Infinitary Expressibility of Necessity in Terms of Contingency

Evgeni E. Zolin

Department of Mathematical Logic

Faculty of Mathematics and Mechanics

Moscow State University, 119899 Moscow, Russia

E-mail: zolin@lpcs.math.msu.su

This paper consists of two parts. In the first part, which is rather technical, we present axiomatizations of versions of some well-known modal logics in the language with non-contingency operator as the sole modal primitive symbol. The second part, having a certain philosophical flavour, is devoted to the old question of possibility to define the necessity operator in terms of the contingency operator.

In general, a notion α is said to be *definable* in terms of a notion β if there exists an expression $A = A(\beta)$ containing β such that A is equal (or equivalent) to α. From this point of view, necessity is *not* definable in terms of contingency (cf. [2, 3]). However, this understanding is rather confined, by the author's opinion, and the more appropriate one is to say that A *behaves* like α, or A subjects the same laws as α. This approach proves to be successful to give a new, affirmative, answer to the above question.

I. The non-contingency operator $\rhd$ is defined in terms of the necessity operator $\Box$ by putting $\rhd A := \Box A \vee \Box\neg A$. This induces a translation of $\rhd$-formulas (i.e., formulas in the propositional modal language with $\rhd$ as the sole modal primitive, $\rhd$-language for short) into $\Box$-formulas. So, to any $\Box$-logic L (i.e., logic in the $\Box$-language) one can associate a *non-contingency logic of* L, denoted by $L^{\rhd}$, consisting of all $\rhd$-formulas whose translations are theorems of L.

Montgomery and Routley [4] axiomatized the non-contingency logics of **T**, **S4**, and **S5** (see also [5, 6]). It is worth noting that in case when L contains **T**, or more specifically, the reflexivity scheme $\Box A \to A$, necessity is definable in terms of non-contingency ($\rhd$-definable, for short) by $\Box A = A \,\&\, \rhd A$. In the logic **Ver**, the same effect is observed: it proves, for any A, a formula $\Box A \leftrightarrow \top$, which can be regarded as a $\rhd$-definition of $\Box$. Cresswell [1] provides an example of logic **H** such that $\mathbf{H} \not\supseteq \mathbf{T}$, $\mathbf{H} \neq \mathbf{Ver}$, but $\Box$ is $\rhd$-definable in **H**.

A systematic study of non-contingency logic was initiated by Humberstone. In his paper [2], a (rather complicated) system axiomatizing the non-contingency logic of **K** was presented. Kuhn [3] succeeded in simplifying this system and proposed a finite axiomatization of the logic $\mathbf{K4}^{\rhd}$.

Let us give some formal definitions. The propositional modal language consists of a denumerable set of variables $\mathrm{Var} = \{p_0, p_1, \ldots\}$, symbols for falsehood $\bot$, implication $\to$, and a unary modal operator $\Box$. Other connectives are taken as standard abbreviations. The set of formulas of this language, $\mathbf{Fm}^{\Box}$, is defined as usual. This language will be referred to as a $\Box$-language and its formulas as $\Box$-formulas. A $\rhd$-language and the set $\mathbf{Fm}^{\rhd}$ of $\rhd$-formulas are defined similarly, just by replacing the symbol $\Box$ by $\rhd$. Fix a natural translation $\mathsf{tr}\colon \mathbf{Fm}^{\rhd} \to \mathbf{Fm}^{\Box}$ which respects boolean connectives and $\mathsf{tr}(\rhd A) := \Box\mathsf{tr}(A) \lor \Box\neg\mathsf{tr}(A)$.

The *minimal normal* modal logic $\mathbf{K}$ has the following axioms and the rules of modus ponens, substitution, and the "necessitation" rule:

$(\mathsf{A}^{\Box}_{\top})$	All classical tautologies in the $\Box$-language	
$(\mathsf{A}^{\Box}_{\mathbf{K}})$	$\Box(p \to q) \to (\Box p \to \Box q)$	(distributivity)

$$(\mathsf{MP})\ \frac{A \quad A \to B}{B} \qquad (\mathsf{Sub})\ \frac{A}{A[B/p]} \qquad (\mathsf{Nec})\ \frac{A}{\Box A}$$

The systems $\mathbf{K\Sigma}$, $\mathbf{\Sigma} \subseteq \{\mathbf{D}, \mathbf{4}, \mathbf{5}\}$, are obtained by adding to $\mathbf{K}$ the axioms $(\mathsf{A}^{\Box}_{\mathfrak{S}})$, $\mathfrak{s} \in \mathbf{\Sigma}$, listed below

$(\mathsf{A}^{\Box}_{\mathbf{D}})$	$\Box p \to \Diamond p$	(seriality)
$(\mathsf{A}^{\Box}_{\mathbf{4}})$	$\Box p \to \Box\Box p$	(transitivity)
$(\mathsf{A}^{\Box}_{\mathbf{5}})$	$\Diamond p \to \Box\Diamond p$	(euclideanness)

The logic **KD45** is known to capture the principles of reasoning involving epistemic judgments: the postulates of this logic are valid under the (informal) interpretation of a sentence of the form $\Box A$ as "A is known (to some idealized person)." In this context, the non-contingency assertion $\rhd A$ means "the truth value of A is known."

Another well-known system is the Gödel-Löb logic $\mathbf{GL} = \mathbf{K} + (\mathsf{A}^{\Box}_{\mathbf{L}})$, where $(\mathsf{A}^{\Box}_{\mathbf{L}})$ is the Löb axiom $\Box(\Box p \to p) \to \Box p$. This system is complete under the formal arithmetical interpretation of a sentence of the form $\Box A$ as "A is provable in **PA**", where **PA** stands for Peano arithmetic (cf. [7]). From this point of view, the assertion $\rhd A$ means "A is decidable in **PA**" (a sentence A is called *decidable* in a formal theory T if either A or $\neg A$ is provable in T).

We shall consider only *normal* modal $\Box$-logics L, i.e., sets L of $\Box$-formulas containing the axioms of $\mathbf{K}$ and closed under the inference rules of $\mathbf{K}$. Given a logic L, a *non-contingency logic of* L (a $\rhd$*-logic of* L, for short), denoted by $L^{\rhd}$, is the set of all $\rhd$-formulas whose translations are theorems of L:

$$L^{\rhd} = \{A \in \mathbf{Fm}^{\rhd} \mid \mathsf{tr}(A) \in L\} = \mathsf{tr}^{-1}(L).$$

Now we formulate our axiomatic systems for $\rhd$-logics of the logics L described above. For notation simplicity, we denote the systems by $\mathbf{L}^{\rhd}$; Theorem 1 below justifies the notation. The logic $\mathbf{K}^{\rhd}$ has the rules (MP) and (Sub) as well as the following axioms and the "noncontingentization" rule (cf. [2]):

$(\mathsf{A}^{\rhd}_{\top})$	All classical tautologies in the $\rhd$-language	
$(\mathsf{A}^{\rhd}_{\mathbf{K}})$	$\rhd(p \leftrightarrow q) \to (\rhd p \leftrightarrow \rhd q)$	(equivalence)
$(\mathsf{A}^{\rhd}_{\neg})$	$\rhd p \leftrightarrow \rhd\neg p$	(mirror axiom)
$(\mathsf{A}^{\rhd}_{\lor})$	$\rhd p \to [\rhd(q \to p) \lor \rhd(p \to r)]$	(dichotomy)

$$(\mathsf{NCR})\ \frac{A}{\rhd A}$$

To obtain the system $\mathbf{K\Sigma}^{\rhd}$, $\mathbf{\Sigma} \subseteq \{\mathbf{D}, \mathbf{4}, \mathbf{5}\}$, we add to $\mathbf{K}^{\rhd}$ the corresponding axioms (note that no axiom corresponds to seriality):

$(\mathsf{A}^{\rhd}_{\mathbf{4}})$ $\rhd p \to \rhd(q \to \rhd p)$ (weak transitivity)

$(\mathsf{A}^{\rhd}_{\mathbf{5}})$ $\neg\rhd p \to \rhd(q \to \neg\rhd p)$ (weak euclideanness)

Finally, $\mathbf{GL}^{\rhd} = \mathbf{K} + (\mathsf{A}^{\rhd}_{\mathbf{L}})$, where $(\mathsf{A}^{\rhd}_{\mathbf{L}})$ is the axiom $\rhd(\rhd p \to p) \to \rhd p$.

Theorem 1 (Completeness) *For any logic* $L \in \{\mathbf{K\Sigma} \mid \mathbf{\Sigma} \subseteq \{\mathbf{D}, \mathbf{4}, \mathbf{5}\}\} \cup \{\mathbf{GL}\}$ *and any* $\rhd$*-formula* A*, the following statements are equivalent:*

(1) $L^{\rhd} \vdash A$*;*

(2) $L \vdash \mathsf{tr}(A)$*;*

(3) A *is valid in all frames validating* L*.*

Note that for $\mathbf{K}^{\rhd}$ and $\mathbf{K4}^{\rhd}$ the theorem is proved in [3], however, the axiomatization of these logics proposed in that paper slightly differs from ours, so we restated the result for our systems. The proof of this theorem follows the standard way, using canonical model argument adapted for $\rhd$-logics by Humberstone [2] and Kuhn [3]. However, analysis of a certain technical trick in the proof has led us to quite unexpected observations discussed in the second part of our paper.

II. Informally speaking, the following "infinitary operator" occurs in the proof of the aforementioned Theorem:

$$\boxtimes A = \bigwedge_{B \in \mathbf{Fm}^{\rhd}} \rhd(B \to A).$$

From this equality one can read off a natural Kripke semantics of the operator $\boxtimes$. So the question immediately arises: What modal principles are valid for this operator? Surprisingly enough, the operator $\boxtimes$ subjects the laws of some normal modal logic, which depends on the normal logic describing the behaviour of the initial necessity operator $\Box$.

To put it in a more precise form, consider the infinitary $\rhd$-language containing the set of variables Var as above, negation $\neg$, infinitary conjunction $\bigwedge$ and a unary modal operator $\rhd$. The set of formulas, $\mathbf{Fm}^{\rhd}_{\infty}$, is defined by induction: every variable p_i is a formula; if A is a formula then so are $\neg A$ and $\rhd A$; if Φ is a finite or coutable set of formulas then $\bigwedge\Phi$ is a formula. Other connectives can be introduced as usual, e.g., $(A \to B) \leftrightharpoons \neg\bigwedge\{A, \neg B\}$; therefore we can assume that $\mathbf{Fm}^{\rhd} \subset \mathbf{Fm}^{\rhd}_{\infty}$. Kripke semantics for this language is defined in an obvious way.

Further, we introduce a $\boxtimes$-language obtained from the $\Box$-language by replacing the symbol $\Box$ by $\boxtimes$. Finally, we define a translation $\mathsf{Tr}{:}\mathbf{Fm}^{\boxtimes} \to \mathbf{Fm}^{\rhd}_{\infty}$ which respects boolean connectives and has the following inductive item:

$$\mathsf{Tr}(\boxtimes A) = \bigwedge_{B \in \mathbf{Fm}^{\rhd}} \rhd(B \to \mathsf{Tr}(A)).$$

This translation induces semantics for the $\boxtimes$-language. One can even define semantics for "mixed" formulas containing $\Box$, $\rhd$, and $\boxtimes$. Note that the implication $\Box A \to \boxtimes A$ is valid in any frame, whereas the converse one is not.

Now, given a $\Box$-logic L, we define a $\boxtimes$*-logic of* L as the set of all $\boxtimes$-formulas valid in any frame validating L:

$$L^{\boxtimes} \leftrightharpoons \{A \in \mathbf{Fm}^{\boxtimes} \mid \text{ for any frame } F\ (F \models L \Rightarrow F \models A)\}.$$

Theorem 2 *If L is a normal $\Box$-logic then $L^{\boxtimes}$ is a normal $\boxtimes$-logic.*

This result implies that the infinitary operator $\boxtimes$ defined in terms of non-contingency behaves like *some*, possibly different from the initial, necessity.

Theorem 3 *For logics $L \in \{\mathbf{K}, \mathbf{K4}, \mathbf{K5}, \mathbf{K45}, \mathbf{GL}\}$, the inclusion $L^{\boxtimes} \supseteq L$ holds (up to replacement of $\Box$ by $\boxtimes$).*

So far, we have not established any equality of the form $L^{\boxtimes} = L$ (of course, again up to replacement of $\Box$ by $\boxtimes$), and the main conjecture here is that $\mathbf{K}^{\boxtimes} = \mathbf{K}$. On the other side, we have an example of inequality, namely $\mathbf{KB}^{\boxtimes} \not\supseteq \mathbf{KB}$, where $\mathbf{KB} = \mathbf{K} + (\mathsf{A}^{\Box}_{\mathbf{B}})$ and $(\mathsf{A}^{\Box}_{\mathbf{B}})$ is the symmetricity axiom $p \to \Box\Diamond p$. One can easily construct a finite symmetrical frame falsifying the formula $p \to \boxtimes\neg\boxtimes\neg p$.

It is worth noting that all the previous reasoning is valid if, in the definition of $\boxtimes$, the infinitary conjunction is taken only over the set of *literals* Ł $:= \{p, \neg p \mid p \in \mathrm{Var}\}$. So, in what follows, we assume that $\boxtimes$ is defined as

$$\boxtimes A := \bigwedge_{\ell \in \text{Ł}} \rhd(\ell \to A).$$

Recall that, starting from $\Box$, we first defined the operator $\rhd$ and then the operator $\boxtimes$. What if we iterate the procedure? Schematically, the next iteration looks like:

$$\blacktriangleright A := \boxtimes A \vee \boxtimes\neg A; \qquad \boxplus A := \bigwedge_{\ell \in \text{Ł}} \blacktriangleright(\ell \to A).$$

Fortunately, this iteration of the construction is redundant.

Theorem 4 *The operators $\boxtimes$ and $\boxplus$ are semantically equivalent, i.e., the formula $\boxtimes p \leftrightarrow \boxplus p$ is valid in any frame.*

Consequently, $\boxtimes$ is to be considered as a distinctive operator, not as "a one in the series", and therefore the problem of what exactly the logic $L^{\boxtimes}$ is, for various modal logics L, is of both technical and philosophical interest.

References

[1] M. J. Cresswell, Necessity and contingency, *Studia Logica*, vol. 47 (1988), pp. 145–149.

[2] I. L. Humberstone, The logic of non-contingency, *Notre Dame Journal of Formal Logic*, 1995, 36(2):214–229.

[3] S. T. Kuhn, Minimal non-contingency logic, *Notre Dame Journal of Formal Logic*, 1995, 36(2):230–234.

[4] H. Montgomery, R. Routley, Contingency and non-contingency bases for normal modal logics, *Logique et analyse*, 9 (1966), 318–328.

[5] H. Montgomery, R. Routley, Non-contingency axioms for $S4$ and $S5$, *Logique et analyse*, 11 (1968), 422–424.

[6] H. Montgomery, R. Routley, Modalities is a sequence of normal non-contingency modal systems, *Logique et analyse*, 12 (1969), 225–227.

[7] R. Solovay, Provability interpretations of modal logics, *Israel Journal of Mathematics*, vol. 25 (1976), pp. 287–304.