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Undecidability of the Problem of Recognizing Axiomatizations of Superintuitionistic Propositional Calculi

Abstract. We give a new proof of the following result (originally due to Linial and Post): it is undecidable whether a given *calculus*, that is a finite set of propositional formulas together with the rules of *modus ponens* and *substitution*, axiomatizes the classical logic. Moreover, we prove the same for every superintuitionistic calculus. As a corollary, it is undecidable whether a given calculus is consistent, whether it is superintuitionistic, whether two given calculi have the same theorems, whether a given formula is derivable in a given calculus. The proof is by reduction from the undecidable halting problem for the so-called *tag systems* introduced by Post. We also give a historical survey of related results.

Keywords: Classical propositional logic, intuitionistic propositional logic, superintuitionistic calculus, implicational calculus, finite axiomatization, tag system.

1. Introduction

Axioms for the classical propositional logic, i.e., formulas from which all tautologies are derivable using the inference rules of modus ponens and substitution, are known since the dawn of formal logic. Axioms for the intuitionistic propositional logic were proposed by Heyting in the late 1920s. A natural and interesting question is how hard is to recognize whether a given finite set of formulas axiomatizes these (and other) logics.

A pioneer work in this direction was due to Linial and Post dating back to 1949. In [14], they announced a result that it is undecidable whether a given finite set of propositional formulas axiomatizes all classical tautologies. Notably, it was only in 1964 that a complete proof of their result appeared in the work of Yntema [32].

Almost at the same time, in 1963, Kuznetsov [13] proved that the same holds for the intuitionistic propositional logic, as well as for each of its finitely axiomatizable extension (including the classical logic and the inconsistent logic). Yntema used semi-Thue systems for her proof, while Kuznetsov's

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proof is based on a calculus for primitive recursive functions specially devised by him for obtaining this result. Both proofs are rather involved.

Recently, a new proof was given in 2009 by Bokov [2] for the undecidability of recognizing axiomatizations of the classical logics (i.e., the result of Linial and Post). It is based on the so-called *tag systems* introduced by Post [23]. The reduction given in [2] is simpler than the earlier ones. This can be attributed to the fact that tag systems are easier to simulate in propositional calculi than semi-Thue systems or primitive recursive functions.

We managed to further simplify the proof from [2] and to adopt it for obtaining an analog of Kuznetsov's result, i.e., the undecidability of recognizing axiomatizations of any fixed superintuitionistic calculus. This yields, to the best of our knowledge, the simplest known proof of this result.

The paper is organized as follows. In the next section we introduce the basic terminology and notation and state our main results. Section 3 gives a historical survey of relevant results. In Section 4, we recall what a tag system is. Section 5 contains the proof of our results, where we give a reduction from the halting problem for tag systems to the problem of recognizing axiomatizations of any superintuitionistic calculus. Finally, in Section 6 we discuss analogues of these results for other sets of connectives and formulate some open problems.

2. Preliminaries and the main result

The language we consider has a denumerable set of propositional variables $\{p_0, p_1, \ldots\}$ and the *signature*, i.e., the set connectives $\{\neg, \wedge, \vee, \rightarrow\}$. *Formulas* are built up according to the following syntax:[1]

$$A, B \ ::= \quad p \ \mid \ \neg A \ \mid \ (A \wedge B) \ \mid \ (A \vee B) \ \mid \ (A \rightarrow B)$$

By a *calculus* we mean a finite set T of formulas (referred to as *axioms* of T); the rules of inference are always assumed to be the same, viz., *modus ponens* and *substitution* (we use A^σ as a notation for the σ-*instance* of A, i.e., the result of applying the substitution σ to the formula A):

$$(\mathsf{MP}) \ \ A,\ A \rightarrow B \ \vdash \ B \qquad\qquad (\mathsf{Sub}) \ \ A \ \vdash \ A^\sigma$$

A *derivation* in (or from) a calculus T is a finite sequence of formulas in which every formula is either an axiom of T or obtained by (MP) or (Sub)

[1] Throughout the paper, we omit the outermost parentheses in formulas, and assume the customary priority of connectives, which allows us to omit even more parentheses.

$(\to 1)$	$p \to (q \to p)$	$(\wedge 1)$	$p \wedge q \to p$
$(\to 2)$	$[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$	$(\wedge 2)$	$p \wedge q \to q$
$(\wedge 3)$	$p \to (q \to p \wedge q)$	$(\vee 1)$	$p \to p \vee q$
$(\vee 3)$	$(p \to r) \to [(q \to r) \to (p \vee q \to r)]$	$(\vee 2)$	$q \to p \vee q$
$(\neg 1)$	$(p \to q) \to [(p \to \neg q) \to \neg p]$	$(\neg 2)$	$p \to (\neg p \to q)$

Table 1. Axioms of the intuitionistic propositional calculus **Int**.

from previous ones. A formula A is said to be *derivable* in (or from) T, or is a *theorem* of T, written as $T \vdash A$, if A is the last formula of some derivation from T. By $[T]$ we denote the set of all theorems of a calculus T.

We say that two calculi S and T are *equivalent* and write $S \sim T$ if they have the same theorems, i.e., if $[S] = [T]$. We write $S \leqslant T$ (or, equivalently, $T \geqslant S$) if $[S] \subseteq [T]$; in other words, if every axiom (and hence every theorem) of S is derivable in T. Clearly, $S \sim T$ iff both $S \leqslant T$ and $S \geqslant T$. Finally, we write $S < T$ if $[S] \subsetneq [T]$.

Two most important calculi are the *intuitionistic* one **Int** (its axioms are listed in Table 1) and the *classical* one **Cl** obtained from **Int** by adding[2] the *law of the excluded middle* (EM) $p \vee \neg p$, or *double negation elimination* (DN) $\neg\neg p \to p$, or *Peirce's law* (PL) $((p \to q) \to p) \to p$. A calculus T is called *superintuitionistic* if $T \geqslant$ **Int**. Examples of superintuitionistic calculi are **Cl** and $\{p\}$. The latter calculus is *inconsistent*: by the rule (Sub), all formulas are derivable in it. In this paper, we will prove the following result.

THEOREM 2.1. *Fix any superintuitionistic calculus S_0. Then the following problems are undecidable:*

(1) *given a calculus T, determine whether $T \sim S_0$;*
(2) *given a calculus T, determine whether $T \geqslant S_0$.*

Note that, for a fixed calculus S_0, the remaining problem "$T \leqslant S_0$" is decidable iff the calculus S_0 itself is decidable (i.e., it is decidable, given a formula A, whether $S_0 \vdash A$). The calculi **Cl** and **Int** are decidable. An undecidable superintuitionistic calculus was first built by Shehtman [24, 25]; later Chagrov [3] did the same using axioms in only 4 variables (for 2 and 3 variables, the question is open, cf. [5, Sections 16.5 and 16.9]).

[2] In fact, (DN) can *replace* the axiom $(\neg 2)$ in **Int** and still yield **Cl**; replacing $(\neg 2)$ in **Int** with (EM) or (PL) yields a weaker calculus, the so-called *minimal classical logic*, cf. [1].

Corollary 2.2. *The following problems are undecidable:*

(3) *given a calculus T, determine if it axiomatizes the classical logic: $T \sim \mathbf{Cl}$;*
(4) *given a calculus T, decide if it axiomatizes the intuitionistic logic: $T\sim\mathbf{Int}$;*
(5) *given a calculus T, determine whether it is superintuitionistic: $T \geqslant \mathbf{Int}$;*
(6) *given a calculus T, determine whether it is inconsistent: $T \sim \{p\}$;*
(7) *given two calculi T and S, determine whether $T \sim S$;*
(8) *given two calculi T and S, determine whether $T \geqslant S$;*
(9) *given a calculus T and a formula A, determine whether $T \vdash A$.*

Here (3)–(8) follow immediately from (1) and (2), while (9) follows from (8), once we observe that checking $T \geqslant S$ amounts to checking $T \vdash A$ for each of the (finitely many) axioms A of S.

In fact, we will prove Theorem 2.1 for every signature extending $\{\to, \wedge\}$ and every calculus S_0 containing the following five axioms of $\mathbf{Int}$:

$$\mathbf{I} \;:=\; \mathbf{Int}(\to, \wedge) \;=\; \{\, (\to 1), (\to 2),\; (\wedge 1), (\wedge 2), (\wedge 3)\,\}.$$

More precisely, the following is the main result obtained in our paper.

Theorem 2.3. *Fix a signature $\mathfrak{S} \supseteq \{\to, \wedge\}$ and a calculus $S_0 \geqslant \mathbf{Int}(\to, \wedge)$ in the signature $\mathfrak{S}$. Then the following problems are undecidable:*

(1) *given a calculus T in the signature $\mathfrak{S}$, determine whether $T \sim S_0$;*
(2) *given a calculus T in the signature $\mathfrak{S}$, determine whether $T \geqslant S_0$.*

Other signatures are discussed in Section 6. Before presenting the proof, we give a survey of relevant results, using the terminology introduced above.

3. Historical survey of related results

In 1949, Linial and Post [14] announced the following result. Note that they deal with the signature $\{\neg, \vee\}$, so that the rule of *modus ponens* must be formulated appropriately. Also recall that a calculus T is said to be an *independent set of axioms* if no axiom of T is derivable from the remaining axioms of T, i.e., if $T \setminus \{A\} \vdash A$ holds for no $A \in T$.

Theorem 3.1 (Linial, Post, 1949). *The following problems are undecidable:*

(a) *given a calculus T, determine if it axiomatizes the classical logic: $T \sim \mathbf{Cl}$;*
(b) *given a calculus T and a formula A, determine whether $T \vdash A$;*
(c) *given a calculus T, determine whether it is an independent set of axioms.*

Comparing this to our results from Section 2, we see that (a) is (3) and (b) is (9), but in a different signature. Note that [14] is only a half-page abstract giving "a hint of a proof idea" (as Kuznetsov wrote later in [13]). In 1958, Davis [7, pp. 137–142] gave a more detailed argument for their results; however, as Yntema [32] remarked, Davis "reaches conclusions that are not immediately obvious". Later in 1965, Gladstone [9] gives an account on where exactly the argument of Davis fails. Finally, in 1964, Yntema [32] presented a complete proof of these results. Her proof uses *semi-Thue systems*, for which derivability problem is known to be undecidable.

In 1963, Kuznetsov [13] (translation in [8, pp. 56–72]) obtained a much stronger result, not only for the classical, but also for every fixed superintuitionistic calculus. He deals with the signature $\{\neg, \wedge, \vee, \to\}$. Recall that two sets X and Y are said to be *recursively inseparable* if there is no recursive (i.e., decidable) set Z such that $X \subseteq Z$ and $Y \cap Z = \varnothing$. Note that any recursively inseparable sets are also undecidable.

Theorem 3.2 (Kuznetsov, 1963). *Fix any superintuitionistic calculus S_0. Then the following two sets of calculi are recursively inseparable:*

$$\{\, T \;\mid\; \mathbf{Int} \not\leqslant T < S_0 \,\} \quad \textit{and} \quad \{\, T \;\mid\; T \sim S_0 \,\}.$$

As Kuznetsov noticed in his paper, this immediately implies result (1), as well as result (2) for decidable calculi S_0, and hence all the consequences (3–9) of Section 2. Kuznetsov's proof is rather involved; he uses a calculus for primitive recursive functions specially devised to obtain this result.

Unfortunately, Springer translated the journal *Algebra and Logic* only starting with volume 7 (1968); and although Kuznetsov's work was translated later by Mendelson and published in 1966 in the book [8], it seems to remain unfamiliar to researchers (it appears to be almost never cited). In [13] Kuznetsov also mentioned the following problem, which seems still to be open. Recall that a formula or a calculus is called *implicational* (or sometimes *implicative*) if '$\to$' is the only connective occurring in it.

Open problem 1 (A. A. Markov (Jr.), 1961). *Is it decidable whether a given implicational calculus axiomatizes all classical implicational tautologies?* The same question for implicational fragments of other superintuitionistic calculi.

Singletary [26] presented (with proof sketches) the following results.

Theorem 3.3 (Singletary, 1964). *There is a calculus T_0 (consisting of tautologies) in the signature $\{\neg, \to\}$, for which the following problems are undecidable: given a formula A, determine whether:*

(d) $T_0 \nvdash A$ (i.e., whether a formula A is *untrue w.r.t.* T_0);
(e) $\forall\sigma\,(T_0 \nvdash A^\sigma)$ (i.e., whether a formula A is *completely untrue w.r.t.* T_0);
(f) $T_0{+}A \vdash p$ (i.e., whether a formula A is *false w.r.t.* T_0);
(g) $\forall\sigma\,(T_0{+}A^\sigma \vdash p)$ (i.e., whether a formula A is *completely false w.r.t.* T_0).

Additionally, it is undecidable, given a calculus T, whether

(h) $\forall A\ [T \nvdash A \;\Rightarrow\; T + A \vdash p]$ (i.e., whether T is *closed*);
(i) $\forall A\ [\forall\sigma\,(T \nvdash A^\sigma) \Rightarrow T + A \vdash p]$ (i.e., whether T is *completely closed*).

Finally, the following problem is undecidable:

(j) *given two calculi S and T, determine if they have a common theorem.*

In 1965, Gladstone [9] and independently Ihrig [12] constructed calculi (consisting of tautologies) for which the problem of derivability of formulas is of any required recursively enumerable degree of unsolvability. Moreover, Gladstone obtained the same result for every signature (i.e., a finite set of Boolean connectives), provided that implication is expressible in it. In 1968, Singletary [28] constructed an undecidable implicational calculus (consisting of tautologies) and also proved the following result.

THEOREM 3.4 (Singletary, 1968). *For every fixed natural number $n \geqslant 1$, the following problem is undecidable, for the signatures $\{\neg, \to\}$ and even $\{\to\}$:*

(k) *given a calculus T, decide if it can be axiomatized by at most n axioms.*

Marcinkowski [19] investigated the entailment problem for first-order Horn clauses and, as a by-product, obtained the following result.

THEOREM 3.5 (Marcinkowski, 1994). *Fix a propositional formula A_0 not of the form $p \to p$, for any variable p. The following problem is undecidable:*

(ℓ) *given a calculus T, determine whether $T \vdash A_0$.*

Several generalizations of the above results have been obtained. Harrop [10, 11] considered similar problems for more general calculi, which allow for additional rules of inference. Singletary [27, 28] investigated calculi with infinite but recursive sets of axioms and showed that the problem of determining whether such a calculus is finitely axiomatizable is undecidable.

Numerous results were obtained on decision problems of the following kind: given a formula A, determine whether the calculus $\mathbf{Int} \cup \{A\}$ has a particular property; if such a problem is decidable, then this property is called *decidable in the class of calculi extending* **Int**. Here we mention several positive results by Maksimova [17, 18] and negative results by Chagrov and Chagrova [6, 4, 3]; see also [5, Chapter 17] for the terminology and proofs.

THEOREM 3.6 (Maksimova, 1972, 1977). *The following problems are decidable in the class of calculi extending* **Int**: *tabularity, pretabularity, interpolation property, consistency, coincidence with a fixed tabular logic.*

Consequently, since **Cl** is tabular, the problem of determining whether adding a given formula A to **Int** yields **Cl**, i.e., $\mathbf{Int} \cup \{A\} \sim \mathbf{Cl}$, is decidable.

THEOREM 3.7 (Chagrov, Chagrova, 1990–1994). *The following problems are undecidable in the class of calculi extending* **Int**: *decidability, finite model property, disjunction property, Kripke completeness, first-order definability (on all or on countable frames), axiomatizability by implicational formulas, axiomatizability by disjunction free formulas, Halldén-completeness.*

Hence, it is undecidable whether an arbitrary given calculus is decidable.

Let us return to the subject of our paper. In 2009, Bokov [2] reestablished the undecidability of the problem of recognizing axiomatizations of the classical logic (i.e., essentially proposition (a) of Theorem 3.1 above). The proof uses *tag systems* (also known as *uniform Post production systems*) introduced in early 1920s by Post [23] (see also [30]), for which the undecidability of the halting problem was established in 1961 by Minsky [21] (see [22, Section 14.6] for a nice exposition). Notably, the reduction used in Bokov's proof is simpler than the one found in earlier works. In particular, the "reduction calculus" (which we present, in our notation, in Table 2 below) uses only two "special" (fixed) axioms, while Yntema [32] and Kuznetsov [13] used 6 and 11 "special" axioms, respectively. However, the proof in [2] contains a serious error (a correction will be published soon). We managed to reveal and eliminate the error, by introducing a wider set of formulas (see our Definition 5.6) and strengthening intermediate propositions. We also further simplified (in fact, significantly reworked) the proof and adapted it for arbitrary superintuitionistic calculus, thus obtaining our Theorems 2.1 and 2.3. The resulting proof is presented in Section 5.

4. Tag systems

Given an alphabet Σ, by Σ^* we denote the set all words in Σ, including the empty word. The *length* of a word w is denoted by $|w|$.

DEFINITION 4.1 (Post, [23]). A *tag system* is a triple $\Pi = (\Sigma, P, \ell)$, where $\Sigma = \{a_1, \dots, a_m\}$ is a finite alphabet, $P\colon \Sigma \to \Sigma^*$ a function, and $\ell \geqslant 1$ an integer. Denote the word $u_i := P(a_i) \in \Sigma^*$, for $1 \leqslant i \leqslant m$.

We say that Π is *applicable* to a word $w \in \Sigma^*$ if $|w| \geqslant \ell$. In this case, the application of Π to w is described as follows: "if the first letter of w is a_i, then remove in w the first ℓ letters and then append the word u_i at the end." Precisely, if $|a_i\,x| = \ell$ and $z \in \Sigma^*$, then Π transforms the word $w = a_i\,x\,z$ into the word $z\,u_i$. For this transformation, we use notation: $a_i\,x\,z \xmapsto{\Pi} z\,u_i$.

A *computation* of a tag system Π on input w_0 is a sequence of words $w_0, w_1, \ldots$ such that $w_i \xmapsto{\Pi} w_{i+1}$ for all i. Note that computations are deterministic. We write $w \overset{\Pi}{\Longmapsto} x$ if there are words $w_0, \ldots, w_n$, $n \geqslant 0$, such that $w_0 = w$, $w_n = x$ and $w_{i-1} \xmapsto{\Pi} w_i$ for all $1 \leqslant i \leqslant n$. In particular, by definition, $w \overset{\Pi}{\Longmapsto} w$. We say that a tag system Π *halts* on input w if the computation of Π on input w reaches a word to which Π is not applicable; in other words, if $w \overset{\Pi}{\Longmapsto} x$ for some word x with $|x| < \ell$.

THEOREM 4.2 (Minsky, [21]). *There is a tag system Π_0 for which it is undecidable, given a word $w \in \Sigma^*$, whether Π_0 halts on input w.*

Moreover, Wang [31] showed that this holds even for some tag system Π_0 with $\ell = 2$ and $1 \leqslant |u_i| \leqslant 3$, for all $1 \leqslant i \leqslant m$. For this reason, to simplify our proof, we will assume that all the words u_i are nonempty. (Otherwise, we only need to split axiom (Π1) in Table 2 into two axioms in an obvious way and consider one more case in Lemma 5.9 accordingly.) In fact, we only need a weaker result, viz., the undecidability of the following problem:

Halting problem: *given a tag system Π (with nonempty words u_i) and a word $w \in \Sigma^*$, determine whether Π halts on input w.*

5. The proof of undecidability

We will effectively reduce the above mentioned Halting problem to problems (1) and (2) from Theorem 2.3. So, fix any signature $\mathfrak{S} \supseteq \{\to, \wedge\}$ and any calculus $S_0 \geqslant \mathbf{I} := \mathbf{Int}(\to, \wedge)$. Given a tag system $\Pi = (\Sigma, P, \ell)$ and a word $w \in \Sigma^*$, we will build a calculus $T = T(\Pi, w; S_0)$ in the signature $\mathfrak{S}$ such that the following equivalences hold:

$$\Pi \textit{ halts on input } w \quad\Longleftrightarrow\quad T \sim S_0 \quad\Longleftrightarrow\quad T \geqslant S_0.$$

First, we encode letters and words as $\{\to, \wedge\}$-formulas. Let us introduce an auxiliary letter $o = a_0 \notin \Sigma$. Let p be a variable not occurring in S_0. Then the *code* of the letter $a_i \in \Sigma$, for $0 \leqslant i \leqslant m$, is defined by induction:

$$\overline{o} = \overline{a_0} \;=\; (p \to p), \qquad \overline{a_{i+1}} \;=\; (p \to \overline{a_i}).$$

The *code* of a nonempty word $u = c_1 \ldots c_n$ is the formula[3] $\overline{u} = \overline{c_1} \wedge \ldots \wedge \overline{c_n}$. Observe that $\overline{a} \wedge \overline{u} = \overline{a\,u}$, for any letter a and any word u. On the contrary,

[3] We will omit parentheses in 'long' conjunctions, assuming implicitly that they are grouped to the right, e.g. $A \wedge B \wedge C \wedge D \wedge E$ is a shortcut for $A \wedge (B \wedge (C \wedge (D \wedge E)))$.

(W)	$\overline{o\,w}$	
(Π1)	$\overrightarrow{\overline{o\,a_i\,x}} \wedge q \;\to\; \overline{o} \wedge q \wedge \overline{u_i}$	for all $x \in \Sigma^*$, $\lvert a_i\,x\rvert = \ell$, $1 \leqslant i \leqslant m$
(Π2)	$\overline{o\,a_i\,x} \;\to\; \overline{o\,u_i}$	for all $x \in \Sigma^*$, $\lvert a_i\,x\rvert = \ell$, $1 \leqslant i \leqslant m$
(Π3)	$\overline{o\,x} \;\to\; A$	for all $x \in \Sigma^*$, $\lvert x\rvert < \ell$, $A \in S_0$
(A1)	$(p \wedge q) \wedge r \;\to\; p \wedge (q \wedge r)$	
(A2)	$(p \to q) \;\to\; (r \wedge p \to r \wedge q)$	

Table 2. The calculus $T = T(\Pi, w; S_0)$.

$\overline{u} \wedge \overline{a} \neq \overline{u\,a}$, unless $|u| = 1$. For this reason, we introduce the following notation. For a nonempty word $u = c_1 \dots c_n$ and a formula A, we write $\overrightarrow{\overline{u}} \wedge A$ as a shortcut for $\overline{c_1} \wedge \dots \wedge \overline{c_n} \wedge A$. Now $\overrightarrow{\overline{u}} \wedge \overline{v} = \overline{u\,v}$, for all words u, v.

Lemma 5.1. $\mathbf{Int}(\to, \wedge) \vdash \overline{u}$, *for every (nonempty) word* $u \in (\Sigma \cup \{o\})^*$.

Proof. First, $\mathbf{I} \vdash p \to p$, so $\mathbf{I} \vdash \overline{a_0}$. Whenever $\mathbf{I} \vdash A$, we have $\mathbf{I} \vdash p \to A$. So, $\mathbf{I} \vdash \overline{a_i}$ implies $\mathbf{I} \vdash \overline{a_{i+1}}$. Finally, $\mathbf{I} \vdash A$ and $\mathbf{I} \vdash B$ imply $\mathbf{I} \vdash A \wedge B$. ■

For a tag system Π and a word w, we build the calculus $T = T(\Pi, w; S_0)$ with axioms listed in Table 2. Let us reveal the intuition behind these axioms. A current word $u \in \Sigma^*$ in a computation of a tag system Π is represented by the formula $\overline{o\,u}$. A transition $u \overset{\Pi}{\longmapsto} v$ is simulated by the axiom (Π2) for $|u| = \ell$, and by (Π1) for $|u| > \ell$.

To see the latter, consider a transition $a_i\,x\,z \overset{\Pi}{\longmapsto} z\,u_i$ with a nonempty word z. Let us substitute the formula $\overline{z}$ for q in (Π1). The premise of the implication will be the formula $\overline{o\,a_i\,x\,z}$ representing the word $a_i\,x\,z$. Its conclusion will be $\overline{o} \wedge \overline{z} \wedge \overline{u_i}$, which is not a code of any word (unless $|z| = 1$). However, Lemma 5.3 guarantees that axioms (A1) and (A2) are sufficient to transform this formula into the formula $\overline{o\,z\,u_i}$ representing the word $z\,u_i$.

Lemma 5.2. $T \leqslant S_0$.

Indeed, using Lemma 5.1, one can check that all axioms of T except (Π3) are derivable in $\mathbf{I}$ and hence in S_0. As for (Π3), note that $p \to (q \to p)$ and hence $A \to (\overline{o\,x} \to A)$ is derivable in $\mathbf{I}$ and so in S_0. Now, if $A \in S_0$, then $S_0 \vdash \overline{o\,x} \to A$ by (MP), so the axiom (Π3) is derivable in S_0.

Lemma 5.2 implies the equivalence: $T \sim S_0$ iff $T \geqslant S_0$. Thus, in order to establish Theorem 2.3, it remains to prove, for every tag system Π and every word $w \in \Sigma^*$, the following equivalence:

$$\Pi \textit{ halts on input } w \quad \Longleftrightarrow \quad T \geqslant S_0.$$

Sections 5.1 and 5.2 below contain the proofs of the two implications.

5.1. Derivability

Here we prove that if Π halts on input w then $T \geqslant S_0$. Consider two subsystems of the calculus T:

$$R = \{ (\mathsf{A1}), (\mathsf{A2}) \}, \qquad T_\Pi = R \cup \{ (\Pi 1), (\Pi 2), (\Pi 3) \}.$$

They are rather weak and not even capable to derive $A \to C$ from $A \to B$ and $B \to C$; for this reason the following notation appears useful. We write $R \vdash A \Rightarrow B$ if there are formulas $A_0, \ldots, A_n$, $n \geqslant 0$, such that $A_0 = A$, $A_n = B$ and $R \vdash A_{i-1} \to A_i$ for all $1 \leqslant i \leqslant n$. By definition, $R \vdash A \Rightarrow A$.

Lemma 5.3 (Concatenation). *$R \vdash \overline{x} \wedge \overline{y} \Rightarrow \overline{x\,y}$, for all nonempty words x, y.*

Proof. Induction on $|x|$. If $|x| = 1$, then simply $\overline{x} \wedge \overline{y} = \overline{x\,y}$.

Now let $|x| \geqslant 2$, so $x = a\,z$, for a letter a and a nonempty word z. Then

$$R \ \vdash \quad (\overline{a} \wedge \overline{z}) \wedge \overline{y} \quad \stackrel{(\mathsf{A1})}{\longrightarrow} \quad \overline{a} \wedge (\overline{z} \wedge \overline{y}) \quad \stackrel{(\mathsf{A2})}{\Longrightarrow} \quad \overline{a} \wedge \overline{z\,y},$$

where the implication '$\Rightarrow$' uses induction hypothesis: $R \vdash \overline{z} \wedge \overline{y} \Rightarrow \overline{z\,y}$ and axiom (A2). In the resulting chain of implications, we have $\overline{x} \wedge \overline{y}$ at the beginning and $\overline{x\,y}$ at the end, as required. ∎

Lemma 5.4 (Simulation). *If $x \stackrel{\Pi}{\longmapsto} y$ then $T_\Pi \vdash \overline{o\,x} \Rightarrow \overline{o\,y}$, for all $x, y \in \Sigma^*$.*

In words: the calculus T_Π can "simulate" transitions of the tag system Π.

Proof. As Π is applicable to x, we have $|x| \geqslant \ell$. So, two cases are possible.

1) $|x| = \ell$. Let the first letter in x be a_i, so $x = a_i\,z$ and $y = u_i$. Then $T_\Pi \vdash \overline{o\,x} \to \overline{o\,y}$ by the axiom (Π2).

2) $|x| > \ell$. Assume that $x = a_i\,z\,v$, where $|a_i\,z| = \ell$ and $|v| \geqslant 1$, so that $y = v\,u_i$. We derive in T_Π:

$$\overline{o\,a_i\,z\,v} \quad \stackrel{(\Pi 1)}{\longrightarrow} \quad \overline{o} \wedge \overline{v} \wedge \overline{u_i} \quad \stackrel{(\mathsf{A2})}{\Longrightarrow} \quad \overline{o} \wedge \overline{v\,u_i},$$

where the implication '$\Rightarrow$' is due to Lemma 5.3 and the axiom (A2). The premise of this chain of implications is $\overline{o\,x}$ and its conclusion is $\overline{o\,y}$. ∎

Corollary 5.5. *If $x \stackrel{\Pi}{\Longmapsto} y$ then $T_\Pi \vdash \overline{o\,x} \Rightarrow \overline{o\,y}$, for all words x, y over Σ.*

We are ready to prove the desired implication. Assume that Π halts on input w. Then $w \stackrel{\Pi}{\Longmapsto} x$, for some word x of length less than ℓ. The formula $\overline{o\,w}$ is an axiom of T. By Corollary 5.5, we have $T_\Pi \vdash \overline{o\,w} \Rightarrow \overline{o\,x}$. Since $|x| < \ell$, axiom (Π3) yields $T_\Pi \vdash \overline{o\,x} \to A$, for all $A \in S_0$. Applying the rule (MP) several times, we obtain $T \vdash A$, for all $A \in S_0$. Thus, $T \geqslant S_0$.

5.2. Non-derivability

Assuming that $T \geqslant S_0$, we will show that the tag system Π halts on input w.[4] Since $S_0 \geqslant \mathbf{I}$ by assumption and $\mathbf{I} \vdash \overline{o\,x}$ by Lemma 5.1, we have that $T \vdash \overline{o\,x}$, for every word $x \in \Sigma^*$. Moreover, $T \vdash \overline{o\,x}^\sigma$, for every substitution σ. Note that any derivation, considered as a tree, can be assigned the *height* $n \geqslant 0$, as specified below.

> **Main idea.** Let us take a word $x \in \Sigma^*$ of length $|x| < \ell$ and a substitution σ such that the formula $\overline{o\,x}^\sigma$ has a derivation from T of the *minimal height* (among all formulas of this kind). Then we claim that $w \overset{\Pi}{\Longmapsto} x$ and therefore the tag system Π halts on input w.

Now we proceed formally. Let T_n be the set of theorems of T with derivations of height at most n, that is: $T_0 = T$, $T_{n+1} = \mathsf{Rules}(T_n)$, where

$$\begin{aligned}\mathsf{Rules}(S) \;=\; & \{\, B \mid A, (A \to B) \in S \text{ for some formula } A \,\} \cup \\ & \{\, A^\sigma \mid A \in S \text{ and } \sigma \text{ is a substitution} \,\}.\end{aligned}$$

It is easily seen that $T_n \subseteq T_{n+1}$ (due to the identical substitution) and the set of all theorems of the calculus T can be represented as $[T] = \bigcup_{n \geqslant 0} T_n$. Now take the minimal $N \geqslant 0$ such that T_N contains at least one formula of the form $\overline{o\,x}^\sigma$ with $|x| < \ell$:

$$N := \min\{\, n \geqslant 0 \mid \overline{o\,x}^\sigma \in T_n \text{ for some word } x \in \Sigma^* \text{ with } |x| < \ell \text{ and some substitution } \sigma \,\}.$$

Key Lemma. *If* $\overline{o\,x}^\sigma \in T_N$ *then* $w \overset{\Pi}{\Longmapsto} x$*, for all* $x \in \Sigma^*$ *and substitutions* σ*.*

This lemma is sufficient to complete our proof. Indeed, by the choice of N, we have $\overline{o\,x}^\sigma \in T_N$, for some word $x \in \Sigma^*$ of length $|x| < \ell$ and some substitution σ. Then, by the Key Lemma, $w \overset{\Pi}{\Longmapsto} x$. So, the computation of Π on input w reaches the word x of length $|x| < \ell$. Thus, Π halts on w.

Our plan is to prove the Key Lemma by induction on the length of a derivation. In the course of this induction, besides formulas of the form $\overline{o\,x}^\sigma$, we need to consider a wider family of formulas Conj defined below.[5]

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[4] To this end, we will show that T derives *only* formulas of a certain kind, hence the title of this subsection.

[5] This is a crucial correction to the proof from [2], where intermediate claims involved only formulas of the form $\overline{o\,x}^\sigma$ and were incorrect. A correction to [2] will appear soon.

Definition 5.6. An *alphabetic formula over the alphabet* Σ (or a Σ*-formula*, for short) is an arbitrary conjunction of the codes of letters from Σ. Formally, $\overline{a}$ is a Σ-formula for each letter $a \in \Sigma$, and if A, B are Σ-formulas then so is $A \wedge B$. The notion naturally extends to the alphabet $\Sigma \cup \{o\}$.

An example of an alphabetic formula is the code $\overline{x}$ of any nonempty word x. Another example is $((\overline{a} \wedge \overline{c}) \wedge \overline{a}) \wedge (\overline{e} \wedge (\overline{c} \wedge \overline{c}))$, where $a, c, e \in \Sigma$.

To every Σ-formula A we associate its $\mathsf{word}(A) \in \Sigma^*$ by induction: $\mathsf{word}(\overline{a}) = a$ for each letter $a \in \Sigma$, and $\mathsf{word}(A \wedge B) = \mathsf{word}(A)\,\mathsf{word}(B)$. For instance, the above example of a Σ-formula yields the word *acaecc*. Observe that $\mathsf{word}(\overline{x}) = x$, for every nonempty word x.

The following interesting generalization of Lemma 5.3 can be easily proved; however, we will not use it later, so we leave the proof to the reader.

Fact. $R \vdash A \Rightarrow \overline{x}$*, for every* Σ*-formula* A *with* $\mathsf{word}(A) = x$*.*

Let us call two formulas A and B *unifiable* if $A^\sigma = B^\pi$, for some (possibly distinct) substitutions σ and π.

Lemma 5.7. *No two distinct alphabetic formulas are unifiable.*

Proof. Easily follows from two properties of our method of coding letters and words: (a) the codes of no two distinct letters are unifiable; (b) the code of a letter is not unifiable with the code of any word of length at least 2. ■

We came to the heart of our proof. Recall that T_Π is the calculus obtained from the calculus T (see Table 2) by removing the axiom (W). Let us consider the set Imp of all formulas of the form

$$A_1 \wedge \ldots \wedge A_s \wedge B \ \rightarrow\ A_1 \wedge \ldots \wedge A_s \wedge C, \qquad (*)$$

where $s \geqslant 0$, A_i are arbitrary formulas, and $B \rightarrow C$ is a substitution instance of some axiom of T_Π. The following lemma shows that *only* formulas of the form $(*)$ are derivable in T_Π.

Lemma 5.8. $[T_\Pi] = \mathsf{Imp}$. *(In fact, we will only need the '$\subseteq$' inclusion later.)*

Proof. ($\supseteq$) First, by the rule (Sub), derive the substitution instance $B \rightarrow C$ of an axiom of T_Π. Then, using the axiom (A2), append conjuncts A_i to both the premise and the conclusion of the resulting implication. This way we can derive every formula of the form $(*)$ from T_Π.

($\subseteq$) Axioms of T_Π are in Imp. The set Imp is closed under (Sub). It remains to check that Imp is closed under (MP). Assume that D and $D \rightarrow E$ are in Imp.

Then $D \to E$ has the form $(*)$, for some $s \geqslant 0$. Moreover, $s = 0$, since the formula D is itself an implication, say, $(F \to G)$. Thus, $(F \to G) \to E$ is a substitution instance of some axiom of T_Π. But (A2) is the only axiom of T_Π whose premise is an implication. Hence, E has the form $(A \wedge F) \to (A \wedge G)$, for some formula A, and hence belongs to Imp, since $F \to G$ is in Imp. ■

Now let us consider another set of formulas:

$$\mathsf{Conj} \;=\; \{\, (\bar{o} \wedge A)^\sigma \mid A \text{ is a } \Sigma\text{-formula, } \sigma \text{ a substitution, } w \overset{\Pi}{\Longmapsto} \mathsf{word}(A) \,\}.$$

The next lemma means that *only* formulas that belong to Imp or Conj are derivable in the calculus T with derivations of height at most N.

LEMMA 5.9. $T_N \subseteq \mathsf{Imp} \cup \mathsf{Conj}$.

PROOF. By induction on $n \leqslant N$, we will show that $T_n \subseteq \mathsf{Imp} \cup \mathsf{Conj}$.

Induction base: $n = 0$, so $T_0 = T$. The axiom (W) is in Conj, all other axioms of T are in Imp.

Induction step: assume that $n < N$. Since both Imp and Conj are closed under (Sub), we only need to consider the case of a formula, say E, been obtained by (MP) from some formulas $D, (D \to E) \in T_n$. By induction hypothesis, $T_n \subseteq \mathsf{Imp} \cup \mathsf{Conj}$. Clearly, $(D \to E) \in \mathsf{Imp}$. If $D \in \mathsf{Imp}$, then also $E \in \mathsf{Imp}$, since Imp is closed under (MP) by (the proof of) Lemma 5.8. So, it remains to consider the case of $D \in \mathsf{Conj}$. We will show that $E \in \mathsf{Conj}$.

Since $D \in \mathsf{Conj}$, we have $D = (\bar{o} \wedge A)^\sigma$ for some substitution σ and some Σ-formula A with $u := \mathsf{word}(A)$ and $w \overset{\Pi}{\Longmapsto} u$. Since the formula $D \to E$ is in Imp, it has the form $(*)$ for some $s \geqslant 0$. Below, we will show that $E = (\bar{o} \wedge B)^\sigma$ for some Σ-formula B. Moreover, if we denote $v := \mathsf{word}(B)$, then the following will be shown: if $s = 0$ then $u \overset{\Pi}{\longmapsto} v$, and if $s > 0$ then $u = v$. In both cases it follows that $w \overset{\Pi}{\Longmapsto} v$ and thus $E \in \mathsf{Conj}$, as required. So, let us consider the following two cases.

Case $s = 0$**.** Then $D \to E$ is a substitution instance of some axiom of T_Π. Which axiom is it?

- (A2): impossible, for the main connective in the premise of (A2) is $\to$, whereas that in D is $\wedge$.
- (A1): impossible, for otherwise D would have the form $(\alpha \wedge \beta) \wedge \gamma$, but we know that the formula $D = (\bar{o} \wedge A)^\sigma$ has the form $(\delta \to \theta) \wedge \xi$.
- (Π3): impossible, for otherwise we would have $D = \overline{o\,x}^\sigma$, for some word $x \in \Sigma^*$ with $|x| < \ell$, which contradicts to that $D \in T_n$ and $n < N$.

(Π2): now $(D \to E) = (\overline{o\, a_i\, x} \to \overline{o\, u_i})^\pi$, for some substitution π. Then $D = (\overline{o} \wedge A)^\sigma = \overline{o\, a_i\, x}^\pi$. By Lemma 5.7, $\sigma = \pi$ and $A = \overline{a_i\, x}$, so $u = \mathsf{word}(A) = a_i\, x$. Furthermore, $E = \overline{o\, u_i}^\pi = (\overline{o} \wedge B)^\sigma$, where $B := \overline{u_i}$ and thus $v = \mathsf{word}(B) = u_i$. Finally, note that $u = a_i\, x \stackrel{\Pi}{\longmapsto} u_i = v$.

(Π1): now $(D \to E) = (\overrightarrow{\overline{o\, a_i\, x}} \wedge q \to \overline{o} \wedge q \wedge \overline{u_i})^\pi$, for some substitution π. Then

$$\begin{array}{llll} (1) & D = (\overrightarrow{\overline{o\, a_i\, x}} \wedge q)^\pi & = & (\overline{o} \wedge A)^\sigma, \\ (2) & E = (\overline{o} \wedge q \wedge \overline{u_i})^\pi. & & \end{array}$$

Comparing the first conjuncts in (1), we see that $\overline{o}^\sigma = \overline{o}^\pi$, hence $p^\sigma = p^\pi$, by Lemma 5.7. Comparing the subsequent $|a_i\, x|$ conjuncts in (1), we obtain that $A = \overrightarrow{\overline{a_i\, x}} \wedge C$, for some Σ-formula C with $C^\sigma = q^\pi$. Now we substitute this in (2) and obtain:

$$E = (\overline{o} \wedge q \wedge \overline{u_i})^\pi = \overline{o}^\pi \wedge q^\pi \wedge \overline{u_i}^\pi = \overline{o}^\sigma \wedge C^\sigma \wedge \overline{u_i}^\sigma = (\,\overline{o} \wedge (C \wedge \overline{u_i})\,)^\sigma.$$

So, as we promised, $E = (\overline{o} \wedge B)^\sigma$ for a Σ-formula $B := (C \wedge \overline{u_i})$. Denote $z := \mathsf{word}(C)$. Then we have:

$$\begin{array}{lllllll} u & = & \mathsf{word}(A) & = & \mathsf{word}(\overrightarrow{\overline{a_i\, x}} \wedge C) & = & a_i\, x\, z, \\ v & = & \mathsf{word}(B) & = & \mathsf{word}(C \wedge \overline{u_i}) & = & z\, u_i. \end{array}$$

It remains to note that $u = a_i\, x\, z \stackrel{\Pi}{\longmapsto} z\, u_i = v$.

Case $s > 0$**.** Then, for some formulas $C_1, \ldots, C_s$ and a substitution instance $F \to G$ of some axiom of the calculus T_Π, we have:

$$\begin{array}{llllll} (1) & D & = & C_1 \wedge \ldots \wedge C_s \wedge F & = & (\overline{o} \wedge A)^\sigma, \\ (2) & E & = & C_1 \wedge \ldots \wedge C_s \wedge G. & & \end{array}$$

Comparing the first s conjuncts in (1), we conclude that $C_1 = \overline{o}^\sigma$ and $A = A_2 \wedge \ldots \wedge A_s \wedge A_0$, where each A_i is a Σ-formula, $A_0^\sigma = F$ and $A_i^\sigma = C_i$ for all $2 \leqslant i \leqslant s$.

Recall that $F \to G$ is a substitution instance of some axiom of T_Π. Which axiom is it? The axiom (A2) is impossible, since its premise has '→' as the principal connective, while that in F is ∧. The axioms (Π1), (Π2), and (Π3) are impossible either, since their premises have the form $\overline{o} \wedge H$, which, by Lemma 5.7, are not unifiable with the Σ-formula A_0, because $o \notin \Sigma$. So, the only possibility is that $F \to G$ is a substitution instance of the axiom (A1). Therefore, for some formulas α, β, γ,

$$\begin{array}{lllll} F & = & (\alpha \wedge \beta) \wedge \gamma & = & A_0^\sigma, \\ G & = & \alpha \wedge (\beta \wedge \gamma). & & \end{array}$$

Then $A_0 = (P \wedge Q) \wedge R$, for some Σ-formulas P, Q, R, and so we have $G = (P \wedge (Q \wedge R))^\sigma$. Thus, the formulas D and E have the form:

$$\begin{array}{rcl} D &=& (\,\overline{o} \wedge A_2 \wedge \ldots \wedge A_s \wedge (P \wedge Q) \wedge R\,)^\sigma \;\;=\;\; (\overline{o} \wedge A)^\sigma, \\ E &=& (\,\overline{o} \wedge A_2 \wedge \ldots \wedge A_s \wedge P \wedge (Q \wedge R)\,)^\sigma \;\;=:\;\; (\overline{o} \wedge B)^\sigma. \end{array}$$

So, the Σ-formulas A and B differ only in how their conjuncts are grouped by parentheses, hence $\mathsf{word}(A) = \mathsf{word}(B)$, i.e., $u = v$. ■

Finally, note that Lemma 5.9 implies the Key Lemma: assume that $\overline{o\,x}^\sigma \in T_N$. By Lemma 5.9, we have $T_N \subseteq \mathsf{Imp} \cup \mathsf{Conj}$. Clearly, the formula $\overline{o\,x}^\sigma$ is not in Imp, so it is in Conj and thus $w \overset{\Pi}{\Longmapsto} \mathsf{word}(\overline{x}) = x$.

This completes the proof of Theorem 2.3.

6. Altering the set of connectives

The main result of our paper, Theorem 2.3, was established for (every extension of) the signature $\{\to, \wedge\}$. Here we address the issue of obtaining similar results for other signatures. We confine ourselves to signatures containing implication (since implication is needed for the rule of modus ponens), leaving the case of signatures without implication outside the scope of our paper.

Our first observation is that the proof remains valid, with minor changes, if we replace $\wedge$ with $\vee$ in all (appropriate) places. Denote the calculus

$$\mathbf{J} \;:=\; \mathbf{Int}(\to, \vee) \;=\; \{\,(\to 1), (\to 2),\; (\vee 1), (\vee 2), (\vee 3)\,\}.$$

Theorem 6.1. *Fix a signature* $\mathfrak{S} \supseteq \{\to, \vee\}$ *and a calculus* $S_0 \geqslant \mathbf{Int}(\to, \vee)$ *in the signature* $\mathfrak{S}$*. Then the following problems are undecidable:*

(1) *given a calculus T in the signature $\mathfrak{S}$, determine whether $T \sim S_0$;*
(2) *given a calculus T in the signature $\mathfrak{S}$, determine whether $T \geqslant S_0$.*

Proof. Let us replace $\wedge$ with $\vee$ in the coding scheme for words $\overline{u}$, in the notation $\overrightarrow{u} \wedge A$, and in all axioms of the calculus T listed in Table 2 (of course, we leave the formulas $A \in S_0$ in the axiom $(\Pi 3)$ unchanged).

Lemma 5.1, now stating that $\mathbf{J} \vdash \overline{u}$, is obvious. In the proof of Lemma 5.2 (which states that $T \leqslant S_0$), the $\vee$-analogues of the axioms $(\Pi 1)$ and $(\Pi 2)$ are derivable in $\mathbf{J}$ (and hence in S_0) simply because their conclusions are. It is easy to derive in $\mathbf{J}$ (using the Deduction Theorem) the $\vee$-analogues of the axioms $(\mathsf{A1})$ and $(\mathsf{A2})$. The remainder of the proof, after replacing $\wedge$ with $\vee$, does not require any further amendments. ■

Let us mentioned some open problems. First, it is natural to ask whether similar results hold for the signatures $\{\to\}$ (this is essentially Markov's open problem formulated in Section 3) and $\{\to, \neg\}$.

Interestingly, the implicational logics $\mathbf{Int}(\to)$ and $\mathbf{Cl}(\to)$ can be axiomatized by the following single formulas, as shown by Łukasiewicz [15] (reprinted in [16]) and Meredith [20], respectively:

$$\begin{array}{rcl} \mathbf{Cl}(\to) & \sim & \{\,[(p \to q) \to r] \to [(r \to p) \to (s \to p)]\,\} \\ \mathbf{Int}(\to) & \sim & \{\,[(p \to q) \to r] \to [s \to ((q \to (r \to t)) \to (q \to t))]\,\} \end{array}$$

In this respect, the following question (seemingly open) also makes sense.

Open problem 2. *Is it decidable whether a given implicational formula A axiomatizes all classical implicational tautologies, i.e., $\{A\} \sim \mathbf{Cl}(\to)$?* The same question for other calculi and signatures.

7. Conclusion and further directions

In this paper, we reestablished the undecidability of the problem of determining whether a given finite set of formulas axiomatizes the classical logic, the intuitionistic logic, the inconsistent logic, or any (fixed) superintuitionistic calculus. These results were obtained for the signatures $\{\to, \wedge\}$ and $\{\to, \vee\}$ and any extensions thereof. The question if this holds for the signature $\{\to\}$ is left open (Markov's problem). It would be natural to investigate similar problems for other formalisms.

In particular, questions of these kinds are interesting in the context of propositional *modal logics*. The latter are formulated in the language considered above augmented with a unary modal operator $\Box$, and typically have three rules of inference: modus ponens, substitution, and necessitation ($A \vdash \Box A$). Recently Chagrov announced the following result: *it is undecidable whether a given finite set of modal formulas axiomatizes the minimal normal modal logic* K (for its definition, see [5, Section 3.6]) *or the inconsistent modal logic.* The proof uses Minsky machines [21] and is similar to the proofs of the results mentioned in Theorem 3.7 (see also [5, Chapter 17]).

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