrak{ue}}$. If $\mathcal{F}' \Vdash \Phi(\vec{x})$ then $\mathcal{F} \Vdash \Phi(\vec{x})$.

Corollary 3.7 (Necessary conditions for modal definability)
Any modally definable class of n-frames is closed under the operations $\uplus, \hookrightarrow, \twoheadrightarrow$ and *reflects* ultrafilter extensions (i.e., its complement is closed under the operation $\mathfrak{ue}$).

4 Main result

For comparison, we give the original (global) and the local versions of the theorem.

Theorem 4.1 (Goldblatt, Thomason, 1974, [7], see also [21] and [1, Sect. 3.8])
Let $\mathbb{K}$ be an elementary class of Kripke frames. Then $\mathbb{K}$ is modally definable iff

- $\mathbb{K}$ is closed under the operations of
  – disjoint unions of frames ($\uplus$),
  – generated subframes ($\hookrightarrow$),
  – p-morphic images of frames ($\twoheadrightarrow$), and
- $\mathbb{K}$ reflects ultrafilter extensions of frames.

Theorem 4.2 (Local Goldblatt–Thomason Theorem)
Let $\mathbb{K}$ be an elementary class of n-frames. Then $\mathbb{K}$ is modally definable iff

- $\mathbb{K}$ is closed under the operations of
  – disjoint extensions of n-frames ($\uplus$),
  – generated n-subframes ($\hookrightarrow$),
  – p-morphic images of n-frames ($\twoheadrightarrow$), and
- $\mathbb{K}$ reflects ultrafilter extensions of n-frames.

Proof. ($\Rightarrow$) By Corollary 3.7. Here we do not use that $\mathbb{K}$ is an elementary class.
($\Leftarrow$) This part requires more notions and results, so we defer it till Section 6. ∎

5 Auxiliary lemmas

To complete the proof of Theorem 4.2, we need notions and facts from modal and first-order model theory, such as ultraproducts and ultrapowers of models, Łoś' Theorem, modally compact models and classes of frames, ω-saturated and modally saturated models, etc. In this section and briefly in Section 8, we recall these notions and supply some of the results with simple proofs.

5.1 Modal equivalence of n-models

Recall that two Kripke models M and N (or two pointed Kripke models (M, a) and (N, c)) are called *modally equivalent* if the same modal formulas are true in them. Notation: $M \equiv N$ and $(M, a) \equiv (N, c)$. For instance, $(M, a) \equiv (M^{\mathfrak{ue}}, a^{\mathfrak{ue}})$, as we mentioned above. Let us generalize the notion of modal equivalence to n-models.

Definition 5.1 (Modal equivalence of n-models)
We say that n-models $(M, \vec{a})$ and $(N, \vec{c})$ are *modally equivalent*, which is written as $(M, \vec{a}) \equiv (N, \vec{c})$, if any of the following two (equivalent) conditions are satisfied:

(1) the same modal n-expressions $\Phi(\vec{x})$ are true in $(M, \vec{a})$ and in $(N, \vec{c})$;
(2) $(M, a_i) \equiv (N, c_i)$, for $1 \leqslant i \leqslant n$.

Conditions (1) and (2) are equivalent. Indeed, (1) implies (2), since $x_i : \varphi$ is a modal expression. Conversely, assuming (2), one can prove (1) by induction on $\Phi(\vec{x})$. The induction base is what (2) says, while the induction steps for $\neg, \wedge, \vee$ are trivial. From known properties of generated submodels, we immediately obtain the following.

Lemma 5.2
Let $(M', \vec{a})$ be a generated n-submodel of an n-model $(M, \vec{a})$. Then $(M', \vec{a}) \equiv (M, \vec{a})$.

Lemma 5.3
Let $(M, \vec{a})$ and $(N, \vec{c})$ be rooted n-models. If $(M, \vec{a}) \equiv (N, \vec{c})$ then $M \equiv N$.

Proof. First, let us prove that if $(M, \vec{a})$ is rooted then, for any modal formula φ,

$$M \models \varphi \quad \Leftrightarrow \quad \forall i = 1 \ldots n \; \forall k \geqslant 0 \quad M, a_i \models \Box^k \varphi.$$

($\Rightarrow$) If $M \models \varphi$, then $M \models \Box^k \varphi$, in particular, $M, a_i \models \Box^k \varphi$.
($\Leftarrow$) Assume that $M, a_i \models \Box^k \varphi$ for all $k \geqslant 0$. Then $\theta(\varphi) \supseteq R^k(a_i)$, for all $k \geqslant 0$, and so $\theta(\varphi) \supseteq R^*(a_i)$. Thus, $\theta(\varphi) \supseteq R^*(a_1) \cup \ldots \cup R^*(a_n) = R^*(\vec{a}) = W$ and so $M \models \varphi$.

Now, if $(M, \vec{a}) \equiv (N, \vec{c})$ and they are both rooted then, for any modal formula φ,

$$\begin{array}{lcll} M \models \varphi & \iff & \forall i = 1 \ldots n \; \forall k \geqslant 0 \; M, a_i \models \Box^k \varphi & \\ & \iff & \forall i = 1 \ldots n \; \forall k \geqslant 0 \; N, c_i \models \Box^k \varphi & \iff \quad N \models \varphi. \end{array}$$

Therefore, $M \equiv N$. ∎

5.2 Reduction to rooted n-frames

Let $\mathbb{K}$ be a class of n-frames and $\Delta(\vec{x})$ a set of modal n-expressions with individual variables among $\vec{x} = (x_1, \ldots, x_n)$. Denote the set of modal n-expressions that are valid in $\mathbb{K}$ and, dually, the class of all n-frames on which $\Delta(\vec{x})$ is valid, by

$$\begin{array}{rcl} \mathsf{Logic}_n(\mathbb{K}) & = & \{\Phi(\vec{x}) \mid \mathbb{K} \Vdash \Phi(\vec{x})\}, \\ \mathsf{Frames}_n(\Delta) & = & \{(F, \vec{a}) \mid F \Vdash \Delta(\vec{a})\}. \end{array}$$

Trivially, we have $\mathbb{K} \subseteq \mathsf{Frames}_n(\mathsf{Logic}_n(\mathbb{K}))$ and $\Delta \subseteq \mathsf{Logic}_n(\mathsf{Frames}_n(\Delta))$. Now, $\mathbb{K}$ is *modally definable* iff $\mathbb{K} = \mathsf{Frames}_n(\Delta)$, for some set of modal n-expressions $\Delta(\vec{x})$.

It is known that any frame is a p-morphic image of the disjoint union of its rooted subframes [1, Ex. 3.3.4]. The following lemma is a local analogue of this statement.

Lemma 5.4
Any n-frame is a p-morphic image of a disjoint extension of its rooted n-subframe:

$$\forall\, \mathcal{F} = (F, \vec{a}) \ \ \exists G: \quad \mathcal{F}_{\vec{a}} \uplus G \twoheadrightarrow \mathcal{F}.$$

Proof. One can take G to be the subframe of F generated by the set $W \setminus \vec{a}$. In fact, a more general result can be easily proved: if the set of worlds of a frame $F = (W, R)$ is partitioned: $W = X \sqcup Y$, then $(F_X \uplus F_Y) \twoheadrightarrow F$, and the p-morphism is $h(\langle 0, a\rangle) = a$ and $h(\langle 1, b\rangle) = b$, for all $a \in R^*(X)$ and $b \in R^*(Y)$. Moreover, h is in fact a p-morphism of n-frames $(F_X, \vec{a}) \uplus F_Y \twoheadrightarrow (F, \vec{a})$, for any tuple $\vec{a} \subseteq X$. ∎

Given a class of n-frames $\mathbb{K}$ and an n-frame $\mathcal{F} = (F, \vec{a})$, we write $\mathbb{K} \models \mathcal{F}$ if, for every modal n-expression $\Phi(\vec{x})$, $\mathbb{K} \Vdash \Phi$ implies $\mathcal{F} \Vdash \Phi$; in other words, if $\mathcal{F} \Vdash \mathsf{Logic}_n(\mathbb{K})$.

Lemma 5.5
The following conditions are equivalent for any class $\mathbb{K}$ of n-frames:

(a) $\mathbb{K}$ is modally definable;
(b) $\mathbb{K} = \mathsf{Frames}_n(\mathsf{Logic}_n(\mathbb{K}))$;
(c) $\mathbb{K} \supseteq \mathsf{Frames}_n(\mathsf{Logic}_n(\mathbb{K}))$;
(d) $\mathbb{K} \models \mathcal{F} \implies \mathcal{F} \in \mathbb{K}$, for all n-frames $\mathcal{F}$.
If, additionally, $\mathbb{K}$ is closed under $\uplus$ and $\twoheadrightarrow$, then (a)–(d) are equivalent to:
(e) $\mathbb{K} \models \mathcal{F} \implies \mathcal{F} \in \mathbb{K}$, for all rooted n-frames $\mathcal{F}$.

Proof. (a) ⇒ (c). If $\mathbb{K} = \mathsf{Frames}_n(\Delta)$, for some set of modal n-expressions Δ, then $\mathbb{K} \Vdash \Delta$, hence $\Delta \subseteq \mathsf{Logic}_n(\mathbb{K})$ and by anti-monotonicity of $\mathsf{Frames}_n(\cdot)$, we obtain $\mathsf{Frames}_n(\mathsf{Logic}_n(\mathbb{K})) \subseteq \mathsf{Frames}_n(\Delta) = \mathbb{K}$.

(c) ⇒ (b). Since the converse inclusion $\subseteq$ in (c) always holds, as noted above.

(b) ⇒ (a). Trivial.

(c) ⇔ (d). Indeed, $\mathsf{Frames}_n(\mathsf{Logic}_n(\mathbb{K})) = \{\mathcal{F} \mid \mathcal{F} \models \mathsf{Logic}_n(\mathbb{K})\} = \{\mathcal{F} \mid \mathbb{K} \models \mathcal{F}\}$.

(d) ⇒ (e). Trivial.

(e) ⇒ (d). Take any n-frame $\mathcal{F} = (F, \vec{a})$. By lemma 5.4, there is a frame G such that $\mathcal{F}_{\vec{a}} \uplus G \twoheadrightarrow \mathcal{F}$. Then we have a chain of implications:

$$\mathbb{K} \models \mathcal{F} \quad \overset{(1)}{\Longrightarrow} \quad \mathbb{K} \models \mathcal{F}_{\vec{a}} \quad \overset{(2)}{\Longrightarrow} \quad \mathcal{F}_{\vec{a}} \in \mathbb{K} \quad \overset{(3)}{\Longrightarrow} \quad (\mathcal{F}_{\vec{a}} \uplus G) \in \mathbb{K} \quad \overset{(4)}{\Longrightarrow} \quad \mathcal{F} \in \mathbb{K}.$$

Here (1) uses that if $\mathsf{Logic}_n(\mathbb{K})$ is valid on $\mathcal{F}$ then it is valid on $\mathcal{F}_{\vec{a}}$; (2) uses (e), since the n-frame $\mathcal{F}_{\vec{a}}$ is rooted; (3) and (4) hold, since $\mathbb{K}$ is closed under $\uplus$ and $\twoheadrightarrow$. ∎

5.3 Modally compact classes of n-frames

We call a modal n-expression $\Phi(\vec{x})$, or a set of modal n-expressions $\Delta(\vec{x})$, *satisfiable* in an n-frame $\mathcal{F}$ if it is true in some n-model based on $\mathcal{F}$. We call $\Phi(\vec{x})$ and $\Delta(\vec{x})$ *satisfiable in a class of n-frames* $\mathbb{K}$ if it is satisfiable in some n-frame from $\mathbb{K}$.

Definition 5.6 (Modally compact classes of n-frames)
A class of n-frames $\mathbb{K}$ is called *modally compact* if, for any set of modal n-expressions $\Delta(\vec{x})$ (over a set of propositional variables $\mathbb{P}$ of arbitrary cardinality), whenever every finite subset of $\Delta(\vec{x})$ is satisfiable in $\mathbb{K}$, the whole set $\Delta(\vec{x})$ is satisfiable in $\mathbb{K}$.

Recall from Section 5.2 that, for a class of n-frames $\mathbb{K}$ and an n-frame $\mathcal{F}$, the notation $\mathbb{K} \models \mathcal{F}$ means that every n-expression valid on $\mathbb{K}$ is also valid on $\mathcal{F}$; equivalently, if every n-expression satisfiable in $\mathcal{F}$ is also satisfiable in $\mathbb{K}$. Let us show that the latter condition holds for any set of n-expressions, if $\mathbb{K}$ is a modally compact class.

Lemma 5.7
Let $\mathbb{K}$ be a modally compact class of n-frames and $\mathcal{F}$ be an n-frame. If $\mathbb{K} \models \mathcal{F}$ then every set of modal n-expressions $\Delta(\vec{x})$ satisfiable in $\mathcal{F}$ is also satisfiable in $\mathbb{K}$.

Proof. Assume that $\Delta(\vec{x})$ is satisfiable in $\mathcal{F}$. Then, for every finite subset $\Gamma \subseteq \Delta$, the n-expression $\bigwedge \Gamma(\vec{x})$ is satisfiable in $\mathcal{F}$ and hence in $\mathbb{K}$, since $\mathbb{K} \models \mathcal{F}$. Thus, every finite subset of Δ is satisfiable in $\mathbb{K}$, hence so is Δ, since $\mathbb{K}$ is modally compact. ∎

5.4 Modally saturated and modally compact Kripke models

This subsection deals with ordinary modal formulas and Kripke models. This is the only part in the proof of GTT that we do not need to "localize". A set of modal formulas Γ is called *satisfiable* in a model M if Γ is true at some world of M; Γ is called *satisfiable in a subset* $X \subseteq W$ in M if Γ is true at some world $a \in X$ in M.

Definition 5.8 (Modally compact model)
A model M is called *modally compact* if, for every set of modal formulas Γ, whenever every finite subset of Γ is satisfiable in M, the whole set Γ is satisfiable in M.

The following notion was first introduced by Fine [6]; see also [1, Def. 2.53].

Definition 5.9 (Modally saturated model)
A model $M = (W, R, \theta)$ is called *modally saturated* if, for every set of modal formulas Γ and every world $a \in W$, whenever every finite subset of Γ is satisfiable in the set $R(a) \subseteq W$ in M, the whole set Γ is satisfiable in the set $R(a)$ in M.

Deep results in first-order model theory (see Section 8.3) imply the following.

Theorem 5.10 (Existence of a modally compact and modally saturated ultrapower)
For every Kripke model M (over a set of variables $\mathbb{P}$ of arbitrary cardinality) there is an ultrafilter U (on some set I) such that the ultrapower M^U is simultaneously a modally compact and modally saturated Kripke model.

6 The proof of the main result

Here we prove the following claim (the 'if' part of Theorem 4.2):

> *if an elementary class $\mathbb{K}$ of n-frames is closed under the operations $\uplus, \hookrightarrow, \twoheadrightarrow$ and reflects ultrafilter extensions, then $\mathbb{K}$ is modally definable.*

Proof. Since $\mathbb{K}$ is elementary, it is closed under ultrapowers an is a modally compact class of n-frames, by Lemmas 8.5 and 8.7. In order to prove that $\mathbb{K}$ is modally definable, we use the equivalence (a)⇔(e) in Lemma 5.5. So, let us take any rooted n-frame $(F, \vec{a})$ such that $\mathbb{K} \models (F, \vec{a})$ and prove that $(F, \vec{a}) \in \mathbb{K}$. Let $F = (W, R)$.

For a start, we introduce a set of fresh propositional variables $\mathbb{P} = \{P_X \mid X \subseteq W\}$ and build a natural model $M = (F, \theta)$ over $\mathbb{P}$ by putting $\theta(P_X) = X$, for all $X \subseteq W$.

Claim 1. *The following formulas are true in M, for all subsets $X, Y \subseteq W$:*

$$P_W, \qquad P_{\overline{X}} \leftrightarrow \neg P_X, \qquad P_{X\cap Y} \leftrightarrow (P_X \wedge P_Y), \qquad P_{\Diamond X} \leftrightarrow \Diamond P_X.$$

Proof. For instance, $\theta(\Diamond P_X) = \Diamond\theta(P_X) = \Diamond X = \theta(P_{\Diamond X})$. (Here $\Diamond X = R^{-1}(X)$.) $\dashv$

Next, consider the "modal n-theory over $\mathbb{P}$" of the n-model $(M, \vec{a})$:

$$\Delta(\vec{x}) \;:=\; \{\, \Phi(\vec{x}) \mid M \models \Phi(\vec{a}) \,\}.$$

This set of modal n-expressions is true in $(M, \vec{a})$ and hence is satisfiable in $(F, \vec{a})$. Since $\mathbb{K} \models (F, \vec{a})$ and $\mathbb{K}$ is a modally compact class, we apply Lemma 5.7 to conclude that the whole set $\Delta(\vec{x})$ is satisfiable in $\mathbb{K}$. This means that there is an n-frame $(G, \vec{c}) \in \mathbb{K}$ and a model $N = (G, \lambda)$ over $\mathbb{P}$ such that $N \models \Delta(\vec{c})$. Let $G = (V, Q)$. Without loss of generality, $(N, \vec{c})$ is rooted (i.e., the frame G is generated by $\vec{c}$); for otherwise, we can take its n-submodel generated by $\vec{c}$ and it will satisfy $\Delta(\vec{x})$ by Lemma 5.2 and will be based on an n-frame from $\mathbb{K}$, since $\mathbb{K}$ is closed under $\hookrightarrow$.

By Theorem 5.10, the model N has an ultrapower $N^U = (V^U, Q^U, \lambda^U)$ that is a modally compact and modally saturated Kripke model. Denote $G^U = (V^U, Q^U)$.

We are going to show that $(G^U, \vec{c}^{\,U}) \twoheadrightarrow (F^{\mathfrak{ue}}, \vec{a}^{\,\mathfrak{ue}})$, for this will complete the proof of the theorem. Indeed, $\mathbb{K}$ is closed under ultrapowers and $\twoheadrightarrow$ and reflects $\mathfrak{ue}$, hence

$$(G, \vec{c}) \in \mathbb{K} \quad\Longrightarrow\quad (G^U, \vec{c}^{\,U}) \in \mathbb{K} \quad\Longrightarrow\quad (F^{\mathfrak{ue}}, \vec{a}^{\,\mathfrak{ue}}) \in \mathbb{K} \quad\Longrightarrow\quad (F, \vec{a}) \in \mathbb{K}.$$

Claim 2. $(M, \vec{a}) \equiv (N, \vec{c})$ *and hence* $M \equiv N \equiv N^U$. (For $\equiv$, see Definition 5.1.)

Proof. For any modal n-expression $\Phi(\vec{x})$ over $\mathbb{P}$, we have:
– $M \models \Phi(\vec{a}) \;\Rightarrow\; \Phi(\vec{x}) \in \Delta(\vec{x}) \;\Rightarrow\; N \models \Phi(\vec{c})$;
– $M \not\models \Phi(\vec{a}) \;\Rightarrow\; M \models \neg\Phi(\vec{a}) \;\Rightarrow\; (\neg\Phi(\vec{x})) \in \Delta(\vec{x}) \;\Rightarrow\; N \models \neg\Phi(\vec{c}) \;\Rightarrow\; N \not\models \Phi(\vec{c})$.

The claim $M \equiv N$ follows from Lemma 5.3, since the n-models $(M, \vec{a})$ and $(N, \vec{c})$ are rooted. Finally, we have $N \equiv N^U$, since, by Lemma 8.4, a model and its ultrapower are indistinguishable by any FO formula, let alone by any modal formula. $\dashv$

Now we build a function $h\colon V^U \to W^{\mathfrak{ue}}$ by putting $h(s) = \{X \subseteq W \mid N^U, s \models P_X\}$, for $s \in V^U$. It remains to show that h is a p-morphism from $(G^U, \vec{c}^{\,U})$ onto $(F^{\mathfrak{ue}}, \vec{a}^{\,\mathfrak{ue}})$. The remainder repeats (with slight simplifications) the proof from [1, Th. 3.19], with additional "local" Claims 6 and 8.

Claim 3. *$h(s)$ is an ultrafilter over W, for any $s \in V^U$.* So, h maps V^U into $W^{\mathfrak{ue}}$.

Proof. Claim 2 implies that the formulas listed in Claim 1 are true not only in M, but also in N^U. Now, $h(s)$ is an ultrafilter, since for all subsets $X, Y \subseteq W$, we have:
– $W \in h(s)$, since $M \models P_W$ and so $N^U \models P_W$;
– $X, Y \in h(s) \;\Rightarrow\; N^U, s \models P_X \wedge P_Y \;\Rightarrow\; N^U, s \models P_{X\cap Y} \;\Rightarrow\; (X \cap Y) \in h(s)$;
– $X \in h(s)$, $X \subseteq Y \;\Rightarrow\; N^U, s \models P_X$, $N^U \models P_X \to P_Y \;\Rightarrow\; N^U, s \models P_Y \;\Rightarrow\; Y \in h(s)$;
– $X \notin h(s) \;\Leftrightarrow\; N^U, s \not\models P_X \;\Leftrightarrow\; N^U, s \models \neg P_X \;\Leftrightarrow\; N^U, s \models P_{\overline{X}} \;\Leftrightarrow\; \overline{X} \in h(s)$. $\dashv$

Claim 4. *The function $h\colon V^U \to W^{\mathfrak{ue}}$ is surjective.*

Proof. Take any ultrafilter $\alpha \in W^{\mathfrak{ue}}$. Denote $\Gamma = \{P_X \mid X \in \alpha\}$. We need to show that $h(s) = \alpha$, i.e., $N^U, s \models \Gamma_\alpha$, for some $s \in V^U$. In other words, we need to show that Γ is satisfiable in N^U. Since N^U is a modally compact Kripke model (see Definition 5.8), it suffices to show that every finite subset of Γ is satisfiable in N^U, or equivalently, in M (because the same modal formulas are satisfiable in any two modally equivalent models). But the latter is easy: every finite set $\{P_{X_1}, \ldots, P_{X_n}\} \subseteq \Gamma$ is satisfiable in M, because $\theta(P_{X_1} \wedge \ldots \wedge P_{X_n}) = X_1 \cap \ldots \cap X_n \neq \varnothing$, due to the fact that the intersection of finitely many sets from an ultrafilter is always nonempty. $\dashv$

Claim 5. *$h(s) = \alpha$ iff $(N^U, s) \equiv (M^{\mathfrak{ue}}, \alpha)$, for all $s \in V^U$ and $\alpha \in W^{\mathfrak{ue}}$.*

Proof. $(\Rightarrow)$ Clearly, for any modal formula φ, the equivalence $\varphi \leftrightarrow P_{\theta(\varphi)}$ holds in M and hence, by Claim 2, in N^U. Then we have:

$$N^U, s \models \varphi \iff N^U, s \models P_{\theta(\varphi)} \iff \theta(\varphi) \in h(s) = \alpha \iff M^{\mathfrak{ue}}, \alpha \models \varphi.$$

$(\Leftarrow)$ We have: $X \in h(s) \Leftrightarrow N^U, s \models P_X \Leftrightarrow M^{\mathfrak{ue}}, \alpha \models P_X \Leftrightarrow X = \theta(P_X) \in \alpha$. $\dashv$

Claim 6. *$h(\vec{s}) = \vec{\alpha}$ iff $(N^U, \vec{s}) \equiv (M^{\mathfrak{ue}}, \vec{\alpha})$, for all n-tuples $\vec{s} \subseteq V^U$ and $\vec{\alpha} \subseteq W^{\mathfrak{ue}}$.*

Proof. Follows from Claim 5, due to the equivalence (1) $\Leftrightarrow$ (2) in Definition 5.1. $\dashv$

Claim 7. *h is a p-morphism from G^U onto $F^{\mathfrak{ue}}$.*

Proof. (zig) Assume $s\, Q^U t$. Let us prove that $h(s)\, R^{\mathfrak{ue}}\, h(t)$. For this, we take any $X \in h(t)$ and show that $\Diamond X \in h(s)$. Since $X \in h(t)$, we have that $N^U, t \models P_X$. Hence $N^U, s \models \Diamond P_X$, since $s\, Q^U t$. Therefore, $N^U, s \models P_{\Diamond X}$ and finally $\Diamond X \in h(s)$.

(zag) Assume $h(s) = \alpha$ and $\alpha\, R^{\mathfrak{ue}}\, \beta$. We need to show that there is $t \in V^U$ such that $s\, Q^U t$ and $h(t) = \beta$. To this end, consider the set of formulas $\Gamma = \{\varphi \mid M^{\mathfrak{ue}}, \beta \models \varphi\}$, i.e., the modal theory (over $\mathbb{P}$) of the world β in $M^{\mathfrak{ue}}$. It suffices to show that Γ is satisfiable at some successor t of s in N^U, for this will imply that $(N^U, t) \equiv (M^{\mathfrak{ue}}, \beta)$ and hence $h(t) = \beta$, by Claim 5.

Since N^U is modally saturated, it suffices to prove that every finite subset of Γ is true at some successor of s in N^U. Moreover, it suffices to prove this property only for singleton subsets of Γ, because Γ is obviously closed under finite conjunctions.

So, take any formula $\varphi \in \Gamma$. This means that $M^{\mathfrak{ue}}, \beta \models \varphi$. Since $\alpha\, R^{\mathfrak{ue}}\, \beta$, we also have that $M^{\mathfrak{ue}}, \alpha \models \Diamond\varphi$, so that $\theta(\Diamond\varphi) \in \alpha = h(s)$, by properties of ultrafilter extension. Now, by definition of h, we have $N^U, s \models P_{\theta(\Diamond\varphi)}$, or equivalently, $N^U, s \models \Diamond\varphi$. This implies that φ holds in some successor of s in N^U, as desired. $\dashv$

Claim 8. *h is a p-morphism from $(G^U, \vec{c}^{\,U})$ onto $(F^{\mathfrak{ue}}, \vec{a}^{\,\mathfrak{ue}})$.*

Proof. $(M^{\mathfrak{ue}}, \vec{a}^{\,\mathfrak{ue}}) \equiv (M, \vec{a}) \equiv (N, \vec{c}) \equiv (N^U, \vec{c}^{\,U})$, so $h(\vec{c}^{\,U}) = \vec{a}^{\,\mathfrak{ue}}$, by Claim 6. $\dashv$

This completes the proof of the theorem. ∎

7 Conclusion and Outlook

In this paper, and implicitly in [10], we investigated definability of classes of n-frames $(F, \vec{a})$ by sets of modal n-expressions $\Phi(\vec{x})$. We already remarked in Introduction that the latter are in essence hybrid formulas of a special kind, namely, Boolean combinations of formulas of the form $@_x\varphi$, where φ is an ordinary modal formula. It would be natural to generalize the framework to arbitrary hybrid formulas [2, Ch. 14].

Specifically, let us consider *hybrid formulas* build up from countable sets of propositional variables $\mathbb{P} = \{p_0, p_1, \ldots\}$ and *nominals* $\mathbb{X} = \{x_0, x_1, \ldots\}$ according to the grammar (where $p \in \mathbb{P}$ and $x \in \mathbb{X}$):

$$\varphi, \psi \quad ::= \quad \bot \mid p \mid x \mid \varphi \to \psi \mid \Box\varphi \mid @_x\varphi.$$

Hybrid formulas are interpreted in Kripke models $M = (W, R, \theta)$ supplemented with an *assignment* π that assigns to every nominal $x \in \mathbb{X}$ a *one-element* set $\pi(x) \subseteq W$.

By definition, $(M,\pi),a \models x$ if $\pi(x)=\{a\}$, and $(M,\pi),a \models @_x\varphi$ if $(M,\pi),\pi(x) \models \varphi$ (notice that the latter does not actually depend on the chosen world a).

Now let $\varphi(\vec{x})$ be a hybrid formula containing nominals only from $\vec{x}=(x_1,\dots,x_n)$. We say that $\varphi(\vec{x})$ is *true* in an n-model $(M,\vec{a})$ if φ is true in M under an assignment π satisfying $\pi(x_i)=\{a_i\}$, for all $1 \leqslant i \leqslant n$. *Validity* of $\varphi(\vec{x})$ on an n-frame $(F,\vec{a})$ is defined in the usual way, as is the notion of *hybrid definability*, i.e., definability of a class of n-frames by a set of hybrid formulas.

For instance, the class of 2-frames (F,a,b) satisfying $a\,R\,b$ is defined, as we saw in Example 3.4, by a modal expression $x\colon \Box p \to y\colon p$, i.e., by a hybrid formula $@_x\Box p \to @_y p$; it is also definable by a simpler, variable-free hybrid formula $@_x\Diamond y$.

Now it is natural to pose the following questions.

- Is hybrid definability (of classes of n-frames) stronger than modal definability?
- What is the analogue of the local GTT for hybrid definability?
  Closely related is the following result obtained in [18, Th. 4.4.1] (see also [2, Th. 22 in Ch. 14]): *an elementary class of frames is definable by a set of hybrid formulas iff it is closed under ultrafilter morphic images and generated subframes.* Note that it is about the definability of classes of frames, not n-frames as defined above.
- Which hybrid formulas define elementary classes of n-frames? This is a request for Sahlqvist-like results. Which hybrid definable classes of n-frames are elementary?
- Is the problem of hybrid definability of conjunctive queries (CQs, see discussion after Example 3.4) decidable? Recall that for modal definability, the problem is decidable in polynomial time, as shown in [10]. Furthermore, is it decidable whether a given CQ is definable by a Boolean combination of hybrid formulas of the form $@_x\varphi$, where φ is @-free? This would have practical consequences for the problem of CQ answering in description logic knowledge bases (see [10, Sect. 7]).

The above questions are also meaningful for richer hybrid languages, e.g. for the above language extended by the *universal modality*, or the *binder operator* $\downarrow$. Recall that the formula $\downarrow x.\varphi$ holds at a world a in (M,π) if φ holds at the same world a in (M,π'), where π' differs from π in only that $\pi'(x)=a$ (so $\downarrow x$ allows one to "name" a current world of evaluation by x and then refer to this world using the given name). The binder operator is rather expressive; for instance, the class of pointed frames (F,a) that satisfy the FO condition $\exists z\,(a\,\mathsf{R}\,z \wedge z\,R\,z)$ is *not* definable by any modal expression $\Phi(x)$ (or equivalently, by any modal formula φ), however, it is definable by a hybrid formula $@_x\Diamond\downarrow y.\Diamond y$ (in which the nominal x is free and y is bound).

There were several variation on the GTT, i.e., on characterization of modally definable classes of frames. In particular, [9, Corol. 3.9] presents an analogue of the GTT for the modal language with universal modality; [3] delivers a version of the GTT for positive modal logic; some generalizations of the GTT were obtained in [13]; [12] puts GTT on coalgebraic rails; [16] extends GTT to the graded modal language; [14] presents a GTT-like result for a nonstandard notion of the modal definability of classes of frames; [8] discusses some results on frame definability in the modal language extended with the modalities for the complement and the inverse of the accessibility relation; a GTT-like theorem for the modal language that corresponds to frames of the form $(W,R,R\cap\neq)$ was obtained in [17] and for its hybrid companion in [15]; [19] discusses an analogue of GTT is obtained with respect to topological semantics for the basic modal language and its hybrid extension. We did not find an explicit reference (but we believe that there is one) for an analogue of GTT for the tense language (with the modalities "always in the future" and "always in the past"). Analogues of the (local) GTT for modal languages combining graded modalities, converse modalities, and nominals (seemingly not obtained yet) would give a more complete picture.

8 Appendix: Facts from first-order model theory

Here we briefly recall the notions and results from first-order model theory that are needed for proving the (local) GTT. For more details, the reader may consult [5] or [1, Appendix A]. We present definitions and results mostly in modal setting, although they hold for more general FO structures.

8.1 Ultraproducts

Let I be a nonempty set (of indices). By an *I-sequence* of elements of some set S we mean a function from I to S; we use notation $a = (a_i)_{i\in I}$ for an I-sequence.

The *product* of sets W_i, $i \in I$, is the set of all I-sequences with i-th element in W_i:

$$\prod_{i\in I} W_i \;:=\; \{\, a = (a_i)_{i\in I} \mid a_i \in W_i \text{ for all } i \in I \,\}.$$

Given an ultrafilter U over I, the following equivalence relation is induced on $\prod_{i\in I} W_i$:

$$a \overset{U}{\sim} b \quad \leftrightharpoons \quad \{i \in I \mid a_i = b_i\} \in U.$$

Let us denote the equivalence class of an I-sequence a by $[a]^U$ or simply by $[a]$. The set of all equivalence classes is called the *ultraproduct* of the sets W_i, $i \in I$, *modulo* U:

$$\prod^U_{i\in I} W_i \;:=\; \{\, [a] \mid a \in \prod_{i\in I} W_i \,\}.$$

Definition 8.1 (Ultraproduct of Kripke models)
The *ultraproduct* of a family of Kripke models $M_i = (W_i, R_i, \theta_i)$, $i \in I$, *modulo* U, where U is an ultrafilter over I, is a Kripke model $M = (W, R, \theta) = \prod^U_{i\in I} M_i$, where

- $W \;=\; \prod^U_{i\in I} W_i$,
- $[a]\, R\, [b] \;\leftrightharpoons\; \{i \mid a_i\, R_i\, b_i\} \in U$,
- $[a] \models p \;\leftrightharpoons\; \{i \mid M_i, a_i \models p\} \in U$, for every variable $p \in \mathbb{P}$.

Theorem 8.2 (Łoś, see e.g.[5, Th. 4.1.9])
Let $M = \prod^U_{i\in I} M_i$. Then for any FO formula $A(x)$ and any element $[a]$ from M,

$$M \models A([a]) \;\text{ iff }\; \{\, i \in I \mid M_i \models A(a_i) \,\} \in U.$$

Similarly for formulas with several and with no free variables.

Corollary 8.3
Let $M = \prod^U_{i\in I} M_i$. Then, for any $[a]$ in M and any modal formula φ,

$$\begin{array}{lll} M, [a] \models \varphi & \text{iff} & \{\, i \in I \mid M_i, a_i \models \varphi \,\} \in U, \\ M \models \varphi & \text{iff} & \{\, i \in I \mid M_i \models \varphi \,\} \in U. \end{array}$$

If all the models M_i are equal to the same model $N = (V, S, \lambda)$, then their ultraproduct is called the *ultrapower* of N modulo U and denoted by $N^U = (V^U, S^U, \lambda^U)$. Any point c in N has its counterpart in N^U, namely, the equivalence class $[a]$ of the constant I-sequence $a = (a_i)_{i\in I}$, $a_i = c$. We have the following consequences.

Lemma 8.4
Any (n-)model is modally equivalent to its ultrapower.
In symbols: $N \equiv N^U$ and $(N, \vec{c}) \equiv (N^U, \vec{c}^U)$.

Lemma 8.5
Any elementary class of (n-)frames is closed under ultraproducts.

8.2 Elementary classes of n-frames are modally compact

Below, we will use the following construction of ultrafilters.

Example 8.6
Let $\mathcal{A}$ be an infinite set. Take all its finite subsets: $I = \{i \subset \mathcal{A} \mid i \text{ is finite}\}$. For every element $a \in \mathcal{A}$, consider the set $\widehat{a} := \{i \in I \mid a \in i\}$ of those finite sets that contain this element. Thus we obtained a family $Z = \{\widehat{a} \mid a \in \mathcal{A}\}$ of subsets of I.

The intersection of finitely many sets from Z is nonempty, because for all elements $a_1, \ldots, a_n \in \mathcal{A}$, we have $(\widehat{a_1} \cap \ldots \cap \widehat{a_n}) \ni \{a_1, \ldots, a_n\}$. Therefore, Z is contained in some ultrafilter U over I (as usual, the latter statement uses Zorn's Lemma).

The next lemma is a modal version of a known first-order fact, see [5, Ex. 4.3.22].

Lemma 8.7
If a class $\mathbb{K}$ of n-frames is closed under ultraproducts, then $\mathbb{K}$ is modally compact.

Proof. We prove the claim for $n = 1$; the proof for $n > 1$ can be obtained by obvious modifications. We need to show that, given an infinite set $\Delta(x)$ of modal 1-expressions, if Δ is finitely satisfiable in $\mathbb{K}$ then it is satisfiable in $\mathbb{K}$.

Take all its finite subsets: $I := \{i \subset \Delta \mid i \text{ is finite}\}$. By assumption, every finite subset of Δ is satisfiable in $\mathbb{K}$. This means that for every finite set of modal expressions $i = i(x) \in I$ there is a pointed frame $\mathcal{F}_i = (F_i, c_i) \in \mathbb{K}$ and a model $M_i = (F_i, \theta_i)$ such that $M_i \models i(c_i)$. Denote the I-sequence $c = (c_i)_{i\in I}$.

Let U be any ultrafilter over I built as in Example 8.6 for the infinite set $\mathcal{A} := \Delta$. Put $M = (F, \theta) = \prod^U_{i\in I} M_i$. Then $(F, [c]) \in \mathbb{K}$, since $\mathbb{K}$ is closed under ultraproducts.

Now we claim that $M \models \Delta([c])$ (and hence Δ is satisfiable in $\mathbb{K}$). Indeed, for any expression $\Phi(\vec{x}) \in \Delta(\vec{x})$, we have $M \models \Phi([c])$, since the set $\{i \in I \mid M_i \models \Phi(c_i)\}$ is in U, because it contains the set $\{i \in I \mid \Phi \in i\} = \widehat{\Phi} \in Z \subseteq U$. ∎

We summarized the relations between properties of classes of (n-)frames in Fig. 1.

- For the implication from 'elementary' to 'ultraproducts', see Lemma 8.5.
- The implication from 'ultraproducts' to 'modally compact' is due to Lemma 8.7.
- The implication from 'ultraproducts' to 'ultrapowers' is trivial.

  The remaining two implications are not essential for proving the (local) GTT, so we only sketch the idea of their proofs.

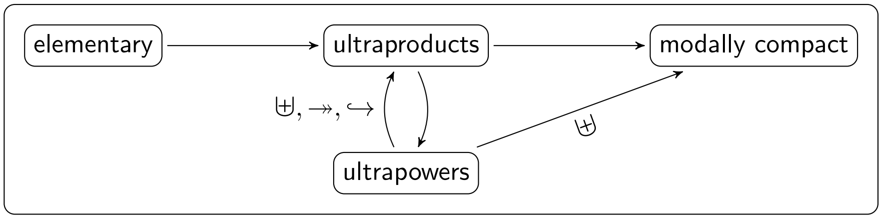

FIG. 1. The relationship between some properties of classes of (n-)frames.

- If a class is closed under ultrapowers and disjoint unions ($\uplus$) then it is modally compact: this is achieved by a slight modification of the proof of Lemma 8.7. Put $M = (\biguplus_{i\in I} M_i)^U$ and take the element $[c']$ of M defined by $c'(i) = \langle i, c_i\rangle$.
  This allows one to slightly strengthen the (local) GTT (see Theorems 4.1 and 4.2) by replacing the condition "the class $\mathbb{K}$ is elementary" with a weaker condition "the class $\mathbb{K}$ is closed under ultrapowers"; cf. [1, Exercise 3.8.4].
- If a class is closed under ultrapowers and the operations $\uplus, \hookrightarrow, \twoheadrightarrow$, then it is closed under ultraproducts, by the following argument. The ultraproduct $\prod_{i\in I}^U F_i$ of frames is isomorphic to a generated subframe of the ultrapower $(\biguplus_{i\in I} F_i)^U$; similarly for n-frames (so, here we only need closure under isomorphisms rather more general p-morphisms). The required isomorphism maps an element $[a]$ from the ultraproduct to the element $[a']$ of the ultrapower, where $a'_i = \langle i, a_i\rangle$.

8.3 ω-saturated first-order structures

Our final remarks are concerned with Theorem 5.10 saying that *any Kripke model has a modally compact and modally saturated ultrapower*. We need this for models over signatures of arbitrary cardinality, since in the proof of the (local) GTT, we considered Kripke models over the set of variables $\mathbb{P} = \{P_X \mid X \subseteq W\}$. This result is usually obtained with the help of a rather involved notion of an *ω-saturated* FO model, which we do not introduce here; for definition, see [1, Def. 2.63] or [5, Sect. 2.3].

For models over *countable* signatures, Theorem 5.10 is the consequence of the following, relatively simple statements:

(a) The so-called *countably incomplete* ultrafilters [5, Def. 4.3.1] exist. In fact, it is easily seen that the ultrafilter in Example 8.6 is countably incomplete.

(b) Let M be a model over a *countable* signature, and U a countably incomplete ultrafilter. Then the ultrapower M^U is ω-saturated [5, Th. 6.1.1].

(c) Any ω-saturated (even 2-saturated) Kripke model is modally saturated [1, Th. 2.65].

(d) Any ω-saturated (even 1-saturated) Kripke model is modally compact. This is a easy exercise mentioned in the proof of the GTT in [1, Th. 3.19], although the notion of a modally compact model is not introduced there explicitly.

For models over signatures of arbitrary infinite cardinality κ, the only known (to us) way of proving Theorem 5.10 is to apply a much harder technique:

(a') The so-called *κ-good* countably incomplete ultrafilters exist [5, Th. 6.1.4]; the proof takes about five pages and is rather involved.

(b') Let M be a model over a signature of cardinality κ, and U a κ-good countably incomplete ultrafilter. Then the ultrapower M^U is κ-saturated [5, Th. 6.1.6].

But even in the case of uncountable signatures, we only need a 2-saturated models in (c) and (d). This leaves the hope that Theorem 5.10 has a much simpler proof.

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