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# Sequential Reflexive Logics with Noncontingency Operator

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**Abstract**—Hilbert systems $L^{\rhd}$ and sequential calculi $[L^{\rhd}]$ for the versions of logics $L = \mathbf{T}, \mathbf{S4}, \mathbf{B}, \mathbf{S5}$, and $\mathbf{Grz}$ stated in a language with the single modal noncontingency operator $\rhd A = \Box A \vee \Box\neg A$ are constructed. It is proved that cut is not eliminable in the calculi $[L^{\rhd}]$, but we can restrict ourselves to analytic cut preserving the subformula property. Thus the calculi $[\mathbf{T}^{\rhd}]$, $[\mathbf{S4}^{\rhd}]$, $[\mathbf{S5}^{\rhd}]$ ($[\mathbf{Grz}^{\rhd}]$, respectively) satisfy the (weak, respectively) subformula property; for $[\mathbf{B}_2^{\rhd}]$, this question remains open. For the noncontingency logics in question, the Craig interpolation property is established.

KEY WORDS: *Hilbert calculi, sequential calculi, cut elimination, noncontingency, Craig interpolation.*

## INTRODUCTION

In the construction of logical calculi in modal logic, it is traditional to choose a language with the necessity $\Box$ (and possibility $\Diamond$) operators. However, systems in which the *noncontingency* operator defined by the equation $\rhd A = \Box A \vee \Box\neg A$ is chosen as the basis operator are of a certain technical and philosophical interest (see [1, 2]).[1] This equation defines translation of $\rhd$-formulas (i.e., formulas of the modal language with the single modal operator $\rhd$ or, in other words, $\rhd$-*language*) into $\Box$-formulas. If a $\Box$-logic $L$ is given (i.e., a logic in the $\Box$-language), then the *noncontingency logic* over $L$ (notation: $L^{\rhd}$) is the set of $\rhd$-formulas whose translations are theorems of $L$.

In [2, 3], various axiomatics of noncontingency logics over the familiar normal logics $\mathbf{T}$, $\mathbf{S4}$, and $\mathbf{S5}$ were proposed (see also [4, 5]). Note that in the case in which a logic $L$ contains $\mathbf{T}$, or more exactly, the reflexivity axiom $\Box A \to A$, the analysis of the logic $L^{\rhd}$ is simplified, because the operator $\Box$ is expressible in terms of $\rhd$ by means of the equation $\Box A = A \& \rhd A$. This makes the construction of Hilbert axiomatics of these logics $L^{\rhd}$ automatic (see Lemma 4.5 of this paper), and so this case is not of considerable interest. On the contrary, for $L^{\rhd}$, the construction of sequential calculi with "good" structural properties (cut eliminability, the subformula property, etc.) is quite meaningful. In [6], a nontrivial example of a logic not containing $\mathbf{T}$ in which, however, $\Box$ is expressible in terms of $\rhd$ is constructed.

Systematic examination of noncontingency logics was started in the paper [7], which contains the first, rather cumbersome axiomatics of the minimal noncontingency logic (i.e., the logic $\mathbf{K}^{\rhd}$). In the subsequent paper [8], it was simplified, and the logic $\mathbf{K4}^{\rhd}$ was axiomatized. In [9], the axiomatics of the noncontingency logic over the "epistemic" logic $\mathbf{KD45}$ was proposed; in addition,

[1] *Editor's note.* In the Russian literature and in the Russian original of the article, the term "razreshimost′" (literally, "decidability") is used; recall that the term "noncontingency" stems from the consideration of the provability interpretation of the $\Box$ operator: a sentence is *noncontingent* in a theory if either the sentence or its negation is provable in this theory.

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elementary equivalents for certain axioms of noncontingency logics were found. Finally, in [10], the logic $\mathbf{GL}^{\triangleright}$ was axiomatized and sequential calculi for $\mathbf{K}^{\triangleright}$, $\mathbf{K4}^{\triangleright}$, and $\mathbf{GL}^{\triangleright}$ were constructed.

The paper continues this line of research. After the statement of the required definitions (Sec. 1), in Sec. 2 we present Hilbert axiomatics for $L^{\triangleright}$ and the sequential calculi $[L_1^{\triangleright}]$ and $[L_2^{\triangleright}]$ for the noncontingency logics over $L \in \{\mathbf{T}, \mathbf{S4}, \mathbf{B}, \mathbf{S5}, \mathbf{Grz}\}$. In Sec. 3, we describe a method to prove the completeness of sequential calculi in the $\triangleright$-language with analytic cut. Sec. 4 is devoted to the proof of completeness of the axiomatics we construct. In Sec. 5, we establish cut ineliminability in the constructed sequential calculi; nonetheless, it follows from the completeness theorem proved in Sec. 4 that the calculi $[\mathbf{T}_2^{\triangleright}]$, $[\mathbf{S4}_2^{\triangleright}]$, $[\mathbf{S5}_2^{\triangleright}]$ ($[\mathbf{Grz}]$, respectively) have (weak, respectively) subformula property (for $[\mathbf{B}_2^{\triangleright}]$ the question remains open); also, in Sec. 5, the Craig interpolation property for the constructed noncontingency logics is proved.

## 1. DEFINITIONS AND FACTS

A propositional modal language ($\Box$-*language*) contains a denumerable set of variables $\mathbb{P} = \{p_0, p_1, \dots\}$, the Boolean connectives $\perp$ (falsehood) and $\to$ (implication), and a unary operator $\Box$. Other connectives are introduced as abbreviations; in particular, $\neg A \leftrightharpoons A \to \perp$, $\Diamond A \leftrightharpoons \neg\Box\neg A$. The set of $\Box$-*formulas* $\mathbf{Fm}^{\Box}$ is defined in the usual way. The *minimal normal logic* $\mathbf{K}$ has the following axioms and inference rules (here $A[B/p]$ is the result of substituting a formula $B$ for all occurrences of a variable $p$ in $A$):

($\mathrm{A}_{\top}^{\Box}$) the classical tautologies in the $\Box$-language,

($\mathrm{A}_{\mathbf{K}}^{\Box}$) distributivity: $\Box(p \to q) \to (\Box p \to \Box q)$,

$$(\mathrm{MP})\frac{A \quad A \to B}{B}, \qquad (\mathrm{Sub})\frac{A}{A[B/p]}, \qquad (\mathrm{Nec})\frac{A}{\Box A}.$$

We shall consider the following normal modal logics:

$$\mathbf{T} = \mathbf{K} + (\mathrm{A}_{\mathbf{T}}^{\Box}), \qquad \mathbf{S4} = \mathbf{T} + (\mathrm{A}_{\mathbf{4}}^{\Box}),$$
$$\mathbf{B} = \mathbf{T} + (\mathrm{A}_{\mathbf{B}}^{\Box}), \qquad \mathbf{S5} = \mathbf{T} + (\mathrm{A}_{\mathbf{5}}^{\Box}),$$
$$\mathbf{S4.1} = \mathbf{S4} + (\mathrm{A}_{\mathbf{1}}^{\Box}), \qquad \mathbf{Grz} = \mathbf{K} + (\mathrm{A}_{\mathbf{G}}^{\Box}),$$

where the additional axioms are given by the formulas

($\mathrm{A}_{\mathbf{T}}^{\Box}$) reflexivity: $\Box p \to p$,

($\mathrm{A}_{\mathbf{B}}^{\Box}$) symmetry: $p \to \Box\Diamond p$,

($\mathrm{A}_{\mathbf{4}}^{\Box}$) transitivity: $\Box p \to \Box\Box p$,

($\mathrm{A}_{\mathbf{5}}^{\Box}$) euclideanness: $\Diamond p \to \Box\Diamond p$,

($\mathrm{A}_{\mathbf{1}}^{\Box}$) the McKinsey axiom: $\Box\Diamond p \to \Diamond\Box p$,

($\mathrm{A}_{\mathbf{G}}^{\Box}$) the Grzegorczyk axiom: $\Box(\Box(p \to \Box p) \to p) \to p$.

In addition, the modal logics mentioned above satisfy the following embedding diagram:

$$\begin{array}{ccccc} \mathbf{T} & \subset & \mathbf{S4} & \subset & \mathbf{S4.1} \\ \cap & & \cap & & \cap \\ \mathbf{B} & \subset & \mathbf{S5} & & \mathbf{Grz} \end{array}.$$

A *sequent* is an expression of the form $\Pi \Rightarrow \Sigma$, where $\Pi$ and $\Sigma$ are finite multisets[2] of formulas. Inclusion of multisets of formulas is defined disregarding multiplicities, i.e., the notation $\Pi \subseteq \Sigma$ means that any formula from $\Pi$ occurs in $\Sigma$. We set $\Pi\Sigma := \Pi \cup \Sigma$ and $\Pi A := \Pi \cup \{A\}$. The set of subformulas of a formula $A$ is denoted by $\mathrm{Sb}\, A$, and if $\Gamma$ is a (multi)set of formulas, then $\mathrm{Sb}\,\Gamma := \cup\{\mathrm{Sb}\, A \mid A \in \Gamma\}$. If the sequent $\Pi \Rightarrow \Sigma$ is denoted by $w$, then its antecedent is denoted by $\langle w| := \Pi$, succedent by $|w\rangle := \Sigma$, and the set of subformulas by $\mathrm{Sb}\, w := \mathrm{Sb}\,\Pi\Sigma$. We write $A \in w$ if $A \in \Pi\Sigma$, $\Gamma \subseteq w$ if $\Gamma \subseteq \Pi\Sigma$, and $w \subseteq \Gamma$ if $\Pi\Sigma \subseteq \Gamma$. If $\mathcal{L}$ is a sequential calculus, then the notation $\mathcal{L} \vdash A \Leftrightarrow B$ means that $\mathcal{L} \vdash A \Rightarrow B$ and $\mathcal{L} \vdash B \Rightarrow A$.

The sequential calculus $[L]$ for a logic

$$L \in \{\mathbf{T}, \mathbf{S4}, \mathbf{B}, \mathbf{S5}, \mathbf{Grz}\}$$

is obtained from the sequential propositional calculus (with cut) by adding to it the rules $(\Box \Rightarrow)$ and $(\Rightarrow^{\Box}_{L})$ given by the formulas

$$(\Box \Rightarrow)\frac{A, \Pi \Rightarrow \Sigma}{\Box A, \Pi \Rightarrow \Sigma}, \qquad (\Rightarrow^{\Box}_{\mathbf{B}})\frac{\Pi \Rightarrow \Box\Sigma, A}{\Box\Pi \Rightarrow \Sigma, \Box A}, \qquad (\Rightarrow^{\Box}_{\mathbf{S5}})\frac{\Box\Pi \Rightarrow \Box\Sigma, A}{\Box\Pi \Rightarrow \Box\Sigma, \Box A},$$

$$(\Rightarrow^{\Box}_{\mathbf{T}})\frac{\Pi \Rightarrow A}{\Box\Pi \Rightarrow \Box A}, \qquad (\Rightarrow^{\Box}_{\mathbf{S4}})\frac{\Box\Pi \Rightarrow A}{\Box\Pi \Rightarrow \Box A}, \qquad (\Rightarrow^{\Box}_{\mathbf{Grz}})\frac{\Box(A \to \Box A), \Box\Pi \Rightarrow A}{\Box\Pi \Rightarrow \Box A}.$$

It is known that cut is eliminable in the calculi for $\mathbf{T}$, $\mathbf{S4}$, and $\mathbf{Grz}$ [11] and not eliminable in the calculi for $\mathbf{B}$ and $\mathbf{S5}$ [12–14]. We can confine ourselves to the *analytic* cut [15] in the last two:

$$\frac{\Pi \Rightarrow \Sigma, A \quad A, \Pi' \Rightarrow \Sigma'}{\Pi\Pi' \Rightarrow \Sigma\Sigma'}, \qquad A \in \mathrm{Sb}(\Pi\Pi'\Sigma\Sigma').$$

The calculus $[\mathbf{S5}]$ thus obtained has the *subformula property* [14]: any deducible sequent $\Pi \Rightarrow \Sigma$ admits a deduction all of whose sequents consist of subformulas of formulas from $\Pi\Sigma$. The rule $(\Rightarrow^{\Box}_{\mathbf{B}})$ can violate the subformula property, but it is known [14] that we can confine ourselves only to its applications in which $\Sigma \subseteq \mathrm{Sb}(\Pi A)$, and even to those in which $\Box\Sigma \subseteq \mathrm{Sb}(\Pi A)$. Thus the subformula property holds for $[\mathbf{B}]$ as well. Finally, the calculus $[\mathbf{Grz}]$ satisfies the *weak* subformula property: any deducible sequent $\Pi \Rightarrow \Sigma$ admits a deduction consisting of sequents of the form $\Gamma \Rightarrow \Delta$, where $\Delta \subseteq \mathrm{Sb}\,\Pi\Sigma$ and

$$\Gamma \subseteq \mathrm{Sb}(\Pi\Sigma \cup \{\Box(A \to \Box A) \mid \Box A \in \mathrm{Sb}\,\Pi\Sigma\}).$$

To describe noncontingency logics, we introduce the $\rhd$-language, which differs from the $\Box$-language only by the replacement of the $\Box$ symbol by $\rhd$, and the set $\mathbf{Fm}^{\rhd}$ of $\rhd$-*formulas*.

Let us specify a $\rhd$-translation $\mathrm{tr}\colon \mathbf{Fm}^{\rhd} \to \mathbf{Fm}^{\Box}$ preserving variables and Boolean connectives such that

$$\mathrm{tr}(\rhd A) = \Box\,\mathrm{tr}(A) \vee \Box\neg\,\mathrm{tr}(A).$$

Instead of $\mathrm{tr}(A)$ we shall often write $A_{\rhd}$. By abuse of notation, sometimes we write $\rhd A$, where $A$ is a $\Box$-formula, meaning $\Box A \vee \Box\neg A$; this usage of the symbol $\rhd$ is easily recognizable by the context. We define the *noncontingency logic* over a logic $L$ as the set of $\rhd$-formulas whose $\rhd$-translations are theorems of the logic $L$:

$$L^{\rhd} := \{A \in \mathbf{Fm}^{\rhd} \mid A_{\rhd} \in L\}.$$

[2] A *multiset* is a set with occurrence multiplicities ($\geq 0$) indicated for each of its elements. Formally, a multiset of $\Box$-formulas is a mapping $\mathbf{Fm}^{\Box} \to \mathbb{N}$.

The Kripke semantics for $\Box$- and $\rhd$-languages is introduced in the usual way. The accessibility relation in a frame and its inverse will be denoted by $\uparrow$ and $\downarrow$, respectively; the quantifiers over points accessible from $w$ are written as $\forall x \downarrow w$ and $\exists x \downarrow w$. In this notation, the modal clause of the truth definition of a $\rhd$-formula at a point of the model is of the form

$$w \models \rhd A \quad \leftrightharpoons \quad (\forall x \downarrow w \quad x \models A) \quad \text{or} \quad (\forall x \downarrow w \quad x \not\models A).$$

Obviously, $w \models A \Leftrightarrow w \models A_{\rhd}$ for any $\rhd$-formula $A$. If $\Gamma$ is a set of formulas, then a $\Gamma$-*frame* is a frame on which $\Gamma$ is valid. The validity of a sequent $\Pi \Rightarrow \Sigma$ is understood as the validity of the formula $\bigwedge \Pi \to \bigvee \Sigma$.

## 2. AXIOMATIC SYSTEMS

The axioms of the minimal reflexive noncontingency logic $\mathbf{T}^{\rhd}$ are all the classical tautologies in the $\rhd$-language and the following axioms:

$(\mathrm{A}^{\rhd}_{\neg})$ reflectivity: $\rhd p \leftrightarrow \rhd \neg p$,

$(\mathrm{A}^{\rhd}_{\mathbf{T}})$ weak distributivity: $p \to [\rhd (p \to q) \to (\rhd p \to \rhd q)]$;

and its inference rules are

$$(\mathrm{MP}), \quad (\mathrm{Sub}), \quad \text{and} \quad (\mathrm{Dec})\frac{A}{\rhd A}.$$

Axioms of other reflexive noncontingency logics are given below (conjecture: the axiom $(\mathrm{A}^{\rhd}_{\mathbf{4}})$ in the statement of the calculus $\mathbf{Grz}^{\rhd}$ is superfluous):

$$\begin{aligned}
\mathbf{B}^{\rhd} &= \mathbf{T}^{\rhd} + (\mathrm{A}^{\rhd}_{\mathbf{B}}), && (\mathrm{A}^{\rhd}_{\mathbf{B}}) \text{ is the axiom } p \to \rhd (\rhd p \to p),\\
\mathbf{S4}^{\rhd} &= \mathbf{T}^{\rhd} + (\mathrm{A}^{\rhd}_{\mathbf{4}}), && (\mathrm{A}^{\rhd}_{\mathbf{4}}) \text{ is the axiom } \rhd p \to \rhd\rhd p,\\
\mathbf{S5}^{\rhd} &= \mathbf{T}^{\rhd} + (\mathrm{A}^{\rhd}_{\mathbf{5}}), && (\mathrm{A}^{\rhd}_{\mathbf{5}}) \text{ is the axiom } \rhd\rhd p,\\
\mathbf{Grz}^{\rhd} &= \mathbf{S4}^{\rhd} + (\mathrm{A}^{\rhd}_{\mathbf{G}}), && (\mathrm{A}^{\rhd}_{\mathbf{G}}) \text{ is the axiom } \rhd (\rhd (p \to \rhd p) \to p) \to \rhd p.
\end{aligned}$$

In what follows, $L$ denotes one of the logics $\mathbf{T}$, $\mathbf{B}$, $\mathbf{S4}$, $\mathbf{S5}$, $\mathbf{Grz}$. Representing the deductions schematically, we can write

$$L \vdash A_0 \overset{1}{-} A_1 \overset{2}{-} \cdots \overset{n}{-} A_n, \qquad \text{where} \quad \overset{k}{-} \in \{\to, \leftrightarrow\},$$

meaning by that

$$L \vdash A_{k-1} \overset{k}{-} A_k, \qquad k = 1, \dots, n.$$

**Lemma 2.1.** (a) *The calculi* $L^{\rhd}$ *are closed with respect to the rule of equivalent replacement*

$$(\mathrm{RE})\frac{A \leftrightarrow B}{\rhd A \leftrightarrow \rhd B}.$$

(b) *We have the deducibility*

$$\mathbf{T}^{\rhd} \vdash \rhd p \,\&\, \rhd q \to \rhd (p \& q).$$

**Proof.** (a) By the (Dec) rule and the $(\mathrm{A}^{\rhd}_{\mathbf{T}})$ axiom, from $A \to B$, we obtain $A \to (\rhd A \to \rhd B)$, and from $\neg A \to \neg B$, we first obtain the formula $\neg A \to (\rhd \neg A \to \rhd \neg B)$, and then, by the axiom $(\mathrm{A}^{\rhd}_{\neg})$, the formula $\neg A \to (\rhd A \to \rhd B)$. In view of $\vdash A \vee \neg A$, from the two formulas obtained above, we derive $\rhd A \to \rhd B$. The inverse implication is proved in a similar way.

(b) By the axiom $(\mathrm{A}^{\rhd}_{\mathbf{T}})$, the tautology $p \to (q \to p\&q)$ implies $p\&q \to [\rhd p \to (\rhd q \to \rhd (p\&q))]$. By the axioms $(\mathrm{A}^{\rhd}_{\mathbf{T}})$ and $(\mathrm{A}^{\rhd}_{\neg})$, the tautology $\neg p \to \neg(p\&q)$ implies $\neg p \to [\rhd p \to \rhd (p\&q)]$ and, all the more, $\neg p \to [\rhd p \to (\rhd q \to \rhd (p\&q))]$. Similarly, we derive

$$\neg q \to [\rhd p \to (\rhd q \to \rhd (p\&q))].$$

Finally, by virtue of the tautology $(p\&q) \vee \neg p \vee \neg q$, we conclude that $\rhd p \to (\rhd q \to \rhd (p\&q))$. $\square$

For each of the logics $L$ in question, let us define two sequential calculi, $[L^{\rhd}_1]$ and $[L^{\rhd}_2]$. The calculus $[L^{\rhd}_1]$ is obtained from the sequential propositional calculus (with cut) by adding to it the rules $(\overset{\rhd}{\neg} \Rightarrow)$, $(\Rightarrow^{\rhd}_{\neg})$, and $(\Rightarrow^{\rhd}_{L})$ defined as follows:

$$(\overset{\rhd}{\neg} \Rightarrow)\frac{\rhd A, \Pi \Rightarrow \Sigma}{\rhd \neg A, \Pi \Rightarrow \Sigma}, \qquad (\Rightarrow^{\rhd}_{\mathbf{T}})\frac{\Pi \Rightarrow A}{\Pi, \rhd \Pi \Rightarrow \rhd A}, \qquad (\Rightarrow^{\rhd}_{\mathbf{B}})\frac{\Pi \Rightarrow (\rhd \Sigma\&\Sigma), A}{\Pi, \rhd \Pi \Rightarrow \Sigma, \rhd A},$$

$$(\Rightarrow^{\rhd}_{\neg})\frac{\Pi \Rightarrow \Sigma, \rhd A}{\Pi \Rightarrow \Sigma, \rhd \neg A}, \qquad (\Rightarrow^{\rhd}_{\mathbf{S4}})\frac{\Pi, \rhd \Pi \Rightarrow A}{\Pi, \rhd \Pi \Rightarrow \rhd A}, \qquad (\Rightarrow^{\rhd}_{\mathbf{S5}})\frac{\Pi, \rhd \Pi \Rightarrow \rhd \Sigma, A}{\Pi, \rhd \Pi \Rightarrow \rhd \Sigma, \rhd A},$$

$$(\Rightarrow^{\rhd}_{\mathbf{Grz}})\frac{\rhd (A \to \rhd A), \Pi, \rhd \Pi \Rightarrow A}{\Pi, \rhd \Pi \Rightarrow \rhd A}.$$

In the statement of the rule $(\Rightarrow^{\rhd}_{\mathbf{B}})$, we used the notation

$$(\rhd \Sigma\&\Sigma) := \{(\rhd \sigma\&\sigma) \mid \sigma \in \Sigma\}.$$

The rules $(\overset{\rhd}{\neg} \Rightarrow)$ and $(\Rightarrow^{\rhd}_{\neg})$ violate the subformula property. Let us introduce the calculus $[L^{\rhd}_2]$ in which these rules are absorbed by others. To obtain this calculus, we add to the sequential propositional calculus (with cut) the rules $(\Rightarrow^{\rhd r}_{L})$, $r \in \{0, 1\}$, defined as follows:

$$(\Rightarrow^{\rhd r}_{\mathbf{T}})\frac{A^{\bar r}, \Pi \Rightarrow \Lambda, A^r}{\Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda, \rhd A}, \qquad (\Rightarrow^{\rhd r}_{\mathbf{B}})\frac{\{A^{\bar r}, \Pi, \Phi' \Rightarrow \Phi, \rhd (\Psi'\Psi), \Lambda, A^r\}^{\Sigma=\Phi\Psi}_{\Sigma'=\Phi'\Psi'}}{\Pi, \rhd (\Pi\Lambda), \Sigma' \Rightarrow \Sigma, \Lambda, \rhd A},$$

$$(\Rightarrow^{\rhd r}_{\mathbf{S4}})\frac{A^{\bar r}, \Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda, A^r}{\Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda, \rhd A}, \qquad (\Rightarrow^{\rhd 0}_{\mathbf{Grz}})\frac{A, \rhd (A \vee \rhd A), \Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda}{\Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda, \rhd A},$$

$$(\Rightarrow^{\rhd r}_{\mathbf{S5}})\frac{A^{\bar r}, \Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda, \rhd \Sigma, A^r}{\Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda, \rhd \Sigma, \rhd A}, \qquad (\Rightarrow^{\rhd 1}_{\mathbf{Grz}})\frac{\rhd (A \to \rhd A), \Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda, A}{\Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda, \rhd A};$$

in these statements, we used the following notation: $\bar r := 1 - r$, $A^0 := \varnothing$, $A^1 := A$. The rules $(\Rightarrow^{\rhd r}_{\mathbf{B}})$, $r \in \{0, 1\}$, have $2^{|\Sigma|+|\Sigma'|}$ antecedents corresponding to all possible partitions of the multisets $\Sigma = \Phi\Psi$ and $\Sigma' = \Phi'\Psi'$.

In Sec. 4 we shall prove cut ineleminability in the calculi $[L^{\rhd}_k]$ constructed above. Now denote by $[L^{\rhd}_2]^-$ the calculi obtained from $[L^{\rhd}_2]$ by replacing the cut rule by *analytic* cut. It will follow from the Completeness Theorem (Theorem 4.1) that the calculi $[L^{\rhd}_2]^-$ and $[L^{\rhd}_2]$ are equivalent. Therefore, the following statement holds.

**Lemma 2.2.** (a) *The calculi* $[L^{\rhd}_2]$, *where* $L \in \{\mathbf{T}, \mathbf{S4}, \mathbf{S5}\}$, *satisfy the subformula property.*

(b) *The calculus* $[\mathbf{Grz}^{\rhd}_2]$ *satisfies the weak subformula property*: *any deducible sequent* $\Pi \Rightarrow \Sigma$ *admits a deduction consisting of sequents of the form* $\Gamma \Rightarrow \Delta$, *where* $\Delta \subseteq \mathrm{Sb}\,\Pi\Sigma$ *and*

$$\Gamma \subseteq \mathrm{Sb}(\Pi\Sigma \cup \{\rhd (A \to \rhd A), \rhd (A \vee \rhd A) \mid \rhd A \in \mathrm{Sb}\,\Pi\Sigma\}).$$

In what follows, we shall need the following fact.

**Lemma 2.3.** (a) *For* $L \in \{\mathbf{T}, \mathbf{S4}, \mathbf{B}, \mathbf{S5}, \mathbf{Grz}\}$, *the calculus* $[L_1^{\rhd}]$ *is closed with respect to the rule* (RE), *i.e.,* $[L_1^{\rhd}] \vdash A \Leftrightarrow B$ *implies* $[L_1^{\rhd}] \vdash \rhd A \Leftrightarrow \rhd B$.

(b) *We have the deducibility* $[\mathbf{T}_2^{\rhd}]^- \vdash \rhd (p \to \rhd p), p \Rightarrow \rhd p$.

(c) *We have the deducibility* $[\mathbf{T}_2^{\rhd}]^- \vdash \rhd (p \vee \rhd p) \Rightarrow p, \rhd p$.

**Proof.** (a) In $[\mathbf{T}_1^{\rhd}]$, we have the deduction

$$\dfrac{\dfrac{\dfrac{A \Rightarrow B}{A, \rhd A \Rightarrow \rhd B}}{\rhd A \Rightarrow \rhd B, \neg A,} \qquad \dfrac{\dfrac{B \Rightarrow A}{\neg A \Rightarrow \neg B}}{\neg A, \rhd \neg A \Rightarrow \rhd \neg B}}{\rhd A, \rhd \neg A \Rightarrow \rhd B, \rhd \neg B.}$$

Applying cuts with the sequent $\rhd A \Rightarrow \rhd \neg A$ (by the formula $\rhd \neg A$) and with the sequent $\rhd \neg B \Rightarrow \rhd B$ (by the formula $\rhd \neg B$), and then abbreviations, we obtain $\rhd A \Rightarrow \rhd B$. The converse sequent is proved similarly.

(b) Using analytic cut, we deduce $[\mathbf{T}_2^{\rhd}]^-$:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow \rhd p, p}{\Rightarrow (p \to \rhd p), p}}{\rhd (p \to \rhd p) \Rightarrow (p \to \rhd p), \rhd p} \qquad \dfrac{p \Rightarrow p \qquad \rhd p \Rightarrow \rhd p}{(p \to \rhd p), \quad p \Rightarrow \rhd p}}{\rhd (p \to \rhd p), p \Rightarrow \rhd p, \rhd p}}{\rhd (p \to \rhd p), p \Rightarrow \rhd p.}$$

(c) This item is similar to (b). $\square$

## 3. THE CLOSURE METHOD

In this section, we describe the method used to prove the completeness of an arbitrary consistent sequential calculus $\mathcal{L}$ (in the $\rhd$-language) with analytic cut.

**Definition 3.1.** A set of formulas $\Gamma$ is *closed* if $\operatorname{Sb}\Gamma \subseteq \Gamma$. A sequent $w$ is called *closed* if $\operatorname{Sb} w \subseteq w$, i.e., any subformula of a formula from $w$ is contained in the antecedent or succedent of the sequent $w$; the sequent $w$ is called *thin* if its antecedent and succedent are sets, i.e., the formulas in them do not repeat.

Obviously, for any finite (multi)set of formulas there exists the smallest *finite* closed set containing it. Let us construct the finite frame $F_{\mathcal{L}}^{\Gamma} := (W_{\mathcal{L}}^{\Gamma}, \uparrow)$ and the model $M_{\mathcal{L}}^{\Gamma} := (F_{\mathcal{L}}^{\Gamma}, \models)$, where $\Gamma \neq \varnothing$ is a finite closed set of formulas. Obviously, the set

$$W_{\mathcal{L}}^{\Gamma} := \{w \subseteq \Gamma \mid w \quad \text{is a closed thin sequent, } \mathcal{L} \nvdash w\}$$

is finite.

**Lemma 3.2** (The Closure Lemma). *Any sequent* $\Pi \Rightarrow \Sigma$ *not deducible in* $\mathcal{L}$ *and consisting of formulas from the set* $\Gamma$ *can be extended to a thin closed sequent not deducible in* $\mathcal{L}$. *Formally, if* $\Pi\Sigma \subseteq \Gamma$ *and* $\mathcal{L} \nvdash \Pi \Rightarrow \Sigma$, *then*

$$\exists w \in W_{\mathcal{L}}^{\Gamma} \colon \Pi \subseteq \langle w|, \Sigma \subseteq |w\rangle.$$

**Proof.** The lemma is proved by the standard closure method: if $\mathcal{L} \not\vdash \Pi \Rightarrow \Sigma$, $A \notin \Pi\Sigma$, and $A \in \mathrm{Sb}\,\Pi\Sigma$, then, by analytic cut in $\mathcal{L}$ (and abbreviation), we have $\mathcal{L} \not\vdash \Pi \Rightarrow \Sigma A$ or $\mathcal{L} \not\vdash A\Pi \Rightarrow \Sigma$; therefore, $A$ can be added to the antecedent or succedent of the sequent $\Pi \Rightarrow \Sigma$. The process continues until the sequent $\Pi \Rightarrow \Sigma$ becomes closed. $\square$

Note that since $\Gamma \neq \varnothing$, we have $\perp \in \Gamma$ or $p \in \Gamma$ for a certain variable $p$. This means that $\Gamma$ contains the sequent $\Rightarrow \perp$ or $\Rightarrow p$, which, obviously, is not deducible in $\mathcal{L}$. By the Closure Lemma, it can be embedded in a certain "world" $w \in W^{\Gamma}_{\mathcal{L}}$. Thus, $W^{\Gamma}_{\mathcal{L}} \neq \varnothing$.

We specify a valuation of variables by setting

$$w \models p \leftrightharpoons p \in \langle w| \qquad \text{for any} \quad w \in W^{\Gamma}_{\mathcal{L}} \quad \text{and} \quad p \in \mathbb{P}.$$

It remains to specify the relation $\uparrow$. Let us state a condition on $\uparrow$ that will suffice for our purposes:

$$\forall w \in W^{\Gamma}_{\mathcal{L}}\, \forall A \in w \qquad w \models A \Leftrightarrow A \in \langle w|. \tag*{$\langle 1^{\rhd}\rangle$}$$

**Lemma 3.3.** *If condition $\langle 1^{\rhd}\rangle$ is satisfied, then for any $\Pi\Sigma \subseteq \Gamma$ the formula $\mathcal{L} \not\vdash \Pi \Rightarrow \Sigma$ implies $M^{\Gamma}_{\mathcal{L}} \not\models \Pi \Rightarrow \Sigma$.*

**Proof.** By the Closure Lemma,

$$\Pi \subseteq \langle w| \qquad \text{and} \qquad \Sigma \subseteq |w\rangle$$

for a certain $w \in W^{\Gamma}_{\mathcal{L}}$. By $\langle 1^{\rhd}\rangle$, we have $w \models \bigwedge \Pi$ and $w \models \bigwedge \neg\Sigma$, i.e., $w \not\models \Pi \Rightarrow \Sigma$. $\square$

Further, let us show that to satisfy $\langle 1^{\rhd}\rangle$, it suffices to impose the following condition on $\uparrow$ (the bracket denotes the disjunction of conditions):

$$\forall w \in W^{\Gamma}_{\mathcal{L}}\, \forall \rhd B \in w \qquad \rhd B \in \langle w| \Leftrightarrow \left[\begin{array}{ll} \forall x \downarrow w & B \in \langle x|, \\ \forall x \downarrow w & B \in |x\rangle. \end{array}\right. \tag*{$\langle 2^{\rhd}\rangle$}$$

**Lemma 3.4.** *We have the implication $\langle 2^{\rhd}\rangle \implies \langle 1^{\rhd}\rangle$.*

**Proof.** The proof will be given simultaneously for all $w \in W^{\Gamma}_{\mathcal{L}}$ by induction on the construction of the formula $A \in w$. For $A \equiv \perp$, the left-hand and right-hand sides of $\langle 1^{\rhd}\rangle$ are false. For $A \equiv p$, the statement follows from the definition of $\models$.

Let $A \equiv (B \to C)$. Since the sequent $w$ is closed, $B, C \in w$, and by the induction hypothesis,

$$\begin{array}{lll} \text{(b)} & w \models B \Leftrightarrow B \in \langle w|, & w \not\models B \Leftrightarrow B \in |w\rangle; \\ \text{(c)} & w \models C \Leftrightarrow C \in \langle w|, & w \not\models C \Leftrightarrow C \in |w\rangle. \end{array}$$

Hence

$$w \models (B \to C) \overset{\mathrm{def}\models}{\Longleftrightarrow} \left[\begin{array}{l} w \not\models B \\ w \models C \end{array}\right. \overset{(b,c)}{\Longleftrightarrow} \left[\begin{array}{l} B \in |w\rangle \\ C \in \langle w| \end{array}\right. \overset{(?)}{\Longleftrightarrow} (B \to C) \in \langle w|.$$

Let us prove the equivalence marked by the question mark (?).

($\Rightarrow$) If $(B \to C) \in |w\rangle$, then $B \notin |w\rangle$ and $C \notin \langle w|$, since the sequents $\Rightarrow B$, $(B \to C)$, and $C \Rightarrow (B \to C)$ are provable in $\mathcal{L}$.

($\Leftarrow$) If $(B \to C) \in \langle w|$, then the conditions $B \in \langle w|$ and $C \in |w\rangle$ cannot hold simultaneously, because the sequent $(B \to C), B \Rightarrow C$ is provable in $\mathcal{L}$.

Finally, suppose that $A \equiv \rhd B$. By the induction hypothesis, for any $x \in W^{\Gamma}_{\mathcal{L}}$, if $B \in x$, then

$$\textbf{(x)} \qquad x \models B \Leftrightarrow B \in \langle x|, \qquad x \not\models B \Leftrightarrow B \in |x\rangle.$$

Hence

$$\rhd B \in \langle w| \overset{\langle 2^{\rhd}\rangle}{\Longrightarrow} \begin{bmatrix} \forall x \downarrow w & B \in \langle x| \\ \forall x \downarrow w & B \in |x\rangle \end{bmatrix} \overset{(\mathbf{x})}{\Longrightarrow} \begin{bmatrix} \forall x \downarrow w & x \models B \\ \forall x \downarrow w & x \not\models B \end{bmatrix} \overset{\mathrm{def}\models}{\Longrightarrow} w \models \rhd B,$$

$$\rhd B \in |w\rangle \overset{\langle 2^{\rhd}\rangle}{\Longrightarrow} \begin{cases} \exists x \downarrow w & B \in \langle x| \\ \exists y \downarrow w & B \in |y\rangle \end{cases} \overset{(\mathbf{x})}{\Longrightarrow} \begin{cases} \exists x \downarrow w & x \models B \\ \exists y \downarrow w & y \not\models B \end{cases} \overset{\mathrm{def}\models}{\Longrightarrow} w \not\models \rhd B.$$

Now the completeness of the logic $\mathcal{L}$ with respect to the class of finite frames $\mathcal{F}$ can be proved as follows. Suppose that $\mathcal{L} \nvdash \Pi \Rightarrow \Sigma$. We construct a finite closed set $\Gamma \supseteq \Pi\Sigma$ and the relation $\uparrow$ so that $F^{\Gamma}_{\mathcal{L}} \in \mathcal{F}$ and condition $\langle 2^{\rhd}\rangle$ holds. By Lemma 3.4, this condition implies $\langle 1^{\rhd}\rangle$; and by Lemma 3.3, we obtain $F^{\Gamma}_{\mathcal{L}} \not\models \Pi \Rightarrow \Sigma$, which was to be proved. $\square$

## 4. COMPLETENESS OF AXIOMATICS

The theorem proved in this section states that the Hilbert and sequential calculi constructed above yield a complete axiomatization of noncontingency logics over $\mathbf{T}$, $\mathbf{S4}$, $\mathbf{B}$, $\mathbf{S5}$, and $\mathbf{Grz}$. At the end of the section, we axiomatize the logic $\mathbf{S4.1}^{\rhd}$.

**Theorem 4.1** (The Joint Completeness Theorem). *For each logic $L \in \{\mathbf{T}, \mathbf{S4}, \mathbf{B}, \mathbf{S5}, \mathbf{Grz}\}$ and any sequent $\Pi \Rightarrow \Sigma$ in the $\rhd$-language, the following statements are equivalent*:

(1) $[L^{\rhd}_2]^- \vdash \Pi \Rightarrow \Sigma$,
(2) $[L^{\rhd}_1] \vdash \Pi \Rightarrow \Sigma$,
(3) $L^{\rhd} \vdash \bigwedge \Pi \to \bigvee \Sigma$,
(4) $L \vdash (\bigwedge \Pi \to \bigvee \Sigma)_{\rhd}$,
(5) $F \models \Pi \Rightarrow \Sigma$ *for any finite $L$-frame $F$*.

**Proof.** The proof will follow the scheme (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longleftrightarrow$ (5) $\Longrightarrow$ (1). The implication (2) $\Longrightarrow$ (3) is proved by induction on the construction of a deduction in $[L^{\rhd}_1]$; in so doing, the steps corresponding to the rules $(^{\rhd}_{\neg} \Rightarrow)$ and $(\Rightarrow^{\rhd}_{\neg})$ are obvious, since logics $L^{\rhd}$ contain the axiom $(\mathrm{A}^{\rhd}_{\neg})$; therefore, we only have to verify the steps corresponding to the rule $(\Rightarrow^{\rhd}_{L})$. Then, it will suffice to prove the implication (3) $\Longrightarrow$ (4) only for $\Pi = \varnothing$ and $\Sigma = \{A\}$, i.e., to verify the deducibility of $\rhd$-translations of the axioms $L^{\rhd}$ in $L$. For the axiom $(\mathrm{A}^{\rhd}_{\neg})$, this verification is trivial, and for $(\mathrm{A}^{\rhd}_{\mathbf{T}})$, $(\mathrm{A}^{\rhd}_{\mathbf{4}})$, and $(\mathrm{A}^{\rhd}_{\mathbf{5}})$, it was carried out in [2, 3]. Further, the equivalence (4) $\Longleftrightarrow$ (5) is the familiar completeness theorem for logics $L$ (see [16, 17]). Finally, in the proof of the implication (5) $\Longrightarrow$ (1), we use the notation $\mathcal{L} := [L^{\rhd}_2]^-$.

(1) $\Longrightarrow$ (2) It will suffice to show that $(\Rightarrow^{\rhd r}_{L})$ are derived rules in $[L^{\rhd}_1]$. For instance, let us deduce the conclusion of the rule $(\Rightarrow^{\rhd 0}_{\mathbf{T}})$ from its premise in the calculus $[\mathbf{T}^{\rhd}_1]$:

$$\cfrac{\cfrac{A, \Pi \Rightarrow \Lambda}{\Pi, \neg\Lambda \Rightarrow \neg A}}{\Pi, \neg\Lambda, \rhd \Pi, \rhd \neg\Lambda \Rightarrow \rhd \neg A.}$$

Applying cut with the sequent $\rhd \neg A \Rightarrow \rhd A$ (by the formula $\rhd \neg A$), and with the sequents $\Rightarrow C$, $\neg C$ (by $\neg C$) and $\rhd C \Rightarrow \rhd \neg C$ (by $\rhd \neg C$) for all $C \in \Lambda$, we obtain $\Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda, \rhd A$.

The consideration of the rule $(\Rightarrow^{\rhd 0}_{\mathbf{Grz}})$ requires the noncontingency of the sequent

$$\rhd (\neg A \to \rhd \neg A) \Rightarrow \rhd (A \vee \rhd A),$$

in $[\mathbf{Grz}^{\rhd}_1]$, which follows from Lemma 2.3 (a).

*Logic* $\mathbf{T}$. (2) $\Longrightarrow$ (3) By Lemma 2.1 (b), $\mathbf{T}^{\rhd} \vdash \wedge \rhd \Pi \to \rhd \wedge \Pi$; therefore, we deduce in $\mathbf{T}^{\rhd}$:

$$\cfrac{\cfrac{\cfrac{\bigwedge\Pi \to B}{\rhd(\bigwedge\Pi \to B)}}{\bigwedge\Pi \to [\rhd \bigwedge\Pi \to \rhd B]}}{\bigwedge\{\Pi, \rhd\Pi\} \to \rhd B.}$$

(5) $\Longrightarrow$ (1) Suppose that $\mathcal{L} \nvdash \Pi \Rightarrow \Sigma$. Let us take the finite closed set $\Gamma := \mathrm{Sb}\,\Pi\Sigma$. On $W_{\mathcal{L}}^{\Gamma}$, define the reflexive relation

$$w \uparrow x \leftrightharpoons \forall C \in \mathbf{Fm}^{\rhd} \qquad \rhd C \in \langle w| \Rightarrow \begin{cases} C \in \langle w| \Rightarrow C \in \langle x|, \\ C \in |w\rangle \Rightarrow C \in |x\rangle. \end{cases} \tag{$3_{\mathbf{T}}^{\rhd}$}$$

Then $F_{\mathcal{L}}^{\Gamma}$ is a finite $\mathbf{T}$-frame, and it remains to verify condition $\langle 2^{\rhd}\rangle$.

**Lemma 4.2.** *We have the implication* $\langle 3_{\mathbf{T}}^{\rhd}\rangle \Longrightarrow \langle 2^{\rhd}\rangle$.

**Proof.** Let us prove the equivalence in $\langle 2^{\rhd}\rangle$. We take $w \in W_{\mathcal{L}}^{\Gamma}$ and $\rhd B \in w$.

($\Rightarrow$) Suppose that $\rhd B \in \langle w|$. By the closure of $w$, two cases are possible:

(1) $B \in \langle w|$; then for all $x \downarrow w$, by $\langle 3_{\mathbf{T}}^{\rhd}\rangle$, we obtain $B \in \langle x|$;
(2) $B \in |w\rangle$; similarly, for all $x \downarrow w$, we obtain $B \in |x\rangle$.

($\Leftarrow$) Suppose that $\rhd B \in |w\rangle$. Let us construct $x, y \downarrow w$ such that $B \in \langle x, B \in |y\rangle$.

*Case* $B \in |w\rangle$. The choice of $y$ is obvious: $y := w$. We set

$$\Pi := \{C \in \langle w| \,\big|\, \rhd C \in \langle w|\}, \qquad \Lambda := \{C \in |w\rangle \,\big|\, \rhd C \in \langle w|\}.$$

Then $\mathcal{L} \nvdash B,\ \Pi \Rightarrow \Lambda$, because otherwise, by the rule $(\Rightarrow_{\mathbf{T}}^{\rhd 0})$, we would obtain $\mathcal{L} \vdash \Pi, \rhd(\Pi\Lambda) \Rightarrow \Lambda, \rhd B$, whence $\mathcal{L} \vdash w$ by weakening. By the Closure Lemma, there exists an $x \in W_{\mathcal{L}}^{\Gamma}$ such that $\Pi \subseteq \langle x|,\ B \in \langle x|,\ \Lambda \subseteq |x\rangle$. Let us prove that $w \uparrow x$. Suppose that $\rhd C \in \langle w|$. If $C \in \langle w|$, then $C \in \Pi \subseteq \langle x|$; and if $C \in |w\rangle$, then $C \in \Lambda \subseteq |x\rangle$.

*Case* $B \in \langle w|$. Setting $x := w$ and using $(\Rightarrow_{\mathbf{T}}^{\rhd 1})$, we construct the desired $y$ in a similar way (for the remaining logics, we shall usually consider only the first case). This proves the lemma. $\square$

*Logic* $\mathbf{S4}$. (2) $\Longrightarrow$ (3) A deduction in $\mathbf{S4}^{\rhd}$:

$$\cfrac{\cfrac{\cfrac{\bigwedge\{\Pi, \rhd\Pi\} \to B}{\bigwedge\{\Pi, \rhd\Pi\} \to [\rhd \bigwedge\{\Pi, \rhd\Pi\} \to \rhd B]}}{\bigwedge\{\Pi, \rhd\Pi, \rhd\rhd\Pi\} \to \rhd B}}{\bigwedge\{\Pi, \rhd\Pi\} \to \rhd B.}$$

(5) $\Longrightarrow$ (1) On $W_{\mathcal{L}}^{\Gamma}$, we introduce the reflexive and transitive relation

$$w \uparrow x \leftrightharpoons \forall C \in \mathbf{Fm}^{\rhd} \qquad \rhd C \in \langle w| \Rightarrow \rhd C \in \langle x| \,\&\, \begin{cases} C \in \langle w| \Rightarrow C \in \langle x|, \\ C \in |w\rangle \Rightarrow C \in |x\rangle. \end{cases} \tag{$3_{\mathbf{S4}}^{\rhd}$}$$

In the proof of $\langle 2^{\rhd}\rangle$, we have $\mathcal{L} \nvdash B, \Pi, \rhd(\Pi\Lambda) \Rightarrow \Lambda$; otherwise, by the rule $(\Rightarrow_{\mathbf{S4}}^{\rhd 0})$, we obtain $\mathcal{L} \vdash w$. For $x \in W_{\mathcal{L}}^{\Gamma}$ such that $\Pi, \rhd(\Pi\Lambda) \subseteq \langle x|$ and $\Lambda \subseteq |x\rangle$, it is obvious that $w \uparrow x$.

*Logic* **B**. (2) $\Longrightarrow$ (3) Setting $\Omega := \neg\Sigma$ and using the formulas $p \to (\rhd\, p \to p)$ and $p \to\rhd (\rhd\, p \to p)$ deducible in $\mathbf{B}^{\rhd}$, we construct the inference in $\mathbf{B}^{\rhd}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bigwedge \Pi \to \bigvee\{(\rhd\, \Sigma \& \Sigma), B\}}{\bigwedge\{\Pi, (\rhd\, \Omega \to \Omega)\} \to B}}{\bigwedge\{\Pi, \rhd\, \Pi, (\rhd\, \Omega \to \Omega), \rhd\, (\rhd\, \Omega \to \Omega)\} \to\rhd B}}{\bigwedge\{\Pi, \rhd\, \Pi, \Omega\} \to\rhd B}}{\bigwedge\{\Pi, \rhd\, \Pi\} \to \bigvee\{\Sigma, \rhd\, B\}.}$$

(3) $\Longrightarrow$ (4) Let us deduce the $\rhd$-translation of the axiom $(\mathrm{A}^{\rhd}_{\mathbf{B}})$ in the logic $\mathbf{B}$. We have

$$\begin{aligned}\mathbf{B} \vdash p &\longrightarrow \Box\Diamond p \longleftrightarrow \Box[\Diamond p \& (\neg p \to \Diamond\neg p)]\\ &\longleftrightarrow \Box[p \vee (\Diamond p \& \Diamond\neg p)] \longleftrightarrow \Box(p \vee \neg \rhd\, p) \longrightarrow\rhd (\rhd\, p \to p).\end{aligned}$$

(5) $\Longrightarrow$ (1) First, by condition $\langle 3^{\rhd}_{\mathbf{T}}\rangle$, we introduce the reflexive relation $\uparrow$ on $W^{\Gamma}_{\mathcal{L}}$, and then we take its symmetrization:

$$w \Uparrow x \leftrightharpoons (w \uparrow x) \& (x \uparrow w). \qquad \langle 3^{\rhd}_{\mathbf{B}}\rangle$$

To prove $\langle 2^{\rhd}\rangle$, we take the same $\Pi$ and $\Lambda$ as before and set

$$\Sigma := \{C \in |w\rangle \big| \rhd\, C \in |w\rangle\}, \qquad \Sigma' := \{C \in \langle w| \big| \rhd\, C \in |w\rangle\}.$$

There exists a partition $\Sigma = \Phi\Psi$, $\Sigma' = \Phi'\Psi'$ such that $\mathcal{L} \nvdash B, \Pi, \Phi' \Rightarrow \Phi, \rhd\, (\Psi'\Psi), \Lambda$; otherwise, starting from all possible sequents of this form by the rule $(\Rightarrow^{\rhd 0}_{\mathbf{B}})$, we would derive[3] the sequent $\Pi, \rhd\, (\Pi\Lambda), \Sigma' \Rightarrow \Sigma, \Lambda, \rhd\, B$ and further, by weakening, $\mathcal{L} \vdash w$.

It remains to check $w \Uparrow x$ for all $x \in W^{\Gamma}_{\mathcal{L}}$ such that $\Pi\Phi' \subseteq \langle x|$ and $\Phi, \rhd\, (\Psi'\Psi), \Lambda \subseteq |x\rangle$. Notice that $x \subseteq w$. The condition $w \uparrow x$ is verified as in the case of the logic $\mathbf{T}$. Let us prove that $x \uparrow w$. Suppose that $\rhd\, C \in \langle x|$. Then $\rhd\, C \in w$ in view of $x \subseteq w$, and $C \in w$ by the closure of $w$.

Further, suppose that $C \in \langle x|$. If $C \in |w\rangle$, then we would have the following cases:

(1) $\rhd\, C \in \langle w|$; then $C \in \Lambda \subseteq |x\rangle$, which is not true;
(2) $\rhd\, C \in |w\rangle$; then $C \in \Sigma = \Phi\Psi$. Now we have: if $C \in \Phi$, then $C \in |x\rangle$, which is not true; and if $C \in \Psi$, then $\rhd\, C \in\rhd \Psi \subseteq |x\rangle$, which is not true either.

Now suppose that $C \in |x\rangle$. If we had $C \in \langle w|$, then the following cases would be possible:

(1) $\rhd\, C \in \langle w|$; then $C \in \Pi \subseteq \langle x|$, which is not true;
(2) $\rhd\, C \in |w\rangle$; then $C \in \Sigma' = \Phi'\Psi'$. Now we have: if $C \in \Phi'$, then $C \in \langle x|$, which is not true; and if $C \in \Psi'$, then $\rhd\, C \in\rhd \Psi' \subseteq |x\rangle$, which is not true either.

*Logic* **S5**. (2) $\Longrightarrow$ (3) We construct a deduction in $\mathbf{S5}^{\rhd}$:

$$\dfrac{\dfrac{\dfrac{\bigwedge\{\Pi, \rhd\, \Pi, \neg \rhd\, \Sigma\} \to B}{\bigwedge\{\Pi, \rhd\, \Pi, \neg \rhd\, \Sigma\} \to [\rhd \bigwedge\{\Pi, \rhd\, \Pi, \neg \rhd\, \Sigma\} \to\rhd B]}}{\bigwedge\{\Pi, \rhd\, \Pi, \neg \rhd\, \Sigma, \rhd\rhd\, \Pi, \rhd\, \neg \rhd\, \Sigma\} \to\rhd B}}{\bigwedge\{\Pi, \rhd\, \Pi, \neg \rhd\, \Sigma\} \to\rhd B.}$$

[3]If, in this application of the rule $(\Rightarrow^{\rhd 0}_{\mathbf{B}})$, we could confine ourselves to the assumption $\Sigma\Sigma' \subseteq \mathrm{Sb}(\Pi\Lambda B)$, then we would incidentally establish the subformula property for the calculus $[\mathbf{B}^{\rhd}_{2}]$.

(5) $\Longrightarrow$ (1) First we introduce a reflexive transitive relation $\uparrow$ on $W^\Gamma_{\mathcal{L}}$ by the condition $\langle 3^{\rhd}_{\mathbf{S4}}\rangle$, and then we take its symmetrization:

$$w \Uparrow x \leftrightharpoons (w \uparrow x)\&(x \uparrow w). \qquad \langle 3^{\rhd}_{\mathbf{S5}}\rangle$$

To prove $\langle 2^{\rhd}\rangle$, we take $\Sigma := \{C \mid \rhd C \in |w\rangle\}$ and chose $\Pi$ and $\Lambda$ as before. If we had $\mathcal{L} \vdash B, \Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda, \rhd \Sigma$, then, by the rule $(\Rightarrow^{\rhd 0}_{\mathbf{S5}})$, we would deduce $\mathcal{L} \vdash w$. It remains to verify $w \Uparrow x$ for $x \in W^\Gamma_{\mathcal{L}}$ such that $\Pi, \rhd (\Pi\Lambda) \subseteq \langle x|$ and $\Lambda, \rhd \Sigma \subseteq |x\rangle$. Note that $x \subseteq w$.

Let us prove that $w \uparrow x$. If $\rhd C \in \langle w|$, then $C \in \Pi\Lambda$ and $\rhd C \in \langle x|$. Further, if $C \in \langle w|$, then $C \in \Pi \subseteq \langle x|$. And if $C \in |w\rangle$, then $C \in \Lambda \subseteq |x\rangle$.

Let us prove that $x \uparrow w$. Suppose that $\rhd C \in \langle x|$. Then $\rhd C \in w$, because $x \subseteq w$, and if we had $\rhd C \in |w\rangle$, then $C \in \Sigma$ and $\rhd C \in |x\rangle$, which is not true; therefore, $\rhd C \in \langle w|$. Further, if $C \in \langle x|$, then $C \in w$, and if we had $C \in |w\rangle$, then, by the inclusion $\rhd C \in \langle w|$ proved above, we would have $C \in \Lambda \subseteq |x\rangle$, which is not true; therefore, $C \in \langle w|$. And if $C \in |x\rangle$, then $C \in w$, and in the case $C \in \langle w|$, in view of $\rhd C \in \langle w|$, we would have $C \in \Pi \subseteq \langle x|$, which is not true; therefore, $C \in |w\rangle$.

*Logic* **Grz**.

(2) $\Longrightarrow$ (3) In $\mathbf{Grz}^{\rhd}$, using the axiom $(\mathrm{A}^{\rhd}_{\mathbf{4}})$ at the last step, we deduce:

$$\dfrac{\dfrac{\dfrac{\bigwedge\{\Pi, \rhd \Pi\} \to (\rhd (B \to\rhd B) \to B)}{\bigwedge\{\Pi, \rhd \Pi\} \to [\rhd \bigwedge\{\Pi, \rhd \Pi\} \to\rhd (\rhd (B \to\rhd B) \to B)]}}{\bigwedge\{\Pi, \rhd \Pi, \rhd\rhd \Pi\} \to\rhd B}}{\bigwedge\{\Pi, \rhd \Pi\} \to\rhd B.}$$

(3) $\Longrightarrow$ (4) Let us prove the $\rhd$-translation of the axiom $(\mathrm{A}^{\rhd}_{\mathbf{G}})$ in **Grz**. On the one hand, $\mathbf{Grz} \vdash \Box p \to p$, and so

$$\mathbf{Grz} \vdash (p \to\rhd p) \longleftrightarrow (p \to \Box p).$$

Hence

$$\begin{aligned}\mathbf{Grz} \vdash \Box\neg(p \to\rhd p) &\longleftrightarrow \Box\neg(p \to \Box p)\\ &\longleftrightarrow [\Box p \& \Box\neg\Box p] \longleftrightarrow \neg[\Box p \to \Diamond\Box p] \longleftrightarrow \bot.\end{aligned}$$

Then we deduce in **Grz**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box[\Box(p \to \Box p) \to p] \to p}{\Box[\Box(p \to\rhd p) \vee \bot \to p] \to p}}{\Box[\Box(p \to\rhd p) \vee \Box\neg(p \to\rhd p) \to p] \to p}}{\Box[\rhd (p \to\rhd p) \to p] \to p}}{\Box[\rhd (p \to\rhd p) \to p] \to \Box p.}$$

On the other hand,

$$\mathbf{Grz} \vdash \Box\neg[\rhd (p \to\rhd p) \to p] \longleftrightarrow [\Box \rhd (p \to\rhd p) \& \Box\neg p] \longrightarrow \Box\neg p \longrightarrow\rhd p.$$

(5) $\Longrightarrow$ (1) We shall slightly modify the method of proof described in Sec. 3. Suppose that $\mathcal{L} \nvdash \Pi \Rightarrow \Sigma$. Let us take $\Gamma := \mathrm{Sb}\,\Pi\Sigma$,

$$\widehat{\Gamma} := \Gamma \cup \mathrm{Sb}\{\rhd (A \to\rhd A), \rhd (A \vee \rhd A) \mid \rhd A \in \Gamma\}.$$

The set

$$W^\Gamma_{\mathcal{L}} := \{w \mid w \text{ is a closed thin sequent}, \ \langle w| \subseteq \widehat{\Gamma},\ |w\rangle \subseteq \Gamma,\ \mathcal{L} \nvdash w\}$$

is finite.

**Lemma 4.3** (The Closure Lemma). *Any sequent $\Pi \Rightarrow \Sigma$ not deducible in $\mathcal{L}$ such that $\Pi \subseteq \widehat{\Gamma}$ and $\Sigma \subseteq \Gamma$, can be extended to a sequent from $W^\Gamma_{\mathcal{L}}$. Formally, if $\mathcal{L} \nvdash \Pi \Rightarrow \Sigma$, with $\Pi \subseteq \widehat{\Gamma}$ and $\Sigma \subseteq \Gamma$, then there exists a $w \in W^\Gamma_{\mathcal{L}}$ such that $\Pi \subseteq \langle w|$, $\Sigma \subseteq |w\rangle$.*

**Proof.** In addition to the proof of Lemma 3.2, we must check that if in the process of closure, the sequent $\Pi' \Rightarrow \Sigma'$, nondeducible in $\mathcal{L}$, is obtained from $\Pi \Rightarrow \Sigma$ by the addition of the formula $A \in \mathrm{Sb}\,\Pi\Sigma$, $A \notin \Pi\Sigma$, to the antecedent or succedent, then $\Pi' \subseteq \widehat{\Gamma}$ and $\Sigma' \subseteq \Gamma$. The first inclusion is obvious. For $A \in \Gamma$, the second one is as well. And if $A \in (\widehat{\Gamma} \setminus \Gamma)$, then in view of $A \notin \Pi\Sigma$, the formula $A$ is either $(B \vee \rhd B)$ or $(B \to \rhd B)$ for a certain $\rhd B \in \Pi\Sigma$, with $\rhd A \in \Pi$. In both cases $\mathcal{L} \vdash \rhd A \Rightarrow A$, which follows from Lemma 2.3 (b, c), and so $\mathcal{L} \vdash \Pi \Rightarrow \Sigma A$. Therefore, the formula $A$ could not be added to the succedent of the sequent $\Pi \Rightarrow \Sigma$, and hence $\Sigma' = \Sigma \subseteq \Gamma$. $\square$

As before, for any $w \in W^\Gamma_{\mathcal{L}}$ and $p \in \mathbb{P}$, we set $w \models p \leftrightharpoons p \in \langle w|$. The statements and proofs of Lemmas 3.3 and 3.4 are carried over to our case without substantial modifications. First, we introduce a transitive relation $\uparrow$ on $W^\Gamma_{\mathcal{L}}$ by the condition $\langle 3^{\rhd}_{\mathbf{S4}}\rangle$; then, the irreflexive transitive relation

$$w \prec x \leftrightharpoons (w \uparrow x) \& (\exists C \in \mathbf{Fm}^{\rhd} \quad \rhd C \notin \langle w| \& \rhd C \in \langle x|) \qquad \langle 3^{\rhd}_{\mathbf{Grz}}\rangle$$

and, finally, the reflexive transitive antisymmetric relation, i.e., a partial order

$$w \preccurlyeq x \leftrightharpoons (w \prec x) \vee (w = x).$$

Thus a finite **Grz**-frame $F^\Gamma_{\mathcal{L}} := (W^\Gamma_{\mathcal{L}}, \preccurlyeq)$ is constructed. It remains to check condition $\langle 2^{\rhd}\rangle$, which now takes the form

$$\forall w \in W^\Gamma_{\mathcal{L}} \forall \rhd B \in w \qquad \rhd B \in \langle w \,| \Leftrightarrow \left[\begin{array}{ll} \forall x \succcurlyeq w & B \in \langle x|, \\ \forall x \succcurlyeq w & B \in |x\rangle. \end{array}\right. \qquad \langle 2^{\rhd}\rangle$$

**Lemma 4.4.** *We have the implication $\langle 3^{\rhd}_{\mathbf{Grz}}\rangle \implies \langle 2^{\rhd}\rangle$.*

**Proof.** Let us prove equivalence in $\langle 2^{\rhd}\rangle$. Let us take any $w \in W^\Gamma_{\mathcal{L}}$ and $\rhd B \in w$.

($\Rightarrow$) Suppose that $\rhd B \in \langle w|$. The following two cases are possible:

(1) $B \in \langle w|$; then for all $x \succcurlyeq w$, we have either $w \prec x$, $w \uparrow x$, and $B \in \langle x|$ by $\langle 3^{\rhd}_{\mathbf{S4}}\rangle$ or $x = w$ and $B \in \langle w| = \langle x|$;

(2) $B \in |w\rangle$; similarly, for all $x \succcurlyeq w$, we obtain $B \in |x\rangle$.

($\Leftrightarrow$) Suppose that $\rhd B \in |w\rangle$. Let us take $\Pi, \Lambda$ as in the proof of Lemma 4.2.

*Case* $B \in |w\rangle$. Set $y := w$. Then we have

$$\mathcal{L} \nvdash B, \rhd (B \vee \rhd B), \Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda\,;$$

otherwise, by the rule $(\Rightarrow^{\rhd 0}_{\mathbf{Grz}})$ and the weakening rules, we obtain $\mathcal{L} \vdash w$. Since $\rhd B \in |w\rangle \subseteq \Gamma$, the antecedent of the sequent written above is contained in $\widehat{\Gamma}$, and the succedent in $\Gamma$. By the Closure Lemma, this sequent can be emdedded into a certain sequent $x \in W^\Gamma_{\mathcal{L}}$. It remains to check that $w \prec x$. The condition $w \uparrow x$ is checked as in the case of the logic **S4**. Further, $\rhd (B \vee \rhd B) \in \langle x|$. But $\rhd (B \vee \rhd B) \notin \langle w|$; otherwise, taking into account that $B, \rhd B \in |w\rangle$, by Lemma 2.3 (c) we would even obtain $[\mathbf{T}^{\rhd}_2]^- \vdash w$.

*Case* $B \in \langle w|$. Now $x := w$, and we similarly have

$$\mathcal{L} \nvdash \rhd (B \to \rhd B), \Pi, \rhd (\Pi\Lambda) \Rightarrow \Lambda, B.$$

As before, we embed this sequent into a certain $y \in W^\Gamma_{\mathcal{L}}$. Obviously, $w \uparrow y$. Finally, $w \prec y$, since $\rhd (B \to \rhd B) \in \langle y|$; but $\rhd (B \to \rhd B) \notin \langle w|$ by Lemma 2.3 (b). $\square$

This completes the proof of the theorem. $\square$

Recall that if reflexivity holds, then the operator $\Box$ can be expressed in terms of $\triangleright$ by the equation $\Box p = p \& \triangleright p$. We shall use this fact to introduce the translation $\mathrm{Tr}\colon \mathbf{Fm}^{\Box} \to \mathbf{Fm}^{\triangleright}$ which preserves variables and Boolean connectives and acts on formulas of the form $\Box A$ as follows:

$$\mathrm{Tr}(\Box A) = \mathrm{Tr}(A) \& \triangleright \mathrm{Tr}(A).$$

Further, for an arbitrary $\triangleright$-logic $M$, we set

$$M^{\boxdot} := \{A \in \mathbf{Fm}^{\Box} \mid \mathrm{Tr}(A) \in M\} = \mathrm{Tr}^{-1}(M).$$

It is readily seen that the translations tr and Tr are mutually inverse in the following sense: $\mathbf{T} \vdash \mathrm{tr}(\mathrm{Tr}(\Box p)) \leftrightarrow \Box p$ and $\mathbf{T}^{\triangleright} \vdash \mathrm{Tr}(\mathrm{tr}(\triangleright p)) \leftrightarrow \triangleright p$. It follows that $(L^{\triangleright})^{\boxdot} = L$ for any $\Box$-logic $L$ containing axiom $(\mathrm{A}^{\triangleright}_{\mathbf{T}})$, and that $(M^{\boxdot})^{\triangleright} = M$ for any $\triangleright$-logic $M$ containing axiom $(\mathrm{A}^{\triangleright}_{\neg})$. Hence the condition $L^{\triangleright} = M$ is equivalent to the conjunction of conditions $[M \subseteq L^{\triangleright}$ and $L \subseteq M^{\boxdot}]$. The last statement makes it possible to construct the axiomatics of noncontingency logic over any normal logic containing $\mathbf{T}$.

**Lemma 4.5.** *Suppose that a normal logic $L$ is axiomatized over $\mathbf{T}$ by the set of axioms $\Gamma \subseteq \mathbf{Fm}^{\Box}$. Then the noncontingency logic over $L$ has the following axiomatics:*

$$L^{\triangleright} = \mathbf{T}^{\triangleright} + \mathrm{Tr}(\Gamma), \qquad \textit{where} \quad \mathrm{Tr}(\Gamma) := \{\mathrm{Tr}(A) \mid A \in \Gamma\}.$$

Using this lemma, it is easy to check that $\mathbf{S4.1}^{\triangleright} = \mathbf{S4}^{\triangleright} + (\mathrm{A}^{\triangleright}_{\mathbf{1}})$, where $(\mathrm{A}^{\triangleright}_{\mathbf{1}})$ is the axiom $\triangleright\triangleright p \to \triangleright p$. Finally, let us show that the transition $L \mapsto L^{\triangleright}$ is an injective homomorphism of the lattice of extensions of the logic $\mathbf{T}$.

**Lemma 4.6.** *If $\Box$-logics $L$ and $M$ contain $\mathbf{T}$, then*

$$L \subset M \Leftrightarrow L^{\triangleright} \subset M^{\triangleright}.$$

**Proof.** It will suffice to verify the preservation of nonstrict inclusion. From $L \subseteq M$, it follows that $L^{\triangleright} \subseteq M^{\triangleright}$. Conversely, if $L^{\triangleright} \subseteq M^{\triangleright}$, then $L = (L^{\triangleright})^{\boxdot} \subseteq (M^{\triangleright})^{\boxdot} = M$. $\Box$

## 5. CUT INELIMINABILITY AND INTERPOLATION

Here we shall establish that in all the sequential calculi $[L^{\triangleright}_k]$, $k = 1, 2$, constructed in Sec. 2 cut is ineliminable, but at the same time, the logics $L^{\triangleright}$ satisfy the Craig interpolation property.

**Theorem 5.1.** *In the calculi $[L^{\triangleright}_k]$, where $L \in \{\mathbf{T}, \mathbf{S4}, \mathbf{S5}, \mathbf{B}, \mathbf{Grz}\}$, $k = 1, 2$, cut is ineliminable.*

**Proof.** (1) The sequent $\triangleright (p \to \triangleright p)$, $p \Rightarrow \triangleright p$ is deducible in $[\mathbf{T}^{\triangleright}_2]^-$ (see Lemma 2.3 (b)), and so, in all the calculi $[L^{\triangleright}_k]$ in question. Let us show that this sequent is not cut free deducible in $[\mathbf{B}^{\triangleright}_1]$ and in $[L^{\triangleright}_k]$ for $L \neq \mathbf{B}$.

Suppose that it can be cut free deduced in one of these calculi. Then the last application of a nonstructural rule in this deduction could only be the application of one of the rules $(\overset{\triangleright}{\neg} \Rightarrow)$, $(\Rightarrow^{\triangleright}_{\neg})$, $(\Rightarrow^{\triangleright}_{L})$, or $(\Rightarrow^{\triangleright r}_{L})$; the first two are excluded immediately, because the formulas of the form $\triangleright \neg A$ are inherited in cut free deductions, and in our sequent there are no such formulas or subformulas. The conclusion of this application must be of the form

$$[\triangleright (p \to \triangleright p)]^l, [p]^m \Rightarrow [\triangleright p]^n, \qquad l, m, n \geq 0,$$

since only the weakening and abbreviation rules were applied after it in this deduction. It is readily seen that for $L \neq \mathbf{B}$, the conclusions of the rules $(\Rightarrow^{\triangleright}_{L})$ and $(\Rightarrow^{\triangleright r}_{L})$ can be of this form only for

$l = m = 0$ and $n > 0$. However, it is clear from semantic considerations (usind the Completeness Theorem proved above) that the sequent $\Rightarrow [\rhd\, p]^n$ is not deducible in the calculi in question.

(2) Let us show that the sequent $\rhd\, p \Rightarrow \rhd\, \neg p$ is deducible even in $[\mathbf{T}_2^{\rhd}]^-$, but is not cut free deducible in $[\mathbf{B}_2^{\rhd}]$. We have the following deduction in $[\mathbf{T}_2^{\rhd}]^-$:

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\Rightarrow p, \neg p}}{\rhd\, p \Rightarrow p, \rhd\, \neg p} \quad \dfrac{\dfrac{p \Rightarrow p}{p, \neg p \Rightarrow}}{p, \rhd\, p \Rightarrow \rhd\, \neg p}}{\rhd\, p \Rightarrow \rhd\, \neg p.}$$

Suppose that this sequent can be cut free deduced in $[\mathbf{B}_2^{\rhd}]$. The last application of a nonstructural rule in this deduction could only be the application of the rule $(\Rightarrow_{\mathbf{B}}^{\rhd r})$. Its conclusion is of the form: $[\rhd\, p]^m \Rightarrow [\rhd\, \neg p]^n$, where $m, n \geq 0$. Comparing this sequent with the notations from the statement of the rule $(\Rightarrow_{\mathbf{B}}^{\rhd r})$, we have: $\Pi = \Lambda = \varnothing$, $\Sigma' = [\rhd\, p]^m$, $\Sigma = [\rhd\, \neg p]^{n-1}$. The premise of this application that corresponds to the partition $\Phi = \Phi' = \varnothing$, $\Psi = \Sigma$, and $\Psi' = \Sigma'$, will be of the form

$$p^{\bar{r}} \Rightarrow p^r, [\rhd\rhd\, p]^m, [\rhd\rhd\, \neg p]^n, \qquad \text{where} \quad r \in \{0, 1\}.$$

Let us show that the last sequent is not deducible in $[\mathbf{B}_2^{\rhd}]$. Otherwise, by the Completeness Theorem, $\mathbf{B}^{\rhd} \vdash p^{\bar{r}} \to (p^r \vee \rhd\rhd\, p \vee \rhd\rhd\, \neg p)$. Suppose that $r = 0$ (the case $r = 1$ is considered similarly). Then by the axiom $(\mathrm{A}_{\neg}^{\rhd})$ and the rule (RE), we obtain $\mathbf{B}^{\rhd} \vdash p \to \rhd\rhd\, p$. Substituting $\neg p$ for $p$, we deduce $\mathbf{B}^{\rhd} \vdash \neg p \to \rhd\rhd\, p$. Hence $\mathbf{B}^{\rhd} \vdash \rhd\rhd\, p$, i.e., $\mathbf{B}^{\rhd} = \mathbf{S5}^{\rhd}$, but by Lemma 4.5, the inclusion $\mathbf{B}^{\rhd} \subset \mathbf{S5}^{\rhd}$ is strict. $\square$

**Definition 5.2.** A logic $L$ has the *Craig* (*interpolation*) *property* if $L \vdash A \to C$ implies the existence of the formula $B$ (*interpolant*) such that

$$L \vdash A \to B, \quad L \vdash B \to C \qquad \text{and} \qquad \operatorname{Var} B \subseteq (\operatorname{Var} A \cap \operatorname{Var} C).$$

**Lemma 5.3.** *A logic $L \subseteq \mathbf{Fm}^{\Box}$ that contains $\mathbf{T}$ possesses the Craig property if and only if so does the logic $L^{\rhd}$.*

**Proof.** $(\Rightarrow)$ Let us use the obvious deducibility $\mathbf{T} \vdash A \leftrightarrow \operatorname{tr}(\operatorname{Tr}(A))$ for any $\Box$-formula $A$. Suppose that $L^{\rhd} \vdash A \to C$, i.e., $L \vdash \operatorname{tr}(A) \to \operatorname{tr}(C)$. By the Craig property for $L$, we have

$$\exists B \in \mathbf{Fm}^{\Box}\colon \operatorname{Var} B \subseteq (\operatorname{Var} A \cap \operatorname{Var} C), \quad L \vdash \operatorname{tr}(A) \to B, \quad B \to \operatorname{tr}(C).$$

Then $L \vdash \operatorname{tr}(A) \to \operatorname{tr}(\operatorname{Tr}(B))$, $\operatorname{tr}(\operatorname{Tr}(B)) \to \operatorname{tr}(C)$. Hence $L^{\rhd} \vdash A \to \operatorname{Tr}(B)$, $\operatorname{Tr}(B) \to C$. Thus $\operatorname{Tr}(B)$ is the interpolant of $A \to C$ in $L^{\rhd}$.

$(\Leftarrow)$ We can repeat the same argument exchanging the roles of the translations $\operatorname{Tr}$ and $\operatorname{tr}$. $\square$

**Corollary 5.4.** *The logics $L^{\rhd}$, where $L \in \{\mathbf{T}, \mathbf{S4}, \mathbf{B}, \mathbf{S5}, \mathbf{Grz}, \mathbf{S4.1}\}$, possess the Craig interpolation property.*

**Proof.** The proof follows from the familiar (see [17, 14]) Craig property for $L$ and Lemma 5.3. $\square$

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