Infinitary Expressibility of Necessity in Terms of Contingency

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Abstract. This paper consists of two parts. In the first part we present an axiomatization of the "epistemic" modal logic **KD45** in the language with non-contingency operator as the sole modal primitive symbol. The second part, having a certain philosophical flavour, is devoted to the old question about the possibility of defining the necessity operator in terms of the contingency operator. Here we give a new, positive answer to this question by constructing an infinitary operator defined in terms of contingency which behaves like some necessity.

1 Introduction

The non-contingency operator $\rhd$ is defined in terms of the necessity operator $\Box$ by putting $\rhd A := \Box A \vee \Box\neg A$. A natural question arises here: is necessity expressible in terms of non-contingency? The answer depends upon the understanding of the notion of expressibility.

In general, a notion α is said to be *definable* (or *expressible*) in terms of a notion β if there exists an expression $A = A(\beta)$ containing β such that A is equal (or equivalent) to α. In our case this means that $\Box$ would be definable in terms of $\rhd$ if there exists a formula $\varphi(p)$ such that all occurrences of $\Box$ in φ are in contexts of the form $\rhd\psi$ and the equivalence $\Box p \leftrightarrow \varphi(p)$ is valid. From this point of view, $\Box$ is *not* definable in terms of $\rhd$ (cf. [3, 4]).

However, this understanding is rather confined, by the author's opinion, and more appropriate is to say that A *behaves* like α, or A subjects the same laws as α. This approach proves to be successful to give a new, positive answer to the above question. In this paper we construct an operator $\boxtimes$ (by giving its infinitary definition in terms of $\rhd$) which behaves like some necessity. To be more exact, we show that, for any normal modal logic L (describing the behaviour of $\Box$) the corresponding logic describing the behaviour of $\boxtimes$ is normal and, for some L, it even contains L (up to replacement of $\Box$ by $\boxtimes$). The operator $\boxtimes$ plays an important rôle in the proof of completeness theorem for non-contingency logic of **KD45**.

2 Preliminaries

The propositional modal language consists of a denumerable set of variables $\mathrm{Var} = \{p_0, p_1, \ldots\}$, symbols for falsehood $\bot$, implication $\to$, and a unary modal operator $\Box$. Other connectives are taken as standard abbreviations. The set of formulas of this language, $\mathbf{Fm}^{\Box}$, is defined as usual, in particular, if A is a formula then so is $\Box A$. This language will be referred to as a $\Box$-language and its formulas as $\Box$-formulas. A $\rhd$-language and the set $\mathbf{Fm}^{\rhd}$ of $\rhd$-formulas are defined similarly, just by replacing the symbol $\Box$ by $\rhd$. Fix a translation $\mathsf{tr}\colon \mathbf{Fm}^{\rhd} \to \mathbf{Fm}^{\Box}$ which respects boolean connectives and sets $\mathsf{tr}(\rhd A) := \Box\mathsf{tr}(A) \vee \Box\neg\mathsf{tr}(A)$.

A (Kripke) *frame* is a structure $\langle W, \uparrow\rangle$, where W is a nonempty set of "worlds" and $\uparrow$ is a binary "accessibility" relation on W. By $\downarrow$ we denote the converse relation of $\uparrow$. Quantification over worlds accessible from a given world $w \in W$ will be written as $\forall x{\downarrow}w$ and $\exists x{\downarrow}w$. A *model* $M = \langle F, \models\rangle$ consists of a frame F and a valuation $\models\, \subseteq (W \times \mathrm{Var})$. The notion "$A$ is true in M at w" (written $M,w \models A$ and M usually omitted) is defined for both $\Box$- and $\rhd$-formulas in the standard way; the modal clauses are as follows:

$$
\begin{array}{lcl}
w \models \Box A & \leftrightharpoons & \forall x{\downarrow}w \quad x \models A; \\
w \models \rhd A & \leftrightharpoons & (\forall x{\downarrow}w \quad x \models A) \text{ or } (\forall x{\downarrow}w \quad x \not\models A).
\end{array}
$$

Obviously, $w \models A \Leftrightarrow w \models \mathsf{tr}(A)$, for any $\rhd$-formula A. A formula A is *valid* in a frame F ($F \models A$, in symbols) if A is true at every world in every model based on F. If Γ is a set of formulas then a Γ-*frame* is a frame validating Γ. A logic L is called *complete* w.r.t. a class of frames $\mathcal{F}$ if, for any formula A, $L \vdash A \Leftrightarrow \mathcal{F} \models A$.

The *minimal normal* modal logic **K** has the following axioms and the inference rules of modus ponens, substitution, and necessitation:

$(\mathsf{A}^{\Box}_{\top})$ All classical tautologies in the $\Box$-language
$(\mathsf{A}^{\Box}_{\mathbf{K}})$ $\Box(p \to q) \to (\Box p \to \Box q)$ (distributivity)

$$
(\mathsf{MP})\ \frac{A \quad A \to B}{B} \qquad (\mathsf{Sub})\ \frac{A}{A[B/p]} \qquad (\mathsf{Nec})\ \frac{A}{\Box A}
$$

The systems $\mathbf{K\Sigma}$, $\mathbf{\Sigma} \subseteq \{\mathbf{D}, \mathbf{4}, \mathbf{5}\}$, are obtained by adding to **K** the axioms $(\mathsf{A}^{\Box}_{\mathfrak{S}})$, $\mathfrak{S} \in \mathbf{\Sigma}$, listed below (the class of frames characterized by an axiom $(\mathsf{A}^{\Box}_{\mathfrak{S}})$ is first-order definable by a formula also shown below).

$(\mathsf{A}^{\Box}_{\mathbf{D}})$	$\Box p \to \Diamond p$	$\forall w\ \exists x \quad w{\uparrow}x$	(seriality)
$(\mathsf{A}^{\Box}_{\mathbf{4}})$	$\Box p \to \Box\Box p$	$\forall w\ \forall x{\downarrow}w\ \forall y{\downarrow}x \quad w{\uparrow}y$	(transitivity)
$(\mathsf{A}^{\Box}_{\mathbf{5}})$	$\Diamond p \to \Box\Diamond p$	$\forall w\ \forall x{\downarrow}w\ \forall y{\downarrow}w \quad x{\uparrow}y$	(euclideanness)

The logic **KD45** is known to capture the principles of reasoning involving epistemic judgments: the postulates of this logic are valid under the (informal) interpretation of a sentence of the form $\Box A$ as "A is known (to some idealized person)." In this context, the non-contingency assertion $\rhd A$ means "the truth value of A is known."

We shall consider only *normal* modal $\Box$-logics, i.e., sets of $\Box$-formulas containing the axioms of **K** and closed under the rules of **K**. Given a logic L, a *non-contingency logic of* L (a $\rhd$*-logic of* L, for short), denoted by $L^{\rhd}$, is the set of all $\rhd$-formulas whose translations are theorems of L:

$$L^{\rhd} = \{A \in \mathbf{Fm}^{\rhd} \mid \mathsf{tr}(A) \in L\} = \mathsf{tr}^{-1}(L).$$

Montgomery and Routley [5, 6, 7] axiomatized the non-contingency logics of **T**, **S4**, and **S5**. It is worth noting that if L contains **T**, or more specifically, the reflexivity scheme $\Box A \to A$, necessity is definable in terms of non-contingency ($\rhd$-definable, for short) by $\Box A = A \,\&\, \rhd A$. In the logic **Ver**, the same effect is observed: it proves, for any A, a formula $\Box A \leftrightarrow \top$, which can be viewed as a $\rhd$-definition of $\Box$. Cresswell [2] provides an example of logic **H** such that $\mathbf{H} \not\supseteq \mathbf{T}$, $\mathbf{H} \neq \mathbf{Ver}$, but $\Box$ is $\rhd$-definable in **H**.

A systematic study of non-contingency logic (in particular, the cases when $\Box$ is counted to be $\rhd$-*un*definable), was initiated by Humberstone. In his paper [3], a (rather complicated) system axiomatizing the non-contingency logic of **K** was presented. Kuhn [4] succeeded in simplifying this system and proposed a finite axiomatization of the non-contingency logic of **K4**.

3 Axiomatizations of non-contingency logics

Now we formulate our systems for $\rhd$-logics of the logics L described above. For notation simplicity, we denote the systems by $L^{\rhd}$; Theorem 3.1 below justifies the notation. The logic $\mathbf{K}^{\rhd}$ has the rules (MP) and (Sub) as well as the following axioms and the "noncontingentization" rule (cf. [3]):

$(\mathsf{A}^{\rhd}_{\top})$ All classical tautologies in the $\rhd$-language		(NCR) $\dfrac{A}{\rhd A}$
$(\mathsf{A}^{\rhd}_{\mathbf{K}})$ $\rhd(p \leftrightarrow q) \to (\rhd p \leftrightarrow \rhd q)$	(equivalence)	
$(\mathsf{A}^{\rhd}_{\neg})$ $\rhd p \leftrightarrow \rhd\neg p$	(mirror axiom)	
$(\mathsf{A}^{\rhd}_{\vee})$ $\rhd p \to [\rhd(q \to p) \vee \rhd(p \to r)]$	(dichotomy)	

To obtain the system $\mathbf{K\Sigma}^{\rhd}$, $\mathbf{\Sigma} \subseteq \{\mathbf{D}, \mathbf{4}, \mathbf{5}\}$, we add to $\mathbf{K}^{\rhd}$ the corresponding axioms (note that no axiom corresponds to seriality):

$(\mathsf{A}^{\rhd}_{\mathbf{4}})$ $\rhd p \to \rhd(q \to \rhd p)$ (weak transitivity)

$(\mathsf{A}^{\rhd}_{\mathbf{5}})$ $\neg\rhd p \to \rhd(q \to \neg\rhd p)$ (weak euclideanness)

The classes of frames characterized by these two axioms strictly contain the classes of transitive (resp. euclidean) frames as well as the class of functional frames (where each world "sees" at most one world); hence their names.

The systems for $\mathbf{K}^{\rhd}$ and $\mathbf{K4}^{\rhd}$ proposed by Kuhn [4] differ from ours: instead of the axiom $(\mathsf{A}^{\rhd}_{\mathbf{K}})$, his systems contain the rule of equivalent replacement $\frac{A \leftrightarrow B}{\rhd A \leftrightarrow \rhd B}$ as well as an additional axiom $\rhd p \,\&\, \rhd q \to \rhd(p \,\&\, q)$. Our systems are similar to the standard axiomatizations of normal logics.

The main result of this section is formulated in Theorem 3.1, stating that the systems $\mathbf{K\Sigma}^{\rhd}$ axiomatize the $\rhd$-logics of $\mathbf{K\Sigma}$, $\mathbf{\Sigma} \subseteq \{\mathbf{4}, \mathbf{5}\}$ (logics containing the seriality axiom will be considered at the end of the section).

Theorem 3.1 (Completeness) *For any $\mathbf{\Sigma} \subseteq \{\mathbf{4}, \mathbf{5}\}$ and a $\rhd$-formula A, the following statements are equivalent:*

(1) $\mathbf{K\Sigma}^{\rhd} \vdash A$*;*

(2) $\mathbf{K\Sigma} \vdash \mathsf{tr}(A)$*;*

(3) *A is valid in all $\mathbf{K\Sigma}$-frames.*

Proof. We follow the scheme (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (1). The equivalence (2) $\Leftrightarrow$ (3) is well-known (cf. [1]) completeness of $\mathbf{K\Sigma}$ w.r.t. $\mathbf{K\Sigma}$-frames. In the rest of the proof we refer to $\rhd$-formulas as just formulas.

<u>(1) $\Rightarrow$ (2)</u> The axioms of $\mathbf{K}^{\rhd}$ are valid in any frame, so the translations thereof are provable in $\mathbf{K}$. For the axiom $(\mathsf{A}^{\rhd}_{\mathbf{4}})$, the proof is in [4]. We give a sketch of a derivation of (the translation of) the axiom $(\mathsf{A}^{\rhd}_{\mathbf{5}})$ in $\mathbf{K5}$:

$$\mathbf{K5} \vdash \neg\rhd p \;\begin{matrix} \nearrow \Diamond p \;\;\;\to \Box\Diamond p \searrow \\ \searrow \Diamond\neg p \to \Box\Diamond\neg p \nearrow \end{matrix}\; \Box\neg\rhd p \to \Box(q \to \neg\rhd p) \to \rhd(q \to \neg\rhd p).$$

<u>(3) $\Rightarrow$ (1)</u> We construct the canonical model $M_{\mathcal{L}} = \langle W_{\mathcal{L}}, \uparrow, \models\rangle$ for the logic $\mathcal{L} = \mathbf{K\Sigma}^{\rhd}$. Its worlds are maximal $\mathcal{L}$-consistent sets of formulas. A valuation is defined in the usual way: $w \models p \Leftrightarrow p \in w$, for any world w and a variable p. Before defining the relation $\uparrow$, we introduce some notation.

For a formula A, denote $\boxtimes A := \{\rhd(B \to A) \mid B \in \mathbf{Fm}^{\rhd}\}$. In the subsequent proof, the symbol $\boxtimes$ plays the rôle similar to that of $\Box$ in the standard canonical model argument for $\Box$-logics. The difference is in their "types": the operator $\Box$ maps a formula to a formula, whereas $\boxtimes$ maps a formula to a set of formulas. Note that semantically $\boxtimes$ is *not* equivalent to $\Box$, i.e., the truth at a world w of the formula $\Box A$ is not equivalent to the truth at w of all formulas in the set $\boxtimes A$. The next section is devoted to investigation of interconnection between the operators $\Box$ and $\boxtimes$.

Now denote $\sharp w := \{A \in \mathbf{Fm}^{\rhd} \mid \boxtimes A \subseteq w\}$. Finally, put $w{\uparrow}x$ iff $\sharp w \subseteq x$.

Lemma 3.2 *For any world $w \in W_{\mathcal{L}}$, the following properties are satisfied:*

1° **(Dichotomy)** *If $\rhd A \in w$ then either $A \in \sharp w$ or $\neg A \in \sharp w$.*

2° *The set $\sharp w$ is closed under (even empty) conjunction (hence $\sharp w \neq \varnothing$).*

3° *The set $\sharp w$ is closed under derivability in $\mathcal{L}$: if $A \in \sharp w$ and $\mathcal{L} \vdash A \to B$, then $B \in \sharp w$.*

4° *The dichotomy property is reversible: if $A \in \sharp w$ then $\rhd A \in w$.*

► **1°.** Suppose $A, \neg A \notin \sharp w$, then by definition of $\sharp w$, for some formulas B, C we have: $\neg\rhd(B \to A) \in w$, $\neg\rhd(C \to \neg A) \in w$. However, using the dichotomy axiom, we derive: $\mathbf{K}^{\rhd} \vdash \rhd A \to [\rhd(B \to A) \vee \rhd(C \to \neg A)]$, hence w is even $\mathbf{K}^{\rhd}$-inconsistent, which contradicts our assumptions.

2°. By definition, the empty conjunction is $\top$. Since $\mathbf{K}^{\rhd} \vdash \rhd(B \to \top)$, for any formula B, we have $\boxtimes\top \subseteq \mathbf{K}^{\rhd} \subseteq \mathcal{L} \subseteq w$ and so $\top \in \sharp w$.

Now let $A, B \in \sharp w$ and prove that $(A\&B) \in \sharp w$, i.e., $\rhd[C{\to}(A \,\&\, B)] \in w$, for any formula C. From $\boxtimes A \subseteq w$ and $\boxtimes B \subseteq w$ it follows that $\rhd(C{\to}A) \in w$ and $\rhd(C{\to}B) \in w$. Then we derive:

$$\begin{aligned}\mathbf{K}^{\rhd} \vdash \rhd(C \to A) \,\&\, \rhd(C \to B) \longrightarrow \rhd[(C \to A) \,\&\, (C \to B)] \longleftrightarrow \\ \longrightarrow \rhd[C \to (A \,\&\, B)].\end{aligned}$$

Since w is closed under conjunction and derivability in $\mathbf{K}^{\rhd}$ (and even in $\mathcal{L}$), we conclude: $\rhd[C \to (A \,\&\, B)] \in w$.

3°. To prove that $B \in \sharp w$, we take an arbitrary formula C and show that $\rhd(C \to B) \in w$. Since $\boxtimes A \subseteq w$, we have $\rhd[\neg(C \to B) \to A] \in w$. The assumption $\mathcal{L} \vdash A \to B$ truth-functionally implies $\mathcal{L} \vdash [\neg(C \to B) \to A] \leftrightarrow [C \to B]$, and this finally yields $\rhd(C \to B) \in w$.

4°. $\boxtimes A \subseteq w$ implies $\rhd(\top{\to}A) \in w$, which is equivalent to $\rhd A \in w$. ◄

Lemma 3.3 **($\models\, = \,\ni$)** *For any formula A and a world w, $w \models A \Leftrightarrow A \in w$.*

► By induction on A. Consider the only interesting case $A = \rhd B$.

$$\begin{array}{lll}(\Leftarrow) & \rhd B \in w & \Rightarrow \text{(by dichotomy } \mathbf{1}^\circ) \\ & B \in \sharp w \ \text{ or } \ \neg B \in \sharp w & \Rightarrow \text{(by definition of } \uparrow) \\ & (\forall x{\downarrow}w\ B \in x) \ \text{ or } \ (\forall x{\downarrow}w\ \neg B \in x) & \Rightarrow \text{(by consistency of } x) \\ & (\forall x{\downarrow}w\ B \in x) \ \text{ or } \ (\forall x{\downarrow}w\ B \notin x) & \Rightarrow \text{(by induction hypothesis)} \\ & (\forall x{\downarrow}w\ x \models B) \ \text{ or } \ (\forall x{\downarrow}w\ x \not\models B) & \Rightarrow w \models \rhd B.\end{array}$$

($\Rightarrow$) Suppose $\rhd B \notin w$. Then the sets $X = \sharp w \cup \{B\}$ and $Y = \sharp w \cup \{\neg B\}$ are $\mathcal{L}$-consistent. For, if Y is not then $\mathcal{L} \vdash (A_1 \,\&\, \ldots \,\&\, A_n) \to B$ for some formulas $A_1, \ldots, A_n \in \sharp w$ and $n \geqslant 0$. By **2°**, $(A_1 \,\&\, \ldots \,\&\, A_n) \in \sharp w$, then $B \in \sharp w$ by **3°** and $\rhd B \in w$ by **4°**, which is not the case. The argument for X is similar except for additional use of the mirror axiom.

Therefore, X and Y are contained in some worlds x and y. Since $\sharp w \subseteq x$ and $\sharp w \subseteq y$, we have $w{\uparrow}x$ and $w{\uparrow}y$; by induction hypothesis, $B \in x$ and $B \notin y$ imply $x \models B$ and $y \not\models B$, thus $w \not\models \rhd B$. ◄

By this lemma, the canonical model falsifies all the nontheorems of L. To conclude the proof, it remains to check that the canonical frame is a $\mathbf{K\Sigma}$-frame. The case $\mathbf{\Sigma} = \varnothing$ is trivial.

Suppose $\mathbf{4} \in \mathbf{\Sigma}$ and prove that $\uparrow$ is transitive. Let $w{\uparrow}x{\uparrow}y$ and show that $w{\uparrow}y$, i.e., $\sharp w \subseteq y$. Take any $A \in \sharp w$, then $\rhd(B \to A) \in w$, for every B. By the axiom $(\mathsf{A}_{\mathbf{4}}^{\rhd})$, $\mathbf{K4}^{\rhd} \vdash \rhd(B \to A) \to \rhd[C \to \rhd(B \to A)]$, for any C. Since w is closed under $\mathbf{K4}^{\rhd}$-derivability, $\rhd[C \to \rhd(B \to A)] \in w$.

Hence $\boxtimes\rhd(B \to A) \subseteq w$ and $\rhd(B \to A) \in \sharp w \subseteq x$, whence $\boxtimes A \subseteq x$ and $A \in \sharp x \subseteq y$, as desired.

Suppose $\mathbf{5} \in \mathbf{\Sigma}$ and prove that $\uparrow$ is euclidean. Let $w{\uparrow}x$, $w{\uparrow}y$ and show that $x{\uparrow}y$, i.e., $\sharp x \subseteq y$. Take any $A \notin y$, then $A \notin \sharp w$ by $\sharp w \subseteq y$, hence $\neg\rhd(B{\to}A) \in w$, for some B. Since w is closed under $\mathbf{K5}^{\rhd}$-derivability, we apply $(\mathsf{A}^{\rhd}_{\mathbf{5}})$ to obtain $\rhd[C \to \neg\rhd(B \to A)] \in w$, for all C, therefore $\boxtimes\neg\rhd(B \to A) \subseteq w$. By $w{\uparrow}x$, we conclude: $\neg\rhd(B \to A) \in x$, thus $\boxtimes A \not\subseteq x$ and $A \notin \sharp x$, hence the claim. ⊣

Now we show, following [3], that adding the axiom $(\mathsf{A}^{\Box}_{\mathbf{D}})$ to some $\Box$-logics does not change $\rhd$-logic thereof. Let $F = \langle W, \uparrow \rangle$ be a frame. We denote the set of worlds accessible from $w \in W$ by $w{\uparrow} := \{x \in W \mid w{\uparrow}x\}$. Turning "blind" worlds into worlds "seeing" only itself yields a frame $\widehat{F} := \langle W, \Uparrow \rangle$, where $\Uparrow := \uparrow \cup \{\langle w, w \rangle \mid w{\uparrow} = \varnothing\}$. For a class of frames $\mathcal{F}$, put $\widehat{\mathcal{F}} := \{\widehat{F} \mid F \in \mathcal{F}\}$. In [3] it is noted that F and $\widehat{F}$ validate the same $\rhd$-formulas.

Theorem 3.4 *Suppose a $\Box$-logic L is complete w.r.t. a class $\mathcal{F}$ and $L\mathbf{D}$ is the smallest logic containing L and $(\mathsf{A}^{\Box}_{\mathbf{D}})$. If $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ then $L\mathbf{D}^{\rhd} = L^{\rhd}$.*

Proof. The inclusion '$\supseteq$' is trivial. Now take any $A \in L\mathbf{D}^{\rhd}$; clearly, $A \in L^{\rhd} \Leftrightarrow \mathsf{tr}(A) \in L \Leftrightarrow \mathcal{F} \models \mathsf{tr}(A) \Leftrightarrow \mathcal{F} \models A$, so it remains to show that $F \models A$, for any frame $F \in \mathcal{F}$. Since $\widehat{\mathcal{F}} \subseteq \mathcal{F}$, we have $\widehat{F} \in \mathcal{F}$ and so $\widehat{F} \models L$; besides, $\widehat{F}$ is serial, hence $\widehat{F} \models (\mathsf{A}^{\Box}_{\mathbf{D}})$. Thus $\widehat{F} \models L\mathbf{D}$, whence $\widehat{F} \models L\mathbf{D}^{\rhd}$, in particular, $\widehat{F} \models A$. By the above, this is equivalent to $F \models A$. ⊣

As a consequence, $\mathbf{KD\Sigma}^{\rhd} = \mathbf{K\Sigma}^{\rhd}$, for any $\mathbf{\Sigma} \subseteq \{\mathbf{4}, \mathbf{5}\}$, since the transitivity and euclideanness properties are preserved as we pass from F to $\widehat{F}$. For the case $\mathbf{\Sigma} = \varnothing$ the result was obtained in [3].

4 Infinitary operator

Roughly speaking, the following "infinitary operator" occurs in the proof of Theorem 3.1 (we replace a set of formulas by a conjunction thereof):

$$\boxtimes A = \bigwedge_{B \in \mathbf{Fm}^{\rhd}} \rhd(B \to A).$$

From this equality one can read off a natural Kripke semantics of the operator $\boxtimes$. The question arises immediately: What modal principles are valid for this operator? Surprisingly enough, the operator $\boxtimes$ subjects the laws of some normal modal logic (which, of course, depends on the normal logic describing the behaviour of the initial necessity operator $\Box$).

To put it in a more precise form, consider the infinitary $\rhd$-language containing the set of variables Var as above, negation $\neg$, infinitary conjunction $\bigwedge$ and a unary modal operator $\rhd$. The set of formulas, $\mathbf{Fm}^{\rhd}_{\infty}$, is defined by induction: every variable p_i is a formula; if A is a formula then so are $\neg A$ and

$\rhd A$; if Φ is a finite or countable set of formulas then $\bigwedge\Phi$ is a formula. Other connectives can be introduced as usual, e.g., $(A \to B) \leftrightharpoons \neg\bigwedge\{A, \neg B\}$; therefore we can assume that $\mathbf{Fm}^{\rhd} \subset \mathbf{Fm}^{\rhd}_{\infty}$. Kripke semantics for this language is defined in an obvious way.

Further, we introduce a $\boxtimes$-language obtained from the $\Box$-language by replacing the symbol $\Box$ by $\boxtimes$. Finally, we define a translation $\mathsf{Tr}{:}\mathbf{Fm}^{\boxtimes} \to \mathbf{Fm}^{\rhd}_{\infty}$ which respects boolean connectives and has the following inductive item:

$$\mathsf{Tr}(\boxtimes A) = \bigwedge_{B \in \mathbf{Fm}^{\rhd}} \rhd(B \to \mathsf{Tr}(A)).$$

This translation induces semantics for the $\boxtimes$-language: $F \models A \leftrightharpoons F \models \mathsf{Tr}(A)$, for any $\boxtimes$-formula A. One can even define semantics for "mixed" formulas containing $\Box$, $\rhd$, and $\boxtimes$. Note that the implication $\Box A \to \boxtimes A$ is valid in any frame, whereas the converse one is not.

Now, given a $\Box$-logic L, we define a *$\boxtimes$-logic of L* as the set of all $\boxtimes$-formulas valid in any L-frame:

$$L^{\boxtimes} \leftrightharpoons \{A \in \mathbf{Fm}^{\boxtimes} \mid \text{ for any frame } F\ (F \models L \Rightarrow F \models A)\}.$$

It is easily seen, for example, that $\mathbf{Ver}^{\boxtimes} = \mathbf{Ver}$ (from now on, we understand such equalities as well as inclusions up to replacement of $\Box$ by $\boxtimes$).

Theorem 4.1 *If L is a normal $\Box$-logic then $L^{\boxtimes}$ is a normal $\boxtimes$-logic.*

Proof. Since $L^{\boxtimes}$ is clearly closed under the rules of $\mathbf{K}$, we only need to verify that $\boxtimes(p \to q) \to (\boxtimes p \to \boxtimes q)$ is valid. Assume the contrary, i.e., there exists a model M and its world w such that

$$w \models \boxtimes(p \to q), \qquad w \models \boxtimes p, \qquad w \not\models \boxtimes q.$$

The latter implies that $w \not\models \rhd(A \to q)$ for some $\rhd$-formula A, and so

$$\begin{array}{ll} \exists x{\downarrow}w & x \models A \to q, \\ \exists y{\downarrow}w & y \models A, \neg q. \end{array}$$

By our assumptions, $w \models \rhd(p \to q), \rhd p$. Hence $w \models \rhd[(p \to q) \,\&\, p]$, or equivalently, $w \models \rhd(p \,\&\, q)$. But the case $w \models \Box(p \,\&\, q)$ is impossible, for $y \not\models p \,\&\, q$, therefore we have $w \models \Box(\neg p \vee \neg q)$. Now consider two cases:

1) $x \models q$. Then $x \not\models p$, since $x \models (\neg p \vee \neg q)$ by the above. Using $w \models \rhd p$, we conclude $y \not\models p$. This yields a contradiction: on the one hand, $w \models \rhd(q \to p)$, since $w \models \boxtimes p$; on the other, $x \not\models q \to p$ and $y \models q \to p$.

2) $x \not\models q$. Then $x \not\models A$, for $x \models A \to q$. Since $w \models \rhd p$, there are two subcases:

2a) $x \models p$ and $y \models p$. Then from $w \models \rhd[A \to (p \to q)]$ it follows that:

- either $w \models \Box[A \to (p \to q)]$, which is not the case, for $y \models A, p, \neg q$;

- or $w \models \Box\neg[A \to (p \to q)]$, so $w \models \Box A$, in contradiction with $x \not\models A$.

2b) $x \not\models p$ and $y \not\models p$. Then from $w \models \rhd[A \to p]$ it follows that:

- either $w \models \Box[A \to p]$, which is not the case, for $y \models A, \neg p$;
- or $w \models \Box\neg[A \to p]$, hence $w \models \Box A$, in contradiction with $x \not\models A$. $\dashv$

This result implies that the infinitary operator $\boxtimes$ defined in terms of non-contingency behaves like *some*, possibly different from the initial, necessity.

Theorem 4.2 *For any* $\mathbf{\Sigma} \subseteq \{\mathbf{4}, \mathbf{5}\}$, *we have* $\mathbf{K\Sigma}^{\boxtimes} \supseteq \mathbf{K\Sigma}$.

PROOF. For $\mathbf{\Sigma} = \varnothing$ the statement follows from Theorem 4.1.
$\underline{\mathbf{4} \in \mathbf{\Sigma}}$. We shall prove that $\boxtimes p \to \boxtimes\boxtimes p$ is valid on any transitive frame. Assume that for a world w of some transitive model we have $w \models \boxtimes p$ and $w \not\models \boxtimes\boxtimes p$. This means that $w \not\models \rhd(A \to \boxtimes p)$ for some $\rhd$-formula A, i.e.,

$$\begin{array}{ll} \exists x{\downarrow}w & x \models A \to \boxtimes p, \\ \exists y{\downarrow}w & y \models A, \neg\boxtimes p. \end{array}$$

The latter, in turn, implies the existence of a $\rhd$-formula B such that

$$\begin{array}{ll} \exists s{\downarrow}y & s \models B \to p, \\ \exists t{\downarrow}y & t \not\models B \to p. \end{array}$$

By transitivity, $w{\uparrow}s$, $w{\uparrow}t$, so $w \not\models \rhd(B \to p)$, in contradiction with $w \models \boxtimes p$.
$\underline{\mathbf{5} \in \mathbf{\Sigma}}$. We show that $\neg\boxtimes p \to \boxtimes\neg\boxtimes p$ is true at any world w of any euclidean model. Let $w \models \neg\boxtimes p$, i.e., $w \models \neg\rhd(A \to p)$ for some $\rhd$-formula A. By the axiom $(\mathsf{A}^{\rhd}_{\mathbf{5}})$, we conclude $w \models \rhd[B \to \neg\rhd(A \to p)]$ for any $\rhd$-formula B, i.e., $w \models \boxtimes\neg\rhd(A \to p)$. Since the implication $\neg\rhd(A \to p) \to \neg\boxtimes p$ is valid in any frame, the formula $\boxtimes\neg\rhd(A \to p) \to \boxtimes\neg\boxtimes p$ is valid too (we can use the monotonicity principle "from $\varphi \to \psi$ it follows that $\boxtimes\varphi \to \boxtimes\psi$", for $\boxtimes$ is a normal modal operator). Thus we have $w \models \boxtimes\neg\boxtimes p$. $\dashv$

This theorem cannot be generalized to all logics. A counterexample is $\mathbf{KB}^{\boxtimes} \not\supseteq \mathbf{KB}$, where $\mathbf{KB} = \mathbf{K} + (\mathsf{A}^{\Box}_{\mathbf{B}})$ and $(\mathsf{A}^{\Box}_{\mathbf{B}})$ is the symmetricity axiom $p \to \Box\Diamond p$. One can easily construct a finite symmetric frame falsifying the formula $p \to \boxtimes\neg\boxtimes\neg p$.

It is worth noting that all the previous reasoning is valid if, in the definition of $\boxtimes$, the infinitary conjunction is taken only over the set of *literals* $\mathrm{Ł} := \{p, \neg p \mid p \in \mathrm{Var}\}$. So, in what follows, we assume that $\boxtimes$ is defined as

$$\boxtimes A := \bigwedge_{\ell \in \mathrm{Ł}} \rhd(\ell \to A).$$

(In fact, this new operator $\boxtimes$ is *not* semantically equivalent to the previous one, as can be easily shown; however, the results obtained above remain true under new definition of $\boxtimes$ as well). Recall that, starting from $\Box$, we have defined the operator $\rhd$ and then the operator $\boxtimes$. What if we iterate the procedure? Schematically, the next iteration looks like:

$$\blacktriangleright A := \boxtimes A \vee \boxtimes\neg A; \qquad \boxplus A := \bigwedge_{\ell\in\mathrm{L}} \blacktriangleright(\ell \to A).$$

Fortunately, this iteration of the construction is redundant.

Theorem 4.3 *The operators $\boxtimes$ and $\boxplus$ are semantically equivalent, i.e., the formula $\boxtimes p \leftrightarrow \boxplus p$ is valid in any frame. Moreover, we have $\models \rhd p \leftrightarrow \blacktriangleright p$.*

PROOF. Validity of the implication $\rhd p \to \blacktriangleright p$ follows from $\models \Box p \to \boxtimes p$. Now, using $\models \boxtimes p \to \rhd p$, we obtain the converse implication:
$\models \blacktriangleright p \longleftrightarrow (\boxtimes p \vee \boxtimes\neg p) \longrightarrow (\rhd p \vee \rhd\neg p) \longleftrightarrow \rhd p.$ ⊣

Let us observe the following distinctive feature of the operator $\boxtimes$. We have established that $\boxtimes$ possesses the following two properties: (a) $\boxtimes$ is a normal modal operator; (b) the operator $\rhd$ is $\boxtimes$-definable by the equality $\rhd A = \boxtimes A \vee \boxtimes\neg A$ (since $\rhd$ and $\blacktriangleright$ are equivalent). It turns out that $\boxtimes$ is the weakest modality possessing (a) and (b) simultaneously (under modality here we mean any unary operator supplied by Kripke semantics; of course, this is not a formal definition; for example, any modal formula of one variable suits for ours purposes). Indeed, assume that $\boxminus$ is a modality satisfying (a) and (b). To prove that $\models \boxminus p \to \boxtimes p$, take any literal ℓ and put $A := (\ell \to p)$. From (b) it follows that $\models (\boxminus A \vee \boxminus\neg A) \to \rhd A$, hence $\models \boxminus A \to \rhd A$. By normality of $\boxminus$, we have $\models \boxminus p \to \boxminus(\ell \to p)$. Therefore $\models \boxminus p \to \rhd(\ell \to p)$, for any ℓ, hence the claim: $\models \boxminus p \to \boxtimes p$.

Theorem 4.3 immediately implies that the infinitary $\Box$- and $\boxtimes$-logics of $\mathbf{K}$ are distinct. Formally, denote by L_∞ the set of all infinitary $\Box$-formulas (defined similarly to infinitary $\rhd$-formulas) that are valid in any L-frame:

$$L_\infty = \{A \in \mathbf{Fm}^{\Box}_\infty \mid \text{ for any frame } F\ (F \models L \Rightarrow F \models A)\}.$$

One can define $L^{\boxtimes}_\infty$ in the same manner. Now observe that $\mathbf{K}_\infty \neq \mathbf{K}^{\boxtimes}_\infty$, for the logic $\mathbf{K}^{\boxtimes}_\infty$ contains the formula $\boxtimes p \leftrightarrow \boxplus p$, or explicitly,

$$\boxtimes p \leftrightarrow \bigwedge_{\ell\in\mathrm{L}} \Big(\boxtimes(\ell \to p) \vee \boxtimes\neg(\ell \to p)\Big),$$

whereas the logic $\mathbf{K}_\infty$ does not contain the corresponding infinitary $\Box$-formula (since $\Box$ and $\boxtimes$ are not equivalent).

5 Conclusion

The aim of this paper was to introduce a new modal operator $\boxtimes$ defined in terms of the non-contingency operator. As we have observed, $\boxtimes$ is a necessity operator (Theorem 4.1), which is similar to the original necessity $\Box$ is some aspects (Theorem 4.2) and different from it is some others ($\mathbf{KB}^{\boxtimes} \not\supseteq \mathbf{KB}$, $\mathbf{K}^{\boxtimes}_{\infty} \neq \mathbf{K}_{\infty}$). The new necessity has several distinctive features (idempotency of the construction $\Box \mapsto \boxtimes$, the fact that $\boxtimes$ is the weakest necessity such that $\rhd$ is $\boxtimes$-definable in a natural manner).

Our main conjecture is that $\mathbf{K}^{\boxtimes} = \mathbf{K}$. If this is the case then the construction of $\boxtimes$ may be regarded as a solution of the problem concerning definability of necessity in terms of contingency. If not then the logic $\mathbf{K}^{\boxtimes}$ is a new modal logic of particular interest, like **K**, **K4** etc. Another interesting issue is axiomatization of infinitary $\boxtimes$-logics $L^{\boxtimes}_{\infty}$ over various modal logics L. This is a rather natural question, for the very definition of $\boxtimes$ is infinitary. Our candidate for $\mathbf{K}^{\boxtimes}_{\infty}$ is $\mathbf{K}_{\infty}[\boxtimes/\Box] + \{\boxtimes p \leftrightarrow \boxplus p\}$. These questions seem to be of both technical and philosophical interest, and the answers may shed a new light on the interconnection between necessity and contingency.

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