Modal Logic of Variable Modalities with Applications to Querying Knowledge Bases

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1 Introduction

One of the central notions in modal logic is that of *validity* of a modal formula on a frame (or at a point of a frame). To mention few examples, Definability Theory explores classes of frames which *validate* a given formula or a set of formulas; axiomatisability issues deal with finding a set of formulas that are *valid* in a given class of frames; in Correspondence Theory, one is interested in conditions under which a class of frames *validating* a given set of modal formulas is first-order definable, etc. Moreover, issues about (un)satisfiability of formulas (or concepts, in Description Logic) are closely related to this notion, since satisfiability is just a notion dual to validity.

The definition of the validity of a modal formula on a frame F involves the quantification over arbitrary valuations of propositional variables (i.e., unary predicates, from the viewpoint of first-order logic) on F, whereas the interpretation of accessibility relations (i.e., binary predicates) is fixed, in a given frame F. As a consequence of this asymmetry, the typical issue in, e.g., the Correspondence Theory is whether a modal formula expresses (locally—at a given point of a frame—or globally) a property of frames definable by some first-order *relational* formula, i.e., a formula involving binary predicates only, and when this first-order formula can be built efficiently. To sum up, the traditional modal logic is a logic of *constant* modalities and propositional *variables*.

However, from theoretical standpoint, as well as for some applications (e.g., querying knowledge bases), it is natural to consider a notion of validity, in which the role of unary and binary predicates is symmetric. In this paper, we restore this balance in quantification. (If we were concerned with a polyadic modal logic, then our new notion of validity would even symmetrically involve predicates of arbitrary arities; however, here we confine ourselves to unary and binary predicates only.) We enrich the standard modal language with *variable* modalities (i.e., modalities whose accessibility relation is quantified over, in the definition of validity) and propositional *constants* (which are not quantified, but rather fixed in a particular frame).

We give some preliminary results on definability and first-order correspondence for this logic (called the *mixed* modal logic henceforth). In particular, we show that reasoning in this logic is not harder than in the standard modal logic. At the same time, the new language turns out to be more expressive than the standard one. We also show that the new language is useful by illustrating its applications to querying Description Logic knowledge bases.

2 Syntax and semantics of the mixed modal logic

The vocabulary of the mixed modal logic consists of the following primitive symbols:

- a countable set of propositional variables $p_0, p_1, \ldots$;
- a finite set of propositional constants $A_0, \ldots, A_m$;
- boolean connectives $\bot$ and $\to$;
- a finite set of constant modalities $\Box_0, \ldots, \Box_n$;
- a countable set of variable modalities $\boxdot_0, \boxdot_1, \ldots$.

The set of well-formed formulas is defined by the following syntax:

$$\varphi \ ::= \ \bot \ \mid \ p_i \ \mid \ A_i \ \mid \ \varphi \to \psi \ \mid \ \Box_i\varphi \ \mid \ \boxdot_i\varphi$$

Other boolean connectives ($\top$, $\neg$, $\wedge$, $\vee$, etc.) as well as dual modalities $\Diamond_i$ and $\diamonddot_i$ are taken as standard abbreviations, e.g., $\diamonddot_i\varphi := \neg\boxdot_i\neg\varphi$.

Definition 2.1 (Semantics). A *Kripke frame* $F = \langle \Delta, \vec{A}^F, \vec{R}^F \rangle$ consists of a non-empty set Δ, a list $\vec{A}^F := \langle A_0^F, \ldots, A_m^F \rangle$ of unary predicates on Δ (i.e., $A_i^F \subseteq \Delta$), which interpret the propositional constants $A_0, \ldots, A_m$, and a list $\vec{R}^F := \langle R_0^F, \ldots, R_n^F \rangle$ of binary relations on Δ (i.e., $R_i^F \subseteq \Delta \times \Delta$), which interpret the constant modalities $\Box_0, \ldots, \Box_n$.

A *Kripke model*[1] $M = \langle F, \vec{p}^{\,M}, \vec{S}^M \rangle$ consists of a frame F and *valuations* of propositional variables $\vec{p}^{\,M} := \langle p_0^M, p_1^M, \ldots \rangle$, where $p_i^M \subseteq \Delta$, and of variable modalities $\vec{S}^M := \langle S_0^M, S_1^M, \ldots \rangle$, where $S_i^M \subseteq \Delta \times \Delta$. The notion 'a modal formula φ is *true* at a point $e \in \Delta$ of a model M' (denoted as $M, e \models \varphi$ and M usually omitted when clear from a context) is defined in the standard way, with the relations R_i and S_j corresponding to the modalities $\Box_i$ and $\boxdot_j$, resp.:

$$\begin{array}{lll}
e \models p_i & \text{iff} & e \in p_i^M \\
e \models A_j & \text{iff} & e \in A_j^M \\
e \models \Box_i\varphi & \text{iff} & \text{for all } d \in \Delta, \text{ if } eR_i^F d \text{ then } d \models \varphi \\
e \models \boxdot_j\varphi & \text{iff} & \text{for all } d \in \Delta, \text{ if } eS_j^F d \text{ then } d \models \varphi
\end{array}$$

What is different in this framework is the notion of validity. A formula φ is *valid at a point* e of a frame F (denoted as $F, e \Vdash \varphi$) if, for any model M based on F, we have $M, e \models \varphi$. A formula φ is *valid* on a frame F if it is valid at all points of this frame. Finally, a formula is called *valid* if it is valid on all frames.

Although the sentence for the definition of validity is literally the same as the standard one, it involves an extra quantification over arbitrary valuations S_j of variable modalities $\boxdot_j$.

Example 2.2. Consider the mixed modal formula $\Box p \to \boxdot p$; denote it by φ. Let us find the condition under which a frame $F = \langle \Delta, R \rangle$ validates this formula (note that a frame for φ contains only one relation). If $F \Vdash \varphi$ holds, then for arbitrary relation S on Δ, the frame $F' := \langle \Delta, R, S \rangle$ validates the formula φ in ordinary sense. Now recall from the standard modal logic that a formula $\Box_1 p \to \Box_2 p$ is valid on a frame $\langle \Delta, R_1, R_2 \rangle$ iff $R_1 \supseteq R_2$. Hence, our formula φ is valid on a frame $F = \langle \Delta, R \rangle$ iff R contains any binary relation on Δ. The latter is obviously equivalent to the condition $R = \Delta \times \Delta$.

Therefore, the formula $\Box p \to \boxdot p$ is valid on a frame iff the first-order condition $\forall x \forall y\, xRy$ holds. It is not hard to check that the same formula $\Box p \to \boxdot p$ is valid *at a point* e of a frame iff e satisfies the condition $\forall y\, eRy$, i.e., e sees all other elements of the frame.

[1] In modal logic, the term *model* refers to what is usually called *interpretation* in other branches of logic.

Example 2.3. Let us find the condition that corresponds to the validity of the formula $p \to \boxdot p$ on a frame $F = \langle \Delta \rangle$ (note that the frame contains no relations at all). If we introduce an equality relation $Id := \{\langle e, e \rangle \mid e \in \Delta\}$ and take a constant modality $\Box$ that is interpreted on F by this relation Id, then we can equivalently rewrite our formula as $\Box p \to \boxdot p$ (because for any formula ψ, $\Box\psi$ is equivalent to ψ). By the above example, this formula is valid on the frame $F' = \langle \Delta, Id \rangle$ iff Id is the total relation on Δ, i.e., $Id = \Delta \times \Delta$. It is easily seen that the equality relation coincides with the total relation on Δ iff $|\Delta| = 1$. In other words, the original formula $p \to \boxdot p$ is valid on a frame $F = \langle \Delta \rangle$ iff the first-order condition $\forall x \forall y\, (x = y)$ holds.

In what follows, we are interested in the following issues:

Decidability How can one verify whether a given mixed modal formula is valid on all frames? And what is the complexity of this problem?

Definability What classes of frames are definable (locally and globally) in the mixed modal language? When these classes are first-order definable? Can one express, in the mixed modal language, any first-order property that is not expressible in the standard modal language?

3 Decidability

Let us define the *minimal* mixed modal logic $\mathbf{K}'$ as the set of all valid mixed modal formulas. Please note that the decidability of this logic is closely related to the decidability of the satisfiability problem for the mixed modal language. Namely, a formula φ is satisfiable (i.e., is true at some point of some model) iff its negation $\neg\varphi$ is not valid.

Theorem 3.1 ($\mathbf{K}' = \mathbf{K}$). *The logic $\mathbf{K}'$ coincides with the standard (multimodal) logic $\mathbf{K}$, where variable modalities and propositional constants are understood as ordinary modalities and propositional variables, resp.*

Proof. A formula φ is valid on all frames of the form $F = \langle \Delta, \vec{A}, \vec{R} \rangle$ iff the same formula φ understood as an ordinary modal formula (i.e., A_i are understood as propositional variables and $\boxdot_i$ as ordinary modalities) is valid on all frames of the form $F' := \langle \Delta, \vec{R}, \vec{S} \rangle$, where the modalities $\boxdot_i$ are interpreted by the relations S_i. ⊣

Since the problem of validity of an ordinary modal formula on all frames (i.e., membership in $\mathbf{K}$) is PSpace-complete, we conclude that the logic $\mathbf{K}'$ is also PSpace-complete. Despite this fact, in what follows we will see that the mixed modal language is more expressive than the standard modal language.

4 Definability and first-order correspondence

When checking the validity of a mixed modal formula on a frame (or at a point of a frame), the variable modalities occurring in the formula can be evaluated by arbitrary binary relations on the domain Δ. In particular, we can instantiate $\boxdot$ with the *minimal* modality $\boxed{\scriptstyle\top}$, which is interpreted as the total relation $\Delta \times \Delta$, or with the *maximal* modality $\boxed{\scriptstyle\bot}$, which is interpreted as the empty relation $\varnothing$. The former ($\boxed{\scriptstyle\top}$) is the so called *universal* modality, whereas the latter ($\boxed{\scriptstyle\bot}$) is just equivalent to $\top$, in the sense that the equivalence $\boxed{\scriptstyle\bot}\varphi \leftrightarrow \top$ is always valid. The words 'minimal' and 'maximal' are explained by the validity of the implications $\boxed{\scriptstyle\top} p \to \boxdot p$ and $\boxdot p \to \boxed{\scriptstyle\bot} p$. As a consequence, we obtain the following lemma, which says that if a variable modality $\boxdot$ occurs only positively or negatively in a formula, then the instantiation of $\boxdot$ with $\boxed{\scriptstyle\top}$ or $\boxed{\scriptstyle\bot}$ is sufficient for checking the validity of the formula. In fact, we have already used this fact implicitly in Examples 2.2 and 2.3.

Lemma 4.1 (Monotonicity). *Let φ be a mixed modal formula. Suppose that a variable modality $\boxdot$ occurs only positively (resp., negatively) in φ. Then φ is valid on a frame (or at a point of a frame) iff the formula obtained from φ by substitution of all occurrences of $\boxdot$ with $\boxed{\scriptstyle\top}$ (resp., $\boxed{\scriptstyle\perp}$) is valid there.*

Now we define the notion of first-order correspondence for the new modal language; as usually, it comes in a local and a global versions. Consider the first-order language with equality, unary predicate symbols[2] $A_0, \dots, A_m$ and binary predicate symbols $R_0, \dots, R_n$. Formulas are built up from atomic ones $x = y$, $x{:}\, A_i$, and xR_jy (we use the notation from Description Logic ABoxes) using boolean operations and quantifiers $\forall x$, $\exists x$. First-order formulas considered below are always assumed to be in this language.

Definition 4.2. A first-order formula $\alpha(x)$ with a single free variable *locally corresponds* to a modal formula φ (written as $\alpha(x) \leftrightsquigarrow \varphi$) if, for any frame F and point $e \in \Delta$, we have: $F \models \alpha(e) \Leftrightarrow F, e \Vdash \varphi$.

A closed first-order formula α *globally corresponds* to a modal formula φ (written as $\alpha \leftrightsquigarrow \varphi$) if, for any frame F, we have: $F \models \alpha \Leftrightarrow F \Vdash \varphi$.

It is not hard to see that local correspondence $\varphi \leftrightsquigarrow \alpha(x)$ implies global correspondence $\varphi \leftrightsquigarrow \forall x\, \alpha(x)$ (but the converse does not hold in general, even for the standard modal language, cf. [2]). From Example 2.2 we conclude that the modal formula $\Box p \to \boxdot p$ locally corresponds to the first-order formula $\forall y\, xRy$ and hence globally corresponds to $\forall x \forall y\, xRy$; and Example 2.3 shows that the formula $p \to \boxdot p$ globally corresponds to $\forall x \forall y\, (x = y)$.

Lemma 4.3. *Mixed modal formulas within the following families locally correspond to first-order formulas. Moreover, given a modal formula, the corresponding first-order formula is efficiently computable.*

1. *Closed formulas, i.e., formulas without propositional variables p_i and variable modalities $\boxdot_i$;*
2. *Uniform formulas, i.e., in which any propositional variable p_i and any variable modality $\boxdot_i$ occurs either only positively, or only negatively.*

Proof. We will give a sketch of the proof; see [2, Sect. 3.5] for details concerning similar results.

1) Validity of any modal formula is equivalent to the truth of the second-order closure of its standard translation into first-order language. However, there is nothing to take the second-order closure over, since the original formula contained no variables or variable modalities.

2) Substitute $\perp$ (resp., $\top$) for all positive (resp., negative) occurrences of propositional variables, and substitute $\boxed{\scriptstyle\perp}$ (resp., $\boxed{\scriptstyle\top}$) for all positive (resp., negative) occurrences of variable modalities. Then we will obtain a closed mixed modal formula (where universal modalities also occur). Its validity is equivalent to the truth of its first-order translation. ⊣

Next we present a number of results about first-order correspondence, in order to illustrate the expressive power of the mixed modal language. Some results are for local, and some for global correspondence; for brievity, we omit F in $F, e \Vdash \varphi$. All the results are collected in Table 1. In addition to that, each of the formulas $\Diamond\boxdot p \to \boxdot\Diamonddot p$, $\Diamonddot\Box p \to \boxdot\Diamonddot p$, $\Diamonddot\boxdot p \to \Box\Diamonddot p$, $\Diamonddot\boxdot p \to \boxdot\Diamond p$, and $\Diamonddot\boxdot p \to \boxdot\Diamonddot p$ is valid on a frame F iff $|W| = 1$. All these results are rather easy to obtain. We will prove only one of them here; this one will be used later for querying knowledge bases.

Lemma 4.4. *$F, e \Vdash \Box p \to \boxdot(A \to p) \Leftrightarrow A \subseteq R(e)$, or explicitly: $\forall y\, (A(y) \to eRy)$.*

Proof. Since $\boxdot$ occurs positively, it can be replaced by the universal modality $\boxed{\scriptstyle\top}$. Given a frame $F = \langle \Delta, R, A \rangle$ and its point e, we prove the following equivalence:

$$F, e \Vdash \Box p \to \boxed{\scriptstyle\top}(A \to p) \qquad \Leftrightarrow \qquad A \subseteq R(e).$$

[2] We denote the predicate symbols by the same letters as propositional constants and binary relations in the modal language above; the meaning of each symbol will always be clear from the context.

$$
\begin{array}{lcl}
e \Vdash p \to \Diamond(A \wedge p) & \Leftrightarrow & A(e)\ \&\ eRe \\
e \Vdash A \wedge p \to \Diamond(A \wedge p) & \Leftrightarrow & A(e) \to eRe \\
e \Vdash A \wedge p \to \Diamond(B \wedge p) & \Leftrightarrow & A(e) \to (B(e)\ \&\ eRe) \\
e \Vdash \Box p \to \boxdot(A \to p) & \Leftrightarrow & A \subseteq R(e) \\
e \Vdash \Diamond p \to \boxdot p & \Leftrightarrow & |W| = 1 \ \vee\ R(e) = \varnothing \\
e \Vdash \Diamond(A \wedge p) \to \boxdot p & \Leftrightarrow & |W| = 1 \ \vee\ A \cap R(e) = \varnothing \\
e \Vdash \Diamond(A \wedge p) \to \boxdot(B \to p) & \Leftrightarrow & \forall y, z\, (eRy\ \&\ A(y)\ \&\ B(z) \to y = z) \\
 & & \\
F \Vdash \Diamonddot p \to \boxdot\Diamonddot p & \Leftrightarrow & |W| = 1 \\
F \Vdash \Diamonddot p \to \Box\Diamonddot p & \Leftrightarrow & R(e) \subseteq \{e\} \\
F \Vdash A \to \boxdot B & \Leftrightarrow & A = \varnothing \ \vee\ B = \Delta \\
F \Vdash \Box p \to \boxdot(A \to p) & \Leftrightarrow & \Delta \times A \subseteq R \\
F \Vdash \Box(A \to p) \to \boxdot p & \Leftrightarrow & R = \Delta \times \Delta \ \&\ A = \Delta \\
F \Vdash \Box(A \to p) \to \boxdot(A \to p) & \Leftrightarrow & \Delta \times A \subseteq R \\
F \Vdash \Box(A \to p) \to \boxdot(B \to p) & \Leftrightarrow & \Delta \times B \subseteq R \cap (\Delta \times A) \\
F \Vdash \Diamond p \to \boxdot p & \Leftrightarrow & |W| = 1 \ \vee\ R = \varnothing \\
F \Vdash \Diamonddot\Box p \to \boxdot\Diamond p & \Leftrightarrow & \forall x, y\, \exists z\, (xRz\ \&\ yRz)
\end{array}
$$

Figure 1: Mixed modal formulas and their first-order correspondents.

($\Rightarrow$) Assume on the contrary that $A \not\subseteq R(e)$. Then, for some $d \in \Delta$, we have $d \in A$ and $d \notin R(e)$. Take the following valuation of the propositional variable p: $p^M := R(e)$. Then $e \models \Box p$, since p is true at any R-successor of e, by construction. But $e \not\models \boxdot(A \to p)$, since for the element d (which is accessible from e by the universal relation) we have $d \models A$ and $d \not\models p$, because $d \notin R(e)$.

($\Leftarrow$) Suppose that $A \subseteq R(e)$. Take arbitrary valuation $p^M \subseteq \Delta$ of the propositional variable p. Assume that $e \models \Box p$. To prove that $e \models \boxdot(A \to p)$, take any $d \in \Delta$. To show that $d \models A \to p$, assume that $d \models A$. Then by inclusion $A \subseteq R(e)$ we have $d \in R(e)$, i.e., eRd, and applying $e \models \Box p$ yields $d \models p$. ⊣

4.1 Frame morphisms

Here we prove a preservation result for the mixed modal language. Namely, we introduce the appropriate notion of frame mappings (usually called *zig-zag morphisms*) and show that it preserves the validity of mixed modal formulas. Suppose we are given two frames $F = \langle \Delta^F, \vec{R}^F, \vec{A}^F \rangle$ and $G = \langle \Delta^G, \vec{R}^G, \vec{A}^G \rangle$. To keep notation simple, we will write $e \in F$ instead of $e \in \Delta^F$.

Definition 4.5. A function $z\colon \Delta^F \to \Delta^G$ is called a *zig-zag morphism* from F to G if the following conditions hold, for all R in $\vec{R}$:

(*zig*) if $eR^F d$ then $z(e)R^G z(d)$, for all $e, d \in F$;

(*zag*) if $z(e)R^G c$, then there exists $b \in F$ such that $eR^F b$ and $z(b) = c$;

(*atom*) e and $z(e)$ satisfy the same propositional constants: $F, e \models A_i \Leftrightarrow G, z(e) \models A_i$.

Lemma 4.6. *If $z\colon F \to G$ is a zig-zag morphism, then for any $e \in F$ and any mixed modal formula φ containing* no variable modalities $\boxdot_i$*, we have:*

$$F, e \Vdash \varphi \quad \Longrightarrow \quad G, z(e) \Vdash \varphi.$$

If, in addition, z is surjective, then the same holds for any *mixed modal formula.*

PROOF. Suppose on the contrary that $G, z(e) \not\Vdash \varphi$, i.e., there is a model $N = \langle G, \vec{p}^N, \vec{S}^N \rangle$ based on G such that $N, z(e) \not\models \varphi$. Then we define a model $M = \langle F, \vec{p}^M, \vec{S}^M \rangle$ based on F as follows: for any p and S,

- $p^M := \{d \in F \mid z(d) \in p^N\}$, or briefly: $p^M := z^{-1}(p^N)$;
- $S^M := \{\langle e, d \rangle \in \Delta^F \times \Delta^F \mid \langle z(e), z(d) \rangle \in S^N\}$, or briefly: $S^M := z^{-1}(S^N)$.

(Of course, if the formula φ contained no variable modalities $\boxdot_i$, then the model N has no $\vec{S}^N$ component, and we omit the definition of S^M.) Now we show, by induction on a modal formula ψ, that for all $d \in F$, the following equivalence holds:

$$M, d \models \psi \quad \Longleftrightarrow \quad N, z(d) \models \psi.$$

This is sufficient for proving our lemma, since it immediately implies that $M, e \not\models \varphi$ and hence $F, e \not\Vdash \varphi$.

Induction base (i.e., when ψ is p_i or A_i, resp.) holds by our definition of the model M and the condition (*atom*) in the definition of zig-zag morphism, resp. Induction steps for booleans are straightforward. Now consider the modal steps.

First, suppose that ψ is $\Diamond\theta$ and we need to prove: $M, d \models \Diamond\theta$ iff $N, z(d) \models \Diamond\theta$. If $M, d \models \Diamond\theta$, then there exists $b \in F$ such that $dR^F b$ and $M, b \models \theta$. Then $z(d)R^G z(b)$, by the condition (*zig*), and $N, z(b) \models \theta$, by induction hypothesis. Hence $N, z(d) \models \Diamond\theta$.

Now assume that $N, z(d) \models \Diamond\theta$, i.e., there exists $c \in G$ such that $z(d)R^G c$ and $N, c \models \theta$. By the condition (*zag*), $c = z(b)$ for some $b \in F$ such that $dR^F b$. Then we rewrite the assertion $N, c \models \theta$ as $N, z(b) \models \theta$ and infer, by induction hypothesis, that $M, b \models \theta$. From this and $dR^F b$ we conclude that $M, d \models \Diamond\theta$.

Finally, suppose that z is surjective, ψ is $\diamonddot\theta$, and let us prove that: $M, d \models \diamonddot\theta$ iff $N, z(d) \models \diamonddot\theta$. If $M, d \models \diamonddot\theta$, then for some $b \in F$ we have $dS^M b$ and $M, b \models \theta$. Then $z(d)S^N z(b)$, by the definition of S^M, and $N, z(b) \models \theta$, by induction hypothesis. Hence $N, z(d) \models \diamonddot\theta$.

Now assume that $N, z(d) \models \diamonddot\theta$, i.e., there exists $c \in G$ such that $z(d)S^N c$ and $N, c \models \theta$. Since z is surjective, $c = z(b)$ for some $b \in F$. By the definition of S^M, from $z(d)S^N z(b)$ we infer that $dR^F b$. Then we rewrite the assertion $N, c \models \theta$ as $N, z(b) \models \theta$ and apply the induction hypothesis to infer $M, b \models \theta$. This together with $dR^F b$ imply that $M, d \models \diamonddot\theta$. ⊣

Surjectivity is inavoidable in this lemma. Indeed, recall from Example 2.3 that the formula $p \to \boxdot p$ is valid at a point of a frame F iff F has exactly one element. Now, if F is a one-element frame and G a two-element frame (with no relations or predicates), then in fact *any* mapping from F to G is a zig-zag morphism, but at the same time, the formula $p \to \boxdot p$ is not valid on G.

5 Application to querying knowledge bases

The mixed modal logic described above is a notational variant of the Description Logic $\mathcal{ALC}$, whose vocabulary consists of concept names $A_0, \dots, A_m, X_0, X_1, \dots$ and role names $R_0, \dots, R_n, S_0, S_1, \dots$ (we will refer to $X_0, X_1, \dots$ and $S_0, S_1, \dots$ as concept and role *variables*. Given a mixed modal formula φ, denote by C_φ the corresponding DL concept obtained from φ by replacing p_i with X_i, $\Box_i$ by $\forall R_i$, and $\boxdot_i$ by $\forall S_i$ (symbols A_i are left unchanged).

In the sequel, first-order formulas will also be called as *queries*. Among them, we distinguish *conjunctive queries*, i.e., formulas of the form $\exists\vec{y} \bigwedge_{i=1}^n term_i$, where each $term_i$ is either of the form $x\colon A_i$ or $xR_j y$, for some variables x and y; its free variables are traditionally called the *distinguished* variables of a query. A query with one distinguished variable will be called *unary*, and a closed formula will be called as a *boolean* query. A query is called *relational* if it contains no symbols A_i.

Definition 5.1. We say that a query $q(x)$ is *answered by* a concept C (written as $q(x) \approx C$) if, for any knowledge base $\mathcal{KB}$ in the vocabulary $\{\vec{A}, \vec{R}\} = \{A_0, \dots, A_m, R_0, \dots, R_n\}$ and any constant a occurring in it, the equivalence holds: $\mathcal{KB} \models q(a) \Leftrightarrow \mathcal{KB} \models a\colon C$.

A boolean query q is *answered by* a concept C (written as $q \approx C$) if, for any knowledge base $\mathcal{KB}$ in the vocabulary $\{\vec{A}, \vec{R}\}$, the equivalence holds: $\mathcal{KB} \models q \Leftrightarrow \mathcal{KB} \models a\colon C$, where a is a fresh individual name (i.e., not occurring in $\mathcal{KB}, q, C$.)

Note that in this definition, a knowledge base $\mathcal{KB}$ is not allowed to contain concept or role *variables*. This is similar to fixing valuations of A_i and R_j in a frame, but quantifying over valuations of p_i and S_j.

In what follows, we will establish a relationship between the notion of local correspondence ($q \leftrightsquigarrow \varphi$) in modal logic and query answering (in the form $q \approx C_\varphi$) in Description Logics. Ideally, it would be desirable to have the equivalence between these two. However, we have not succeeded in proving this yet (and at the same time we have no counterexamples to this equivalence). Below, we first prove 50% of the equivalence (that $q \leftrightsquigarrow \varphi$ implies $q \approx C_\varphi$). Secondly, we prove another 25% of the equivalence ($q \approx C_\varphi$ implies that, for any frame F and its element e, the implication $F \models q(e) \Rightarrow F, e \Vdash \varphi$ holds). The converse implication, i.e., the remaining 25% of the desired equivalence, is not yet proved in general case, but we have established it for finitely branching frames, i.e., say, another 10%. After that, we prove the similar results for boolean queries.

5.1 Answering unary queries

Theorem 5.2 (Unary queries, 50%). *If a unary query $q(x)$ locally corresponds to a mixed modal formula φ, then $q(x)$ is answered by the $\mathcal{ALC}$-concept C_φ. In symbols: $q(x) \leftrightsquigarrow \varphi \;\Rightarrow\; q(x) \approx C_\varphi$.*

Proof. Suppose that $q(x) \leftrightsquigarrow \varphi$. Then, given a knowledge base $\mathcal{KB}$ in the vocabulary $\{\vec{A}, \vec{R}\}$ and a constant a, we will prove the following equivalence: $\mathcal{KB} \models q(a) \Longleftrightarrow \mathcal{KB} \models a\colon C_\varphi$.

($\Rightarrow$) Suppose that $\mathcal{KB} \models q(a)$. Take any model[3] $\mathcal{I}$ of $\mathcal{KB}$. By assumption, $\mathcal{I} \models q(a)$. We need to show that $\mathcal{I} \models a\colon C_\varphi$ (independently of how the concept variables X_i and role variables S_j occurring in C_φ are interpreted in $\mathcal{I}$). Let $F = \langle \Delta, \vec{A}^{\mathcal{I}}, \vec{R}^{\mathcal{I}} \rangle$ be the frame[4] underlying $\mathcal{I}$ and denote $e := a^{\mathcal{I}}$. By definition, from $q(x) \leftrightsquigarrow \varphi$ it follows that, for these F and e, we have: $F \models q(e) \Leftrightarrow F, e \Vdash \varphi$. But we also know that $F \models q(e)$, because $\mathcal{I} \models q(a)$ and $q(x)$ contains only symbols from $\{\vec{A}, \vec{R}\}$. Hence $F, e \Vdash \varphi$, i.e., $M, e \models \varphi$, for any model M based on F.

Now we apply this to the model $M := \langle F, \vec{p}^{\,M}, \vec{S}^M \rangle$ that is "read-off" from our interpretation $\mathcal{I}$ by putting $p_i^M := X_i^{\mathcal{I}}$ and $S_i^M := S_i^{\mathcal{I}}$. It is easily seen that, for any element $d \in \Delta$ and any mixed modal formula ψ, $M, d \models \psi \Leftrightarrow d \in C_\psi^{\mathcal{I}}$, since C_ψ is just a notational variant of ψ, whereas M and $\mathcal{I}$ are essentially the same. As shown above, $M, e \models \varphi$ and so $a^{\mathcal{I}} = e \in C_\varphi^{\mathcal{I}}$, hence $\mathcal{I} \models a\colon C_\varphi$.

($\Leftarrow$) Suppose that $\mathcal{KB} \models a\colon C_\varphi$. Take any model $\mathcal{I}$ of $\mathcal{KB}$. By assumption, $\mathcal{I} \models a\colon C_\varphi$. Let F be a frame underlying $\mathcal{I}$ and put $e := a^{\mathcal{I}}$. We need to show that $\mathcal{I} \models q(a)$; since $q(x)$ contains only symbols from $\{\vec{A}, \vec{R}\}$, it suffices to show that $F \models q(e)$. By the assumption $q(x) \leftrightsquigarrow \varphi$, it remains to show that $F, e \Vdash \varphi$.

To this end, take any model $M = \langle F, \vec{p}^{\,M}, \vec{S}^M \rangle$ based on F and show that $M, e \models \varphi$. Let $\mathcal{J}$ be an interpretation that differs from $\mathcal{I}$ only in how it interprets concept variables $\vec{X}$ and role variables $\vec{S}$, namely it is "read-off" from the model M by putting $X_i^{\mathcal{J}} := p_i^M$ and $S_i^{\mathcal{J}} := S_i^M$. Then, for any $d \in \Delta$ and formula ψ, we have $d \in C_\psi{}^{\mathcal{J}} \Leftrightarrow M, d \models \psi$, since C_ψ is just a notational variant of ψ, whereas M and $\mathcal{J}$ are essentially the same. Since $\mathcal{J}$ and $\mathcal{I}$ agree on interpretation of all symbols from $\mathcal{KB}$, we conclude that $\mathcal{J} \models \mathcal{KB}$. Now we use our assumption $\mathcal{KB} \models a\colon C_\varphi$ to infer that $\mathcal{J} \models a\colon C_\varphi$ and so $e = a^{\mathcal{I}} = a^{\mathcal{J}} \in C_\varphi^{\mathcal{J}}$, whence $M, e \models \varphi$.

This completes the proof of Theorem 5.2. ⊣

[3] Note that $\mathcal{I}$ interprets all concept and role names in our DL language, in particular, X_i and S_j.

[4] This is indeed a frame for the mixed modal language.

In [7], this theorem was proved for the standard modal language (without A_i and $\boxdot_j$) and hence for relational queries only. Note that Theorem 5.2 holds for *arbitrary* first-order formula $q(x)$, and not only for DL knowledge bases $\mathcal{KB}$, but for arbitrary first-order theories in the vocabulary $\{\vec{A}, \vec{R}\}$.

Lemma 5.3 (Unary queries, 25%). *Let $q(x)$ be a conjunctive query, φ a mixed modal formula. If $q(x) \approx C_\varphi$, then for any frame F and its element e, $F \models q(e)$ implies $F, e \Vdash \varphi$.*

Proof. The query $q(x)$ has the form $\exists \vec{y} \bigwedge_{i=1}^{n} term_i(x, \vec{y})$. Suppose that $q \approx C_\varphi$, take any frame F and its element e, and assume that $F \models q(e)$. Then there exist $\vec{o} \in \Delta$ such that $F \models term_i(e, \vec{o})$ for all $i \leqslant n$. Now we build the *canonical* ABox for $q(x)$: $\mathcal{A}_q := \{term_i(a, \vec{c}) \mid 1 \leqslant i \leqslant n\}$, where a and $\vec{c}$ are fresh constants, and consider the knowledge base $\mathcal{KB}_q := \langle \emptyset, \mathcal{A}_q \rangle$. Since $\mathcal{KB}_q \models \bigwedge_{i=1}^{m} term_i(a, \vec{c})$, we have $\mathcal{KB}_q \models \exists \vec{y} \bigwedge_{i=1}^{m} term_i(a, \vec{y})$, and hence $\mathcal{KB}_q \models q(a)$. Applying our assumption ($q(x) \approx C_\varphi$), we obtain that $\mathcal{KB}_q \models a\colon C_\varphi$.

Now to prove that $F, e \Vdash \varphi$, take any model M based on the frame F and show that $M, e \models \varphi$. Let $\mathcal{I}$ be an interpretation based on F such that it is "read-off" from M by putting $X_i^{\mathcal{I}} := p_i^M$ and $S_i^{\mathcal{I}} := S_i^M$, and extended it to the new constants as $a^{\mathcal{I}} := e$ and $\vec{c}^{\,\mathcal{I}} := \vec{o}$. Since $\mathcal{I}$ is based on F and $F \models term_i(o, \vec{e})$, we have $\mathcal{I} \models term_i(a, \vec{c})$, for all $i \leqslant n$, and hence $\mathcal{I} \models \mathcal{KB}_q$. As shown above, $\mathcal{KB}_q \models a\colon C_\varphi$. Therefore, $\mathcal{I} \models a\colon C_\varphi$, and so $e = a^{\mathcal{I}} \in C_\varphi^{\mathcal{I}}$. Finally, it is easily seen that for any $d \in \Delta$ and formula ψ, we have $d \in C_\psi^{\mathcal{I}} \Leftrightarrow M, d \models \psi$, since C_ψ is just a notational variant of ψ, whereas M and $\mathcal{I}$ are essentially the same. Hence $M, e \models \varphi$. ⊣

This lemma is easily generalised to the case of $q(x)$ being a disjunction $q(x) \equiv q_1(x) \vee \ldots \vee q_s(x)$ of conjunctive queries $q_i(x)$. Indeed, in its proof, from $F \models q(e)$ we first derive that $F \models q_i(e)$, for some i, and then proceed the same proof. Note that in this case, DL knowledge bases (in the definition of $\approx$) are still enough to complete the proof. However, they are not enough if we want to generalise Lemma 5.3 to the case of arbitrary first-order formulas $q(x)$. In that case, we can use arbitrary first-order theories instead of KBs, and the proof of Lemma 5.3 for this case remains literally the same. The only difference is that, instead of the canonical knowledge base $\mathcal{KB}_q$, one should take the first-order theory $T_q := \{q(a)\}$.

5.2 Answering boolean queries

Now we realise the same scenario for boolean queries.

Theorem 5.4 (Boolean queries, 50%). *If a boolean query q globally corresponds to a mixed modal formula φ, then q is answered by the $\mathcal{ALC}$-concept C_φ. In symbols: $q \leftrightsquigarrow \varphi \;\Rightarrow\; q \approx C_\varphi$.*

Proof. Suppose that $q \leftrightsquigarrow \varphi$. Take any knowledge base $\mathcal{KB}$ in the vocabulary $\{\vec{A}, \vec{R}\}$ and a fresh constant a and prove the equivalence: $\mathcal{KB} \models q \Leftrightarrow \mathcal{KB} \models a\colon C_\varphi$.

($\Rightarrow$) Assume that $\mathcal{KB} \models q$. Take any model $\mathcal{I} \models \mathcal{KB}$ and show that $\mathcal{I} \models a\colon C_\varphi$. Since $\mathcal{I} \models \mathcal{KB}$, we have $\mathcal{I} \models q$. But q contains only symbols from the vocabulary $\{\vec{A}, \vec{R}\}$, hence $F \models q$, where F is the frame underlying $\mathcal{I}$. Using the assumption $q \leftrightsquigarrow \varphi$, we infer that $F \Vdash \varphi$, in particular, $F, a^{\mathcal{I}} \Vdash \varphi$, which implies that $\mathcal{I} \models a\colon C_\varphi$.

($\Leftarrow$) Assume that $\mathcal{KB} \models a\colon C_\varphi$. Take any model $\mathcal{I} \models \mathcal{KB}$ and show that $\mathcal{I} \models q$. Since $\mathcal{I} \models \mathcal{KB}$, we have $\mathcal{I} \models a\colon C_\varphi$. Moreover, if we vary the interpretation of the constant a and of the concept and role variables occurring in C_φ, then $\mathcal{I}$ remains to be a model of $\mathcal{KB}$. This shows that $F \Vdash \varphi$, where F is the frame underlying $\mathcal{I}$. By the assumption $q \leftrightsquigarrow \varphi$, we infer that $F \models q$. Finally, since $\mathcal{I}$ is based on F, we conclude that $\mathcal{I} \models q$. ⊣

As above, this theorem holds for arbitrary (closed) first-order formulas q and arbitrary first-order theories in the vocabulary $\{\vec{A}, \vec{R}\}$ in place of DL knowledge bases.

Lemma 5.5 (Boolean queries, 25%). *Let q be a boolean conjunctive query, φ a mixed modal formula. If $q \approx C_\varphi$, then for any frame F, $F \models q$ implies $F \Vdash \varphi$.*

Proof. The query $q(x)$ has the form $\exists \vec{y} \bigwedge_{i=1}^{n} term_i(\vec{y})$. Suppose that $q \approx C_\varphi$ and take any frame F such that $F \models q$. Then there exist $\vec{o} \in \Delta$ such that $F \models term_i(\vec{o})$, for all $i \leqslant n$. Now we build the *canonical* ABox for q: $\mathcal{A}_q := \{term_i(\vec{c}) \mid 1 \leqslant i \leqslant n\}$, where $\vec{c}$ are frech constants, and consider the knowledge base $\mathcal{KB}_q := \langle \emptyset, \mathcal{A}_q \rangle$. Since $\mathcal{KB}_q \models \bigwedge_{i=1}^{m} term_i(\vec{c})$, we have $\mathcal{KB}_q \models \exists \vec{y} \bigwedge_{i=1}^{m} term_i(\vec{y})$, and so $\mathcal{KB}_q \models q$. Applying our assumption ($q \approx C_\varphi$), we obtain that $\mathcal{KB}_q \models a\colon C_\varphi$, where a is another fresh constant.

Now to prove that $F \Vdash \varphi$, take any model M based on the frame F and show that $M, e \models \varphi$, for all elements $e \in \Delta$. To this end, take the interpretation $\mathcal{I}$ based on F that is "read-off" from M by putting $X_i^{\mathcal{I}} := p_i^M$ and $S_i^{\mathcal{I}} := S_i^M$, and extended it to the new constants as $a^{\mathcal{I}} := e$ and $\vec{c}^{\,\mathcal{I}} := \vec{o}$. Since $\mathcal{I}$ is based on F and $F \models term_i(\vec{e})$, we have $\mathcal{I} \models term_i(\vec{c})$, for all $i \leqslant n$, and hence $\mathcal{I} \models \mathcal{KB}_q$. As shown above, $\mathcal{KB}_q \models a\colon C_\varphi$. Therefore, we have $\mathcal{I} \models a\colon C_\varphi$, so that $e = a^{\mathcal{I}} \in C_\varphi^{\mathcal{I}}$. Finally, it is easily seen that for any $d \in \Delta$ and formula ψ, we have $d \in C_\psi{}^{\mathcal{I}} \Leftrightarrow M, d \models \psi$, since C_ψ is just a notational variant of ψ, whereas M and $\mathcal{I}$ are essentially the same. Hence $M, e \models \varphi$. ⊣

Again, Lemma 5.5 is easily generalised to the case of a disjunction of boolean conjunctive queries. But if we want to generalise it further to the case of an arbitrary (closed) first-order formula q, then we need to resort to first-order theories in place of DL knowledge bases, and use the theory $T_q := \{q\}$ instead of the knowledge base $\mathcal{KB}_q$ in the proof.

5.3 Another 10%, or From Query Answering back to Modal Logic

In this section, we confine ourselves to the language without variable modalities $\boxdot_i$ (in fact, we do not know how to extend the result presented obtained below to the language involving variable modalities). We will show (see Theorem refThmAnswLoc) that, under conditions of Lemma 5.3, if $q(x) \approx C_\varphi$, then for any *finitely generated* frame F and its element e, $F, e \Vdash \varphi$ implies $F \models q(e)$. Whether this holds for arbitrary frames is an open question.

First we need the means of representing a frame $F = \langle \Delta^F, \vec{R} \rangle$ by a (possibly infinite) knowledge base $\mathcal{KB}$ in some DL, in the sense that any model of that $\mathcal{KB}$ has a subframe "similar" to F. A natural way to build such a KB is to introduce individual names (constants) for all nodes of F, so we do: $\mathsf{INames}_F := \{a_e \mid e \in \Delta^F\}$. The mapping from e to (the interpretation of) a_e will serve the desired embedding of F into models of $\mathcal{KB}$. The notion of homomorphism is too weak for our purposes, whereas the notion of isomorphism is too strong. What will suit us is the notion of bounded morphism, or *zig-zag morphism* introduced in Section 4.1. We are interested in transferring the local validity of modal formulas on a frame to logical entailment from a knowledge base. To this end, we introduce the following notion.

Definition 5.6. A (possibly infinite) knowledge base $\mathcal{KB}$ the vocabulary $\{\vec{A}, \vec{R}\}$ is called *characteristic* for a frame $F = \langle \Delta^F, \vec{R}^F, \vec{A}^F \rangle$ if, for any model $\mathcal{I} \models \mathcal{KB}$, the mapping $z\colon e \mapsto a_e^{\mathcal{I}}$ is a zig-zag morphism from F to $\mathcal{I}$. (Note that an interpretation $\mathcal{I}$ of the vocabulary $\{\vec{A}, \vec{R}\}$ can be considered as a modal frame.)

We are ready to prove our main lemma. Recall that if φ is a modal formula, then C_φ denotes its DL counterpart obtained from φ by replacing $\Box_k$ with $\forall R_k.$ and propositional letters p_i with *fresh* concept names X_i (which cannot occur in any KBs under consideration).

Lemma 5.7. *Let $\mathcal{KB}$ be a characteristic knowledge base for a frame F. Then for any modal formula φ and element $e \in F$:*

$$F, e \Vdash \varphi \quad \Longrightarrow \quad \mathcal{KB} \models a_e\colon C_\varphi.$$

Proof. Suppose that $F, e \Vdash \varphi$. To prove that $\mathcal{KB} \models a_e\colon C_\varphi$, take any model $\mathcal{I} \models \mathcal{KB}$. Since $\mathcal{KB}$ is characteristic for F, the mapping $z\colon e \to a_e^{\mathcal{I}}$ is a zig-zag morphism from F to $\mathcal{I}$. By Lemma 4.6, $\mathcal{I}, z(e) \models \varphi$ (independently of how $\mathcal{I}$ interprets the propositional variables p_i occurring in φ). But since φ is just a notational variant of C_φ and $z(e) = a_e^{\mathcal{I}}$, we conclude that $\mathcal{I} \models a_e\colon C_\varphi$. ⊣

Now it is the right time to show how to build a characteristic knowledge base for any finitely branching frame (and explain why it is related to query answering). Recall that a frame $F = \langle \Delta^F, \vec{R} \rangle$ is called *finitely branching* if, for any $e \in F$ and any R from $\vec{R}$, the set $R^F(e) := \{d \in F \mid eR^F d\}$ is finite.

Lemma 5.8. *Every finitely branching frame has a (possibly infinite) characteristic knowledge base in the Description Logic $\mathcal{ALCO}$.*

Proof. Given such a frame $F = \langle \Delta^F, \vec{R} \rangle$, consider the knowledge base $\mathcal{KB}_F = \langle \emptyset, \mathcal{A}_F \rangle$ in the Description Logic $\mathcal{ALCO}$ consisting of the following ABox only:

$$\begin{aligned}\mathcal{A}_F := &\{\, a_e\colon A_i \mid e \in A_i^F \,\} \cup \{\, a_e\colon \neg A_i \mid e \notin A_i^F \,\} \cup \\ &\{\, a_e R a_d \mid e, d \in F, eR^F d \,\} \cup \{\, a_e\colon \forall R.\{a_d \mid d \in R^F(e)\} \mid e \in F,\ R \in \vec{R} \,\}.\end{aligned}$$

Since F is finitely branching, then each expression of the form $\{a_d \mid d \in R^F(e)\}$ is finite and hence an $\mathcal{ALCO}$-concept (if this set is empty, then the correspondind assertion in the ABox is $a_e\colon \forall R.\bot$). To see that $\mathcal{KB}_F$ is characteristic for F, observe that the condition (*atom*) is satisfied due to the first and second parts of the ABox $\mathcal{A}_F$, (*zig*) is guaranteed by the third part of the ABox $\mathcal{A}_F$. To show that the condition (*zag*) also holds, assume that $\mathcal{I}$ is a model of $\mathcal{A}_F$, and $\mathcal{I} \models a_e R b$, for some element $b \in \Delta^{\mathcal{I}}$. Then the condition $\mathcal{I} \models a_e\colon \forall R.\{a_d \mid d \in R^F(e)\}$ implies that $b = a_d^{\mathcal{I}}$ for some $d \in R^F(e)$, so we are done. ⊣

This construction of a characteristic knowledge base can be used for proving the following theorem.

Theorem 5.9 (10% for finitely branching frames). *Let $q(x)$ be a conjunctive query, φ a mixed modal formula (without $\boxdot_i$). If $q(x)$ is answered by the concept C_φ (in symbols: $q(x) \approx C_\varphi$), then $q(x)$ locally corresponds to the modal formula φ over the class of finitely branching frames: $F, e \Vdash \varphi \Leftrightarrow F \models q(e)$.*

Proof. Suppose that $q(x) \approx C_\varphi$, take a finitely branching frame F and its node $e \in F$. We have already proven that $F \models q(e)$ implies $F, e \Vdash \varphi$ (see Lemma 5.3), so it remains to prove the converse implication. Assume that $F, e \Vdash \varphi$. Then build a (possibly infinite) characteristic knowledge base $\mathcal{KB}_F$ in the language $\mathcal{ALCO}$, as in Lemma 5.8. By Lemma 5.7, we have $\mathcal{KB}_F \models a_e\colon C_\varphi$.

Claim. *The Description Logic $\mathcal{ALCO}$ is compact: if $\Gamma \models A$ (where $\Gamma \cup \{A\}$ is a set of KB assertions in the language of $\mathcal{ALCO}$), then there exists a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models A$.*

Indeed, $\Gamma \cup \{A\}$ can be expounded as set of closed formulas in the first-order language with equiality, for which the **Compactness Theorem** (cf. [4]) holds.

This statement implies that there exists a finite subset $\mathcal{KB} \subseteq \mathcal{KB}_F$ such that $\mathcal{KB} \models a_e\colon C_\varphi$. Since $q(x)$ is answered by C_φ (w.r.t. any KB), we conclude that $\mathcal{KB} \models q(a_e)$.

Finally, take an interpretation $\mathcal{I}$ based on the frame F with constants interpreted as $a_d^{\mathcal{I}} := d$, for all $d \in \Delta^F$. Then $\mathcal{I} \models \mathcal{KB}_F$ and hence $\mathcal{I} \models \mathcal{KB}$. Therefore $\mathcal{I} \models q(a_e)$. But the formula $q(x)$ contains only the symbols in the vocabulary $\{\vec{A}, \vec{R}\}$, thus $F \models q(a_e^{\mathcal{I}})$, i.e., $F \models q(e)$. ⊣

5.4 Examples of query answering

Let us illustrate how these theorems can be applied to querying DL knowledge bases.

Example 5.10 (Querying for properties of roles). Suppose we are given a knowledge base $\mathcal{KB}$; in the context of applications, it is reasonable to assume here that the ABox part of $\mathcal{KB}$ is empty, however all the results obtained below hold even without this assumption as well. We are interested in checking whether some properties of roles are entailed from the knowledge base. Let us start with the properties of transitivity and role inclusions. The following statement is easy to show.

Lemma 5.11. *Assume that the constants a, b, c do not occur in a knowledge base $\mathcal{KB}$. Then:*

1. $\mathcal{KB} \models R \sqsubseteq S \iff$ *a knowledge base* $\mathcal{KB} \cup \{aRb, \neg aRb\}$ *is inconsistent.*
2. $\mathcal{KB} \models \mathsf{Trans}(R) \iff$ *a knowledge base* $\mathcal{KB} \cup \{aRb, bRc, \neg aRc\}$ *is inconsistent.*

An assertion of the form $\neg aRb$ can be expressed as $a\colon \neg\exists R.\{b\}$. But can we achieve the same goal without introducing nominals? The answer is Yes, due to the following lemma.

Lemma 5.12. *Assume that a constant c and a concept name X do not occur in $\mathcal{KB}$. Then:*

1. $\mathcal{KB} \models R \sqsubseteq S \iff \mathcal{KB} \models c\colon (\neg\exists R.X \sqcup \exists S.X)$.
2. $\mathcal{KB} \models \mathsf{Trans}(R) \iff \mathcal{KB} \models c\colon (\neg\exists R.\exists R.X \sqcup \exists R.X)$.

Proof. This is a consequence of Theorem 5.4, since role inclusion is expressed by a modal formula $\Diamond_R p \to \Diamond_S p$, whereas transitivity by a modal formula $\Diamond\Diamond p \to \Diamond p$. ⊣

Moreover, a similar statement holds for any property of roles that is first-order local correspondent of some modal property (in the r.h.s. of the statement we should put the DL counterpart of that modal property). Therefore, we can query a knowledge base for various properties of roles (reflexivity, transitivity, symmetry, euclideanness, being an equivalence relation, Church-Rosser, density, etc.), by reducing the problem to the knowledge base satisfiability. A few more examples:

3. $\mathcal{KB} \models \mathsf{Func}(R) \iff \mathcal{KB} \models c\colon (\forall R.X \sqcup \forall R.\neg X)$.
4. $\mathcal{KB} \models \mathsf{Symm}(R) \iff \mathcal{KB} \models c\colon (\neg X \sqcup \forall R.\exists R.X)$.
5. $\mathcal{KB} \models \mathsf{Dense}(R) \iff \mathcal{KB} \models c\colon (\neg\exists R.X \sqcup \exists R.\exists R.X)$.
6. $\mathcal{KB} \models \mathsf{ChRoss}(R) \iff \mathcal{KB} \models c\colon (\forall R.\exists R.\neg X \sqcup \forall R.\exists R.X)$.

Neither asymmetry ($\forall x, y \neg(xRy \;\&\; yRx)$) nor antisymmetry ($\forall x, y\,(xRy \;\&\; yRx \to x = y)$) can be expressed by a modal formula (and even by a graded modal formula). However, they *can* be expressed by a hybrid modal formulas (cf. [1]): $@_i \neg\Diamond\Diamond i$ and $@_i \Box(\Diamond i \to i)$, resp. Question: can we extend our results so that to obtain the following consequences? Again, a constant c does not occur in $\mathcal{KB}$:

7. $\mathcal{KB} \models \mathsf{Asymm}(R) \iff \mathcal{KB} \models c\colon \neg\exists R.\exists R.\{c\}$.
8. $\mathcal{KB} \models \mathsf{Antisym}(R) \iff \mathcal{KB} \models c\colon \forall R.(\neg\exists R.\{c\} \sqcup \{c\})$.

Remark. One would argue that role inclusion and role transitivity in, for instance, the DL $\mathcal{SHIQ}$ are checked before giving a definition of concepts, because we need to know which roles are simple (i.e., do not contain a transitive subrole). However, in this way we only check whether a role inclusion or a transitivity of a role follows from an RBox only (i.e., from a set of role inclusion and transitivity axioms). Moreover, this is sufficient for restricting the language to a decidable one. But it appears that role inclusions and transitivity may also follow from concept inclusion axioms (and even from ABox part of a knowledge base, but we do not consider these cases here).

For example, it is easily seen that the role isAuthorOf that links a person with his/her publications is transitive. Moreover, its transitivity is entailed by the following TBox:

$$\exists \mathsf{isAuthorOf}.\top \subseteq \mathsf{Person}, \qquad \top \subseteq \forall \mathsf{isAuthorOf}.\mathsf{Paper}, \qquad \mathsf{Person} \sqcap \mathsf{Paper} \subseteq \bot.$$

The same holds for any role with disjoint domain and range. Note that in this case it is reasonable *not* to add the transitivity axiom to the knowledge base, even though it is its consequence, because adding such an axiom would disable us to use expressions of the form ($\geqslant 3$ isAuthorOf.$\top$). To conclude the remark, here is an example of role inclusion $R \subseteq$ entailed by concept inclusions:

$$\mathrm{Ran}(R) \subseteq \{a\}, \qquad \mathrm{Ran}(S) \subseteq \{a\}, \qquad \mathrm{Dom}(R) \subseteq \mathrm{Dom}(S).$$

Example 5.13 (Mary likes all cats). Suppose that we have a knowledge base $\mathcal{KB}$, an individual Mary, a concept Cat and a role name Likes. If we want to query this knowledge base of whether Mary likes all cats, then this would be usually formulated as the following concept subsumption: $\mathsf{Cat} \sqsubseteq \exists \mathsf{Likes}^{-}.\{\mathsf{Mary}\}$. In this formulation, we need role inverses and nominals, even if the knowledge base $\mathcal{KB}$ did not contain some of these two constructors. But it appears that, using a fresh concept name SomeConc and fresh role name SomeRel, we can do this within the logic $\mathcal{ALC}$. Indeed, from Theorem 5.2 and Lemma 4.4 it follows that $\mathcal{KB}$ entails the above subsumption iff

$$\mathcal{KB} \models \mathsf{Mary}\colon(\exists \mathsf{SomeRel}.(\mathsf{SomeConc} \sqcap \mathsf{Cat}) \to \exists \mathsf{Likes}.\mathsf{SomeOne}),$$

where we have applied the contraposition to the modal formula from Lemma 4.4.

The solution to this problem proposed in [6] involves extension of the DL language with role negation. Namely, the assertion "Mary likes all cats" can be expressed as: $\mathsf{Mary}\colon \forall \neg \mathsf{Likes}.\neg \mathsf{Cat}$. In that paper it was shown that such an extension results, for the logic $\mathcal{ALC}$, the increase of the complexity from PSpace to ExpTime. Here we succeeded to express the same statement without increase in complexity. However, please note that our approach has the following deficiency: the usage of fresh concept and role names makes sense only on the right hand side of the entailment sign ($\models$), since it is only in this case that the entailment implicitly assumes the *universal* quantification over arbitrary interpretations of the fresh names. If we would want to add to a knowledge base (i.e., to the left hand side of the entailment) an assertion with fresh concept or role names, then we will have to prefix this assertion with the universal (second-order) quantifiers over all fresh names, which takes us beyond the DL scope.

Example 5.14. This example is similar to the previous one, but involves a boolean query. Given a knowledge base $\mathcal{KB}$, a concept C and a role R, one can query the KB of whether C contains or is contained in the domain or the range of the relation R. Three of these four queries can be formulated as subsumption between $\mathcal{ALC}$-concepts, whereas the fourth cannot (*Can we prove this negative statement?*):

$$\begin{array}{ll} \mathrm{Dom}(R) \subseteq C \;\Leftrightarrow\; \exists R.\top \sqsubseteq C & \quad C \subseteq \mathrm{Dom}(R) \;\Leftrightarrow\; C \sqsubseteq \exists R.\top \\ \mathrm{Ran}(R) \subseteq C \;\Leftrightarrow\; \top \sqsubseteq \forall R.C & \quad C \subseteq \mathrm{Ran}(R) \;\Leftrightarrow\; C \sqsubseteq \exists R^{-}.\top \end{array}$$

Now, by Lemma 4.4, we have that the formula $\Box p \to \boxdot(A \to p)$ globally corresponds to the query $\forall x \forall y\, (A(y) \to xRy)$. However, we need $C \subseteq \mathrm{Ran}(R)$, which is $\forall y \exists x\, (A(y) \to xRy)$. Ooops...

6 Discussion on the "10%" result

Analysing the proof of this theorem, we come up to a conclusion that, in order to prove the same for the class of *all* frames, we must be able to build a characteristic KB for an arbitrary frame. Note that if we apply the same construction as above for arbitrary frame F, then we will obtain a knowledge base $\mathcal{KB}_F$ in the language $\mathcal{ALCO}^{\infty}$, where in addition to ordinary syntax of $\mathcal{ALCO}$, it is allowed to form a concept from an infinite number of constants: if a_i $(i \in I)$ are constants, then $\{a_i \mid i \in I\}$ is a concept expression. The resulting KB will indeed be characteristic for F, which is easily verified.

However, there are two difficulties here: first, $q(x) \approx C_\varphi$ does not imply that $q(x)$ has the same answer set as C_φ w.r.t. KB in the logic $\mathcal{ALCO}^{\infty}$ (this holds only for finitary logics). Secondly, the logic $\mathcal{ALCO}^{\infty}$

is *not* compact. Indeed, take Γ consisting of the assertions $a{:}\neg\exists R.\{b_i\}$ for all $i \geqslant 0$, and let A be the assertion $a{:}\neg\exists R.\{b_0, b_1, \ldots\}$. Then it is easily seen that $\Gamma \models A$, but no finite subset of Γ entails A.

A candidate of characteristic KB for arbitrary frame $F = \langle \Delta^F, \vec{R} \rangle$: for each $e \in \Delta^F$ and $R_i \in \vec{R}$, introduce a concept name $A_{e,i}$, which will be thought of as the set $R_i^F(e)$ of all R_i-successors of e. If we could express in an ordinary DL the inclusion $A_{e,i} \sqsubseteq \{a_d \mid d \in R_i^F(e)\}$ (the converse inclusion is expressible by taking, for all $d \in R_i^F(e)$, the assertions $a_d{:}\, A_{e,i}$), then our task would be accomplished. But this seems impossible, so we need to find a roundabout way to do this. Let an $\mathcal{ALCIO}$-KB be as follows: $\mathcal{KB}_F := \langle \mathcal{T}_F, \mathcal{A}_F \rangle$, where

$$\begin{array}{rcl} \mathcal{T}_F & := & \{ A_{e,i} \sqsubseteq \exists R_i^-.\{a_e\} \text{ or } \exists R_i.A_{e,i} \sqsubseteq \{a_e\} \mid e \in \Delta^F,\ R_i \in \vec{R} \}, \\ \mathcal{A}_F & := & \{ a_e R a_d \mid e, d \in \Delta^F, eR^F d \} \cup \{ a_e{:}\, \forall R_i.A_{e,i} \mid e \in \Delta^F,\ R_i \in \vec{R} \} \\ & \cup & \{ a_d{:}\, A_{e,i} \mid F \models eR_i d \} \cup \{ a_d{:}\, \neg A_{e,i} \mid F \not\models eR_i d \} \\ & \cup & \{ a_e{:}\, \neg\exists R_i.(A_{e,i} \setminus \{a_{d_1}, \ldots, a_{d_n}\}) \mid n \leqslant \mathsf{Card}(R_i(e)),\ \mathcal{I} \models a_e R_i a_{d_j} \} \end{array}$$

1. To prove the full converse of Theorem 5.2 we need in fact the following construction: given a frame F, its node $e \in F$, and a modal formula φ such that $F, e \Vdash \varphi$, build a $\mathcal{KB}$ such that $\mathcal{KB} \models a_e{:}\, C_\varphi$. $\mathcal{KB}$ is supposed to be formulated in a DL with the set of constants including $\mathsf{INames}_F = \{a_e \mid e \in \Delta^F\}$. If not in DL, then at least in a language with the compactness property.

2. A sufficient (but probably not necessary) construction for 1): given a frame F, build a (possibly infinite) $\mathcal{KB}$ such that $\forall e \in \Delta^F\ \forall \varphi$ if $F, e \Vdash \varphi$ then $\mathcal{KB} \models a_e{:}\, C_\varphi$.

3. A sufficient (but probably not necessary) construction for 2): given a frame F, build a characteristic $\mathcal{KB}$, i.e., for any $\mathcal{I} \models \mathcal{KB}$ the mapping $z{:}\, e \mapsto a_e^{\mathcal{I}}$ is a zig-zag morphism from F to $\mathcal{I}$.

Example 6.1. To see that 3) is not necessary for 2), consider a Countable Hedgehog: $F = \langle \mathbb{N}, R \rangle$, where $R = \{\langle 0, i \rangle \mid i > 0\}$. In this case $\mathsf{INames}_F = \{a_0, a_1, \ldots\}$. Now take the knowledge base $\mathcal{KB}_F = \langle \varnothing, \mathcal{A}_F \rangle$, where the ABox is:

$$\mathcal{A}_F = \{ a_0 R a_i \mid i > 0 \} \cup \{ a_0{:}\, \forall R.\neg\exists R.\top \}.$$

Note that this frame can be turned into finitely branching one by inversing the role R.

Claim 1. For any modal formula φ and $i \in \mathbb{N}$, if $F, i \Vdash \varphi$ then $\mathcal{KB}_F \models a_i{:}\, C_\varphi$.

Claim 2. There is a model $\mathcal{I} \models \mathcal{KB}_F$ such that $z{:}\, i \mapsto a_i^{\mathcal{I}}$ is not a zig-zag morphism from F to $\mathcal{I}$. Take $\mathcal{I}$ to be an Uncountable Hedgehog. Do the Countable and Uncountable Hedgehogs have the same modal formulas valid at its root? Is it equal to $\mathbf{K} + \Diamond\top, \Box\Box\bot$?

Example 6.2. An example of not finitely branching frame for which we have built a characteristic KB. Consider the frame $F = \langle \mathbb{N}, < \rangle$. Then take $\mathcal{KB}_F = \langle \mathcal{T}_F, \mathcal{A}_F \rangle$, where $\mathcal{T}_F = \{\mathsf{Trans}(R)\}$ and

$$\mathcal{A} = \{a_i R a_{i+1}, \neg a_{i+1} R a_i, a_i \neg\exists R.\exists R.\{a_{i+1}\}, a_i{:}\, (\leqslant 1\, R).(\exists R^-.\{a_i\} \sqcap \neg\exists R^-.\exists R^-.\{a_i\}) \mid i \geqslant 0\}$$

Claim. This $\mathcal{KB}_F$ is characteristic for F.

More generally: given a frame $F = \langle \Delta^F, \vec{R} \rangle$, let $Th^1(F)$ be the first-order theory of F, i.e., the set of all closed FO formulas in the language with equality, binary relations from $\vec{R}$, and constants from INames_F (and no unary predicates) that are true on F. This theory, in particular, covers KBs in most standard DLs, say $\mathcal{SHOIQ}$. Then is it the case that, for any modal formula φ, if $F, e \Vdash \varphi$ then $Th^1(F) \models a_e{:}\, C_\varphi$? (The sentence $a_e{:}\, C_\varphi$ can be regarded as a first-order closed formula.)

Yet more generally: for a given frame F, take the monadic second-order theory of F, to be more exact, the 2-order translations of ABox assertions that are valid on F: $Th^2(F) = \{ST^2_\varphi(a_e) \mid F, e \Vdash \varphi\}$, where $ST^2_\varphi(x)$ is the stadnard translation of the modal formula φ into a first-order formula with 1 variable, universally quantified over all unary predicates occurring in it. For example, if φ is $\Box p \to \Box\Box p$, then $ST^2_\varphi(x)$ is

$$\forall P\,[\,\forall y(xRy \to P(y)) \to \forall y(xRy \to \forall z(yRz \to P(z)))\,].$$

Then it is straightforward to show that if $F, e \Vdash \varphi$ then $Th^2(F) \models a_e\colon C_\varphi$.

Note that we could not avoid using second-order quantifiers here: if we take only the first-order sentences of the form $a_e\colon C_\psi$ for all e, ψ such that $F, e \Vdash \psi$ (where all C_ψ use concept names different from those used in φ), then this theory does not entail $a_e\colon C_\varphi$. Hint: $ST^1_{\Box p \to p}(a) \not\models ST^1_{\Box q \to q}(a)$.

Question. Is MSO compact? If not, is the "guarded", i.e., $Th^2(F)$-sublanguage of MSO compact? Is it the same as the compactness of the notion of validity of a modal formula in a frame?

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