Sequent logic of arithmetic decidability*

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Abstract

Our paper continues, on the one hand, the study of modal logics that have arithmetical semantics, and on the other, the investigation of decidability (or "non-contingency") logics. We present Hilbert-style axiomatic system for the non-contingency logic of the Gödel-Löb provability logic **GL**, in other words, for the logic that is complete under the interpretation of a formula $\rhd A$ as 'a sentence A is decidable in the Peano Arithmetic **PA**'. We also present sequent calculi for the non-contingency logics of **K**, **K4**, and **GL**.

Keywords: modal logic, provability logic, non-contingency logic.

1 Introduction

The study of a central notion in mathematical logic, the provability, by means of modal logic dates back to the works of Orlov [1] and Gödel [2]. Independently of each other, they formulated a system, now known as **S4**, and left open the problem of its provability interpretation (now we know that the logic **S4** is incomplete and even incompatible with the "right" provability logic discussed below). So they raised the problem of describing of all modal principles that are valid under interpretation of modal formulas of the form '$\Box A$' as 'the proposition A is provable in the Peano Arithmetic **PA**'. Later Löb [3] suggested a new correct principle of provability, now known as Löb's axiom, and thus a modal system appeared, now known as the Gödel–Löb logic **GL**. It was conjectured that **GL** describes exactly the laws of provability in **PA**. Finaly, Solovay [4] confirmed the conjecture, thus proving the arithmetical completeness of **GL**.

An important notion related to the provability is the notion of (formal, or deductive) decidability:[1] a proposition A is called *decidable* in a theory T if either T proves A or T proves $\neg A$. For example, Gödel's Second Incompleteness Theorem states that there exists sentences that are undecidable in **PA**. The *decidability operator*, which we will denote by $\rhd$, is expressed in terms of the provability operator $\Box$ by the equality $\rhd A = \Box A \vee \Box\neg A$. A natural question raises of describing the 'laws' that the operator $\rhd$ obeys in a some modal logic L; in other words, this is the problem of finding an axiomatization of the *decidability logic* over L (for precise definitions, see below). (In modal logics not related to provability, the operator $\rhd$ is usually called *non-contingency.*)

Such a problem, not related to the provability interpretation, has already been investigated for some modal logics. In [5, 6], the non-contingency logics were axiomatized over the modal logics **T**, **S4**, **S5**; in [7, 8] the non-contingency logics over **K** and **K4** were axiomatized.

In this paper we present an axiomatization of the decidability logic over **GL**, that is, the logic that is complete under interpretation of formulas of the form $\rhd A$ as 'the proposition A is decidable in **PA**'.

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[1]Not to be confused with the same word, but denoting different notion, in the theory of algorithms, where there are decidable sets of numbers or sets of words in the sense that there is no algorithm for recognizing this set.

We also build sequent calculi for the non-contingency logics over **K**, **K4**, **GL**. Our axiomatizations of the non-contingency logics over **K** and **K4** differ from the systems proposed in [7, 8] and are more resemble the axiomatizations of **K** and **K4**.

2 Definitions

The propositional modal language (or $\Box$-language) has a countable set of variables $\mathbb{P} = \{p_0, p_1, \ldots\}$, Boolean connectives $\bot$ and $\to$ and a unary operator $\Box$. Other connectives are introduced as abbreviations. The set of $\Box$-*formulas*, $\mathbf{Fm}^\Box$, is defined in the usual way:

$$A, B \quad ::= \quad \bot \mid p \mid A \to B \mid \Box A.$$

The *minimal normal modal logic* **K** has the following axioms and rules of inference (here $A[B/p]$ is obtained from A by substituting the formula B for all occurrences of a variable p):

$(\mathsf{A}^\Box_\top)$	classical tautologies in the $\Box$-language
$(\mathsf{A}^\Box_\mathbf{K})$	$\Box(p \to q) \to (\Box p \to \Box q)$ (distributivity)

$$(\mathsf{MP})\ \frac{A \quad A \to B}{B} \qquad (\mathsf{Sub})\ \frac{A}{A[B/p]} \qquad (\mathsf{Nec})\ \frac{A}{\Box A}$$

We will be interested in the following systems: $\mathbf{K4} = \mathbf{K} + (\mathsf{A}^\Box_\mathbf{4})$, $\mathbf{GL} = \mathbf{K} + (\mathsf{A}^\Box_\mathbf{L})$, where

$(\mathsf{A}^\Box_\mathbf{4})$	$\Box p \to \Box\Box p$	(transitivity)
$(\mathsf{A}^\Box_\mathbf{L})$	$\Box(\Box p \to p) \to \Box p$	(Löb's axiom)

The following strict inclusions hold: $\mathbf{K} \subset \mathbf{K4} \subset \mathbf{GL}$.

Definition 2.1. A *sequent* is an expression of the form $\Pi \Rightarrow \Sigma$, where Π and Σ are finite multisets[2] of formulas. Inclusion between multisets will be understood without taking into account the multiplicities; that is, the notation $\Pi \subseteq \Sigma$ will mean that every formula that occurs (at least onces) in Π occurs (at least onces) in Σ. The union of multisets Π and Σ is denoted by $\Pi\Sigma$ (of course, it regards multiplicities); we also denote $\Pi A := \Pi \cup \{A\}$ for short. The set of subformulas of a formula A is denoted by $\operatorname{Sb} A$. For a multiset of formulas Γ, the set of its subformulas is denoted by $\operatorname{Sb}\Gamma := \bigcup\{\operatorname{Sb} A \mid A \in \Gamma\}$.

We will often denote a sequent $\Pi \Rightarrow \Sigma$ by w; in this case, we will denote its *antecedent* by $\langle w| := \Pi$, and its *succedent* by $|w\rangle := \Sigma$; the set of subformulas of w is denoted by $\operatorname{Sb} w := \operatorname{Sb} \Pi\Sigma$. We write $A \in w$ if $A \in \Pi\Sigma$; also we write $\Gamma \subseteq w$ if $\Gamma \subseteq \Pi\Sigma$, and $w \subseteq \Gamma$ if $\Pi\Sigma \subseteq \Gamma$. If $\mathcal{L}$ is a sequent calculus, then $\mathcal{L} \vdash A \Leftrightarrow B$ means that $\mathcal{L} \vdash A \Rightarrow B$ and $\mathcal{L} \vdash B \Rightarrow A$.

The sequent calculus $[L]$ for the logic $L \in \{\mathbf{K}, \mathbf{K4}, \mathbf{GL}\}$ is obtained by adding to the sequent calculus for the classical propositional logic (with the Cut rule) the following rule $(\Rightarrow^\Box_L)$:

$$(\Rightarrow^\Box_\mathbf{K})\ \frac{\Pi \Rightarrow A}{\Box\Pi \Rightarrow \Box A} \qquad (\Rightarrow^\Box_\mathbf{K4})\ \frac{\Pi, \Box\Pi \Rightarrow A}{\Box\Pi \Rightarrow \Box A} \qquad (\Rightarrow^\Box_\mathbf{GL})\ \frac{\Box A, \Pi, \Box\Pi \Rightarrow A}{\Box\Pi \Rightarrow \Box A}$$

2.1 Decidability (or non-contingency) logics

Let us consider the $\rhd$-*language*, which differs from the $\Box$-language just by replacing the symbol $\Box$ with $\rhd$. The set of $\rhd$-formulas is denoted by $\mathbf{Fm}^\rhd$. Sometimes we abuse the notation and, for a $\Box$-formula A, write $\rhd A$ as a denotation for $\Box A \vee \Box\neg A$; such occurrences of $\rhd$ can be easily recognized by context. Next, we define the $\rhd$-*translation* $(\cdot)_\rhd \colon \mathbf{Fm}^\rhd \to \mathbf{Fm}^\Box$ that respects variables

[2] By a *multiset* we mean a set with an indication of the 'multiplicity' ($\geqslant 0$) of each of its element. Formally, a multiset of $\Box$-formulas is a function $\mathbf{Fm}^\Box \to \mathbb{N}$.

and Boolean connectives and satisfies the equality $(\rhd A)_\rhd = \Box(A_\rhd) \vee \Box\neg(A_\rhd)$. Finally, the *non-contingency logic* over a $\Box$-logic L is, by definition, the set of all $\rhd$-formulas whose $\rhd$-translations are theorems of L:

$$L^\rhd := \{A \in \mathbf{Fm}^\rhd \mid A_\rhd \in L\}.$$

The Kripke semantics for $\Box$- and $\rhd$-languages is introduced as usual. The accessibility relation and its converse will be denoted by $\uparrow$ and $\downarrow$, respectively; in this case, the quantifiers over the points accessible from a given point w (in other words, over the *successors* of w) will be written as $\forall x{\downarrow}w$ and $\exists x{\downarrow}w$. In this notation, the modal clause in the definition of the *truth* of a $\rhd$-formula A at a point w looks as follows:

$$w \models \rhd A \quad \leftrightharpoons \quad \big(\forall x{\downarrow}w \quad x \models A\big) \text{ or } \big(\forall x{\downarrow}w \quad x \not\models A\big).$$

Clearly, $w \models A \Leftrightarrow w \models A_\rhd$, for every $\rhd$-formula A. If Γ is a set of formulas, then by a Γ-*frame* we mean a frame on which all formulas from Γ are valid. A sequent $\Pi \Rightarrow \Sigma$ is said to be true (valid) somewhere if the formula $\bigwedge\Pi \to \bigvee\Sigma$ is true (valid).

3 Axiomatization

The calculus for $\mathbf{K}^\rhd$ has the following axioms and the rules (MP), (Sub), and (Dec):

$(\mathsf{A}^\rhd_\top)$	Classical tautologies in the $\rhd$-language		(Dec) $\dfrac{A}{\rhd A}$
$(\mathsf{A}^\rhd_\neg)$	$\rhd p \leftrightarrow \rhd\neg p$	(mirror axiom)	
$(\mathsf{A}^\rhd_\leftrightarrow)$	$\rhd(p \leftrightarrow q) \to (\rhd p \leftrightarrow \rhd q)$	(equivalent replacement)	
$(\mathsf{A}^\rhd_\vee)$	$\rhd p \to [\rhd(q \to p) \vee \rhd(p \to r)]$	(dichotomy)	

The axioms of the logics $\mathbf{K4}^\rhd$ and $\mathbf{GL}^\rhd$ are:[3] $\mathbf{K4}^\rhd = \mathbf{K}^\rhd + (\mathsf{A}^\rhd_\mathbf{4})$, $\mathbf{GL}^\rhd = \mathbf{K4}^\rhd + (\mathsf{A}^\rhd_\mathbf{L})$, where

$(\mathsf{A}^\rhd_\mathbf{4})$	$\rhd p \to \rhd(q \to \rhd p)$	(weak transitivity)
$(\mathsf{A}^\rhd_\mathbf{L})$	$\rhd(\rhd p \to p) \to \rhd p$	(weak Löb's axiom)

The sequent calculi[4] $[L^\rhd]$ for the non-contingency logics over $L \in \{\mathbf{K}, \mathbf{K4}, \mathbf{GL}\}$ are obtained by adding to the seqent calculus for the classical propositional logic (with the Cut rule) the rules $({}^\rhd_\neg\!\Rightarrow)$, $(\Rightarrow^\rhd_\neg)$, $(\Rightarrow^\rhd_\vee)$, $(\Rightarrow^\rhd_\leftrightarrow)$ and $(\Rightarrow^\rhd_L)$ shown on Fig. 1, where we denote: $(\Pi \vee A) := \{(\pi \vee A) \mid \pi \in \Pi\}$.

$$(\Rightarrow^\rhd_\vee)\ \frac{\Pi \Rightarrow \Sigma, \rhd A}{\Pi \Rightarrow \Sigma, \rhd(B \to A), \rhd(A \to C)} \qquad (\Rightarrow^\rhd_\leftrightarrow)\ \frac{\Pi, A \Rightarrow B, \Sigma \quad \Pi, B \Rightarrow A, \Sigma}{\Pi, \rhd A \Rightarrow \rhd B, \Sigma}$$

$$({}^\rhd_\neg\!\Rightarrow)\ \frac{\rhd A, \Pi \Rightarrow \Sigma}{\rhd\neg A, \Pi \Rightarrow \Sigma} \qquad (\Rightarrow^\rhd_\neg)\ \frac{\Pi \Rightarrow \Sigma, \rhd A}{\Pi \Rightarrow \Sigma, \rhd\neg A} \qquad (\Rightarrow^\rhd_\mathbf{K})\ \frac{\Pi \Rightarrow A}{\rhd(\Pi \vee A) \Rightarrow \rhd A}$$

$$(\Rightarrow^\rhd_\mathbf{K4})\ \frac{\Pi, \rhd\Sigma \Rightarrow A}{\rhd(\Pi \vee A), \rhd\Sigma \Rightarrow \rhd A} \qquad (\Rightarrow^\rhd_\mathbf{GL})\ \frac{\rhd A, \Pi, \rhd\Sigma \Rightarrow A}{\rhd(\Pi \vee A), \rhd\Sigma \Rightarrow \rhd A}$$

Figure 1: Rules of the sequent calculi $[\mathbf{K}^\rhd]$, $[\mathbf{K4}^\rhd]$, and $[\mathbf{GL}^\rhd]$.

[3] **Conjecture:** the axiom $(\mathsf{A}^\rhd_\mathbf{4})$ in the calculus $\mathbf{GL}^\rhd$ is redundant (recall that $\Box p \to \Box\Box p$ is redundant in $\mathbf{GL}$).

[4] Perhaps, it is more natural to formulate the sequent calculi if $\neg$, $\vee$ are primitive rather than defined connectives.

4 Completeness

The completeness proof is based, on the one hand, on the canonical model construction that was adapted in [7, 8] to deal with the $\rhd$-logics; on the other hand, it is based on the 'sequent saturation' method that is customarily used in the completeness proofs for (ordinary) sequent calculi. First we prove two auxiliary lemmas.

Lemma 4.1. $\mathbf{K}^{\rhd} \vdash \rhd p \,\&\, \rhd q \to \rhd(p \,\&\, q)$.

Proof. . The following implications and equivalences derivable in $\mathbf{K}^{\rhd}$:

$\mathbf{K}^{\rhd} \vdash \rhd[p \to q] \stackrel{1}{\longleftrightarrow} \rhd[p \leftrightarrow (p \,\&\, q)] \stackrel{2}{\longrightarrow} [\rhd p \to \rhd(p \,\&\, q)]$.

Here '$\stackrel{1}{\longleftrightarrow}$' is obtained from the tautology $[p \to q] \leftrightarrow [p \leftrightarrow (p \,\&\, q)]$ by the axiom $(\mathsf{A}^{\rhd}_{\leftrightarrow})$, while '$\stackrel{2}{\longrightarrow}$' is a substitution instance of the axiom $(\mathsf{A}^{\rhd}_{\leftrightarrow})$. Similarly:

$\mathbf{K}^{\rhd} \vdash \rhd[q \to p] \to [\rhd q \to \rhd(p \,\&\, q)]$. Finally, using the Dichotomy axiom $(\mathsf{A}^{\rhd}_{\vee})$, we obtain:

$$\begin{aligned}\mathbf{K}^{\rhd} \vdash \rhd p &\longrightarrow \{\rhd(q \to p) \vee \rhd(p \to q)\}\\ &\longrightarrow \{[\rhd p \to \rhd(p \,\&\, q)] \vee [\rhd q \to \rhd(p \,\&\, q)]\}\\ &\longleftrightarrow \{(\rhd p \,\&\, \rhd q) \to \rhd(p \,\&\, q)\}.\end{aligned}$$

Now note that the first premise $\rhd p$ in the formula $\rhd p \to \{(\rhd p \& \rhd q) \to \rhd(p \& q)\}$ is redundant. $\square$

Lemma 4.2. $\mathbf{K4}^{\rhd} \vdash \rhd(p \vee q) \to \rhd[p \vee (\rhd q \to q)]$.

Proof. . Since the completeness of $\mathbf{K4}^{\rhd}$ will already be proved by the time we will use this Lemma[5] we can build a derivation in $\mathbf{K4}$, not in $\mathbf{K4}^{\rhd}$. On the one hand, by the monotonicity:

$\mathbf{K} \vdash \Box(p \vee q) \longrightarrow \Box[p \vee (\rhd q \to q)] \longrightarrow \rhd[p \vee (\rhd q \to q)]$. On the other hand:

$$\begin{aligned}\mathbf{K4} \vdash\ & \Box\neg(p \vee q) \longleftrightarrow \Box(\neg p \,\&\, \neg q) \longleftrightarrow [\Box\neg p \,\&\, \Box\neg q] \longrightarrow [\Box\neg p \,\&\, \Box\neg q \,\&\, \Box\Box\neg q] \longrightarrow\\ &\to [\Box\neg p \,\&\, \Box\neg q \,\&\, \Box\rhd q] \longleftrightarrow \Box(\neg p \,\&\, \neg q \,\&\, \rhd q) \longleftrightarrow \Box\neg[p \vee (\rhd q \to q)] \longrightarrow \rhd[p \vee (\rhd q \to q)].\end{aligned}$$ $\square$

Theorem 4.3 (Completeness)**.** *For every logic $L \in \{\mathbf{K}, \mathbf{K4}, \mathbf{GL}\}$ and for any sequent $\Pi \Rightarrow \Sigma$ in the $\rhd$-language, the following statements are equivalent:*

(1) $[L^{\rhd}] \vdash \Pi \Rightarrow \Sigma$,
(2) $L^{\rhd} \vdash \bigwedge\Pi \to \bigvee\Sigma$,
(3) $L \vdash (\bigwedge\Pi \to \bigvee\Sigma)_{\rhd}$,
(4) $F \models \Pi \Rightarrow \Sigma$, *for every finite L-frame F.*

Proof. We follow the schema (1)⇒(2)⇒(3)⇔(4)⇒(1). Here the equivalence (3)⇔(4) is the well-known (see [9, 10]) completeness theorem for our three logics L with respect to the class of finite L-frames.[6] In the sequel, we refer to $\rhd$-formulas as just formulas.

(1) ⇒ (2) In the derivations presented below, we use the following simple fact from the classical propositional logic: if the implication $P \to A$ is derivable, then the equivalence $[P \vee A] \leftrightarrow A$ is derivable as well. Let us assume that $\Pi = \{\pi_1, \ldots, \pi_m\}$ and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$, where $m, n \geqslant 0$.

$L = \mathbf{K}$. Assume that $\mathbf{K}^{\rhd} \vdash \bigwedge\Pi \to A$. Using Lemma 4.1, the axiom $(\mathsf{A}^{\rhd}_{\leftrightarrow})$, and the above mentioned fact, we derive:

$\mathbf{K}^{\rhd} \vdash \bigwedge\rhd(\pi_i \vee A) \longrightarrow \rhd[\bigwedge(\pi_i \vee A)] \longleftrightarrow \rhd(\bigwedge\Pi \vee A) \longleftrightarrow \rhd A$.

$L = \mathbf{K4}$. Assume that $\mathbf{K4}^{\rhd} \vdash \bigwedge(\Pi, \rhd\Sigma) \to A$. Using the axiom $(\mathsf{A}^{\rhd}_{\mathbf{4}})$ rewritten as $\rhd\sigma \to \rhd(A \vee \rhd\sigma)$, we derive:

$$\begin{aligned}\mathbf{K4}^{\rhd} \vdash\ & [\bigwedge\rhd(\pi_i \vee A) \,\&\, \bigwedge\rhd\sigma_j] \longrightarrow [\rhd(\bigwedge\pi_i \vee A) \,\&\, \bigwedge\rhd(A \vee \rhd\sigma_j)] \longrightarrow\\ &\to [\rhd(\bigwedge\Pi \vee A) \,\&\, \rhd(\bigwedge\rhd\Sigma \vee A)] \longrightarrow \rhd\,[\bigwedge(\Pi, \rhd\Sigma) \vee A] \longleftrightarrow \rhd A.\end{aligned}$$

[5] This lemma will be used only in the completeness proof for $\mathbf{GL}^{\rhd}$. Note that only for this Lemma we included the axiom $(\mathsf{A}^{\rhd}_{\mathbf{4}})$ in the axiomatization of $\mathbf{GL}^{\rhd}$.

[6] Recall that the logic $\mathbf{K}$ is valid on all frames; the class of (finite) $\mathbf{K4}$-frames is the class of (finite) transitive frames; the class of finite $\mathbf{GL}$-frames is exactly the class of finite irreflexive transitive frames.

$\underline{L = \mathbf{GL}.}$ Assume that $\mathbf{GL}^{\triangleright} \vdash \bigwedge(\Pi, \triangleright\Sigma) \to (\triangleright A \to A)$. We have already proved above that

$\mathbf{K4}^{\triangleright} \vdash [\bigwedge\triangleright(\pi_i \vee A) \,\&\, \bigwedge\triangleright\sigma_j] \longrightarrow \triangleright\, [\bigwedge(\Pi, \triangleright\Sigma) \vee A]$. Now, using Lemma 4.2, we derive:

$\mathbf{K4}^{\triangleright} \vdash \triangleright\, [\bigwedge(\Pi, \triangleright\Sigma) \vee A] \longrightarrow \triangleright\, [\bigwedge(\Pi, \triangleright\Sigma) \vee (\triangleright A \to A)]$. Finally, using the axiom $(\mathsf{A}^{\triangleright}_{\mathbf{L}})$, we obtain:

$\mathbf{GL}^{\triangleright} \vdash \triangleright\, [\bigwedge(\Pi, \triangleright\Sigma) \vee (\triangleright A \to A)] \longleftrightarrow \triangleright(\triangleright A \to A) \longrightarrow \triangleright A$.

$\underline{(2) \Rightarrow (3)}$ The axioms of $\mathbf{K}^{\triangleright}$ are valid on every frame, hence their $\triangleright$-translations are derivable in $\mathbf{K}$.

Let us derive the $\triangleright$-translation of the axiom $(\mathsf{A}^{\triangleright}_{\mathbf{4}})$ in the logic $\mathbf{K4}$. On the one hand:

$\mathbf{K4} \vdash \Box p \longrightarrow \Box\Box p \longrightarrow \Box\triangleright p \longrightarrow \Box(q \to \triangleright p) \longrightarrow \triangleright(q \to \triangleright p)$.

On the other hand: $\mathbf{K4} \vdash \Box\neg p \to \triangleright(q \to \triangleright p)$.

Let us derive the $\triangleright$-translation of the axiom $(\mathsf{A}^{\triangleright}_{\mathbf{L}})$ in $\mathbf{GL}$.

Using the tautology $(\triangleright p \to p) \to (\Box p \to p)$, we obtain:

$\mathbf{GL} \vdash \Box(\triangleright p \to p) \longrightarrow \Box(\Box p \to p) \longrightarrow \Box p \longrightarrow \triangleright p$. At the same time:

$\mathbf{GL} \vdash \Box\neg(\triangleright p \to p) \longleftrightarrow \Box(\triangleright p \,\&\, \neg p) \longrightarrow \Box\neg p \longrightarrow \triangleright p$.

$\underline{(4) \Rightarrow (1)}$ Let us denote the sequent calculus for L by $\mathcal{L} := [L^{\triangleright}]$. Denote $\overline{A} := B$ if $A = \neg B$, for some formula B; otherwise denote $\overline{A} := \neg A$. For a set of formulas Γ, denote $\overline{\Gamma} := \{\overline{A} \mid A \in \Gamma\}$. Let us call a set of formulas Γ *closed* if $\mathrm{Sb}\,\Gamma \subseteq \Gamma$ and $\overline{\Gamma} \subseteq \Gamma$ (in other words, if it is closed under subformulas and 'economical' negation). In it easily seen that any finite set of formulas is contained in some, again finite, closed set of formulas. Let us also say that a sequent w is Γ-*saturated* if $\Gamma \subseteq w$; a sequent w is called *thin* if both $\langle w|$ and $|w\rangle$ are sets, in the sense that every formula in them has the multiplicity 1.

For the sake of contradiction, assume that $\mathcal{L} \nvdash \Pi \Rightarrow \Sigma$. We will build a finite counter-model $M^{\Gamma}_{\mathcal{L}}$ for $\Pi \Rightarrow \Sigma$ based on an L-frame. To this end, we put:

$$\Gamma := \mathrm{Sb}\,\Pi\Sigma,\ \natural\Gamma := \{A, \overline{A} \mid \triangleright A \in \Gamma\},\ \beta := \Gamma \cup \mathrm{Sb}\{\triangleright(A \vee B) \mid A, B \in \natural\Gamma\},\ \widehat{\Gamma} := \beta \cup \overline{\beta}.$$

Clearly, the set $\widehat{\Gamma}$ is closed.

Here we introduce an important notation. For an arbitrary formula $A \in \natural\Gamma$, let us denote:

$$\boxtimes A := \{\triangleright(B \vee A) \mid B \in \natural\Gamma\} \subseteq \beta.$$

In the subsequent proof, the symbol $\boxtimes$ will play the rôle that is similar to the role of the operator $\Box$ in the standard construction of the canonical model of a normal logic. The difference, however, is in their "types": the $\Box$ was an operator that, when applied to a formula, yields again a formula; at the same time, when $\boxtimes$ is applied to a formula, it yields a finite set of formulas.[7] One should keep in mind, however, that semantically $\boxtimes$ is not equivalent to $\Box$ (we will not prove this here).

We are ready to construct a finite counter-model $M^{\Gamma}_{\mathcal{L}} = (W^{\Gamma}_{\mathcal{L}}, \uparrow, \models)$ for the sequent $\Pi \Rightarrow \Sigma$. Put

$$W^{\Gamma}_{\mathcal{L}} := \{w \mid\ w \text{ is a thin}^{8}\ \widehat{\Gamma}\text{-saturated sequent},\ \ w \subseteq \widehat{\Gamma},\ \text{ and } \mathcal{L} \nvdash w\}.$$

It is clearly a finite set. Due to the presence in $\mathcal{L}$ of the Cut rule (and the Contraction rule), every sequent that is underivable in $\mathcal{L}$ and consists of formulas from Γ can be extended to a thin $\widehat{\Gamma}$-saturated sequent that is again underivable in $\mathcal{L}$. In particular, since the sequent $\Pi \Rightarrow \Sigma$ is not derivable in $\mathcal{L}$, by our assumption, $\exists z \in W^{\Gamma}_{\mathcal{L}}\colon\ \ \Pi \subseteq \langle z|,\ \Sigma \subseteq |z\rangle$, and hence $W^{\Gamma}_{\mathcal{L}} \neq \varnothing$.

Next, we define the valuation of variables "canonically":

$$w \models p \ \leftrightharpoons\ p \in \langle w|, \text{ for every } w \in W^{\Gamma}_{\mathcal{L}} \text{ and } p \in \mathbb{P}.$$

[7] Remark in translation: note that the symbol $\boxtimes$ is used in other papers of ours to denote the infinite set of formulas $\{\triangleright(B \vee A) \mid B \in \mathbf{Fm}^{\triangleright}\}$ or even the (infinitary) conjunction of this infinite set. Here we will only need "normality" of $\boxtimes$ with respect to the formulas in $\natural\Gamma$, and for this reason, it suffices to take into account only formulas $\triangleright(B \vee A)$ with $B \in \natural\Gamma$.

[8] We need to confine to thin sequents, otherwise we can repeat the same formula as many times as we want and thus obtain an infinite set $W^{\Gamma}_{\mathcal{L}}$.

It remains to construct an accessibility relation $\uparrow$ on $W^\Gamma_{\mathcal{L}}$. We will approach this task gradually. To begin with, it is easily seen that the following condition (on the relation $\uparrow$) will be sufficient to complete the proof of our Theorem:

$$\langle 1^\rhd\rangle \qquad \forall w \in W^\Gamma_{\mathcal{L}}\ \forall A \in \Gamma. \qquad w \models A \Leftrightarrow A \in \langle w|.$$

Indeed, for the point z we found above, we will obtain, by $\langle 1^\rhd\rangle$, that $z \models \bigwedge\Pi$ and $z \models \bigwedge\neg\Sigma$, therefore, $z \not\models \Pi \Rightarrow \Sigma$ and so $M^\Gamma_{\mathcal{L}} \not\models \Pi \Rightarrow \Sigma$.

The condition $\langle 1^\rhd\rangle$ on the relation $\uparrow$ is formulated as the relationship between the truth relation $\models$ and the membership of formulas to the antecedents of sequents from $W^\Gamma_{\mathcal{L}}$. Now recall that, in the definition of the truth relation, only the modal clause (for $\rhd$) depends on the relation $\uparrow$. Therefore it suffices to require the following condition for $\uparrow$; here the square bracket means the disjunction:

$$\langle 2^\rhd\rangle \qquad \forall w \in W^\Gamma_{\mathcal{L}}\ \forall \rhd A \in \Gamma. \qquad \rhd A \in \langle w| \ \Leftrightarrow \ \left[\begin{array}{ll} \forall x{\downarrow}w & A \in \langle x|, \\ \forall x{\downarrow}w & A \in |x\rangle. \end{array}\right.$$

Indeed, the implication $\langle 2^\rhd\rangle \Rightarrow \langle 1^\rhd\rangle$ can be easily proved, as usually, simultaneously for all $w \in W^\Gamma_{\mathcal{L}}$, by induction on the complexity of the formula A (taken from Γ); the proof in our case (for $\rhd$-formulas) literally repeats the same proof for $\Box$-logics.

Given a subset $\Phi \subseteq \natural\Gamma$, let us denote $\sharp\Phi := \{A \in \natural\Gamma \mid \boxtimes A \subseteq \Phi\}$. It is easily seen that the following inclusions hold: $\boxtimes\sharp\Phi \subseteq \Phi \subseteq \sharp\boxtimes\Phi$. Furthermore, for any sequent $w \in W^\Gamma_{\mathcal{L}}$, let us denote $\sharp w := \sharp\langle w|$.

We came to the crux of the proof. We define the relation $\uparrow$ on $W^\Gamma_{\mathcal{L}}$ as follows, where the curly bracket means the conjunction:[9]

$$\begin{array}{lll} \langle 3^\rhd_{\mathbf{K}}\rangle & w{\uparrow}x & \risingdotseq\ \sharp w \subseteq \langle x|; \\ \langle 3^\rhd_{\mathbf{4}}\rangle & w{\uparrow}x & \risingdotseq\ \sharp w \subseteq \big(\sharp x \cap \langle x|\big); \\ \langle 3^\rhd_{\mathbf{L}}\rangle & w{\uparrow}x & \risingdotseq\ \left\{\begin{array}{l} \sharp w \subseteq \big(\sharp x \cap \langle x|\big), \\ \exists A \in \natural\Gamma. \quad \rhd A \in |w\rangle\ \ \&\ \ \rhd A \in \langle x|. \end{array}\right. \end{array}$$

Lemma 4.3.1 (Dichotomy). *If* $\rhd A \in \Gamma$ *and* $\rhd A \in \langle w|$, *then* $\big(A \in \sharp w$ *or* $\overline{A} \in \sharp w\big)$.

► Assume the contrary. Then, since $A \in \natural\Gamma$, we obtain:

1) $\boxtimes A \not\subseteq \langle w|$, i.e., $\exists B \in \natural\Gamma\colon\ \rhd(B \vee A) \notin \langle w|$; but since $\rhd(B \vee A) \in \beta$, we have $\rhd(B \vee A) \in |w\rangle$.

2) $\boxtimes\overline{A} \not\subseteq \langle w|$, i.e., $\exists C \in \natural\Gamma\colon\ \rhd(C \vee \overline{A}) \notin \langle w|$, but since $\rhd(C \vee \overline{A}) \in \beta$, we have $\rhd(C \vee \overline{A}) \in |w\rangle$.

However $\mathcal{L} \vdash \rhd A \Rightarrow \rhd(B \vee A), \rhd(C \vee \overline{A})$; indeed, this sequent is obtained by the rule $(\Rightarrow^\rhd_\vee)$, if we take into account that from $\mathcal{L} \vdash (B \vee A) \Leftrightarrow (\neg B \to A)$ by the rule $(\Rightarrow^\rhd_\leftrightarrow)$ one can derive that $\mathcal{L} \vdash \rhd(B \vee A) \Leftrightarrow \rhd(\neg B \to A)$, and similarly $\mathcal{L} \vdash \rhd(C \vee \overline{A}) \Leftrightarrow \rhd(A \to C)$. Therefore, $\mathcal{L} \vdash w$, which contradicts to that $w \in W^\Gamma_{\mathcal{L}}$. ◄

Lemma 4.3.2 (Main). $\langle 3^\rhd_{\mathfrak{S}}\rangle \Rightarrow \langle 2^\rhd\rangle$, *for every* $\mathfrak{S} \in \{\mathbf{K}, \mathbf{4}, \mathbf{L}\}$.

► We need to prove the equivalence in $\langle 2^\rhd\rangle$, for every $w \in W^\Gamma_{\mathcal{L}}$ and $\rhd A \in \Gamma$. One implication is easy:

($\Rightarrow$) Assume that $\rhd A \in \langle w|$. By the Dichotomy lemma, two cases are possible:

[9] To understand this definition, it is instructive to recall how the relation is defined in the usual $\Box$-language. Let us give these definitions for the case when we deal with maximal consistent subsets of $\widehat{\Gamma}$, in order not to overwhelm the reader with antecedents and succedents of sequents.

For **K**, one puts $w{\uparrow}x$ if, for every formula of the form $\Box A$, if $\Box A \in w$ then $A \in x$.

For **K4**, one puts $w{\uparrow}x$ if, for every formula of the form $\Box A$, if $\Box A \in w$ then $A, \Box A \in x$.

For **GL**, one puts $w{\uparrow}x$ if a) for every formula of the form $\Box A$, if $\Box A \in w$ then $A, \Box A \in x$; and b) there is a formula $\Box A \notin w$ such that $\Box A \in x$. Item (a) guarantees transitivity, item (b) irreflexivity.

When we return to sequents, then membership $B \in w$ really means $B \in \langle w|$, and non-membership $B \notin w$ (for formulas $B \in \widehat{\Gamma}$ and for $\widehat{\Gamma}$-saturated sequents) means $B \in |w\rangle$. Finally, $\Box A \in w$ is the same as saying that $A \in \sharp w$.

1) $A \in \sharp w$, then by the condition $\langle 3_L^{\triangleright}\rangle$ we conclude: $\forall x{\downarrow}w\ A \in \langle x|$.

2) $\overline{A} \in \sharp w$, then similarly $\forall x{\downarrow}w\ \overline{A} \in \langle x|$. But $A \in \natural\Gamma \subseteq \widehat{\Gamma}$, hence $A \in x$, because x is $\widehat{\Gamma}$-saturated; however, the case $A \in \langle x|$ is impossible, since otherwise $\mathcal{L} \vdash x$. Therefore, $A \in |x\rangle$.

($\Leftarrow$) This implication is proved for each of our three logics individually. Assume that $\triangleright A \notin \langle w|$. Since $A \in \widehat{\Gamma}$ and the sequent w is $\widehat{\Gamma}$-saturated, this means that $\triangleright A \in |w\rangle$. We need to build two points $x, y \in W^{\Gamma}_{\mathcal{L}}$ such that $w{\uparrow}x$, $w{\uparrow}y$ and $A \in \langle x|$ and $A \in |y\rangle$.

<u>Case $\mathfrak{s} = \mathbf{K}$.</u> Consider $\Pi := \sharp w$. Let us prove that $\mathcal{L} \nvdash \Pi \Rightarrow A$. Assume the contrary, then by the rule $(\Rightarrow^{\triangleright}_{\mathbf{K}})$ we derive: $\mathcal{L} \vdash \triangleright(\Pi \vee A) \Rightarrow \triangleright A$. Then $\forall \pi \in \Pi$, we have $\pi \in \sharp w$, which means that $\boxtimes\pi \subseteq \langle w|$. Hence $\forall \alpha \in \natural\Gamma$: $\triangleright(\alpha \vee \pi) \in \langle w|$. Since it is clear that $\mathcal{L} \vdash \triangleright(\alpha \vee \pi) \Leftrightarrow \triangleright(\pi \vee \alpha)$, we obtain: $\forall \alpha \in \natural\Gamma$: $\triangleright(\pi \vee \alpha) \in \langle w|$. In particular, taking $\alpha := A$, we conclude that $\triangleright(\pi \vee A) \in \langle w|$. Therefore, $\triangleright(\Pi \vee A) \subseteq \langle w|$, $\triangleright A \in |w\rangle$ and $\mathcal{L} \vdash \triangleright(\Pi \vee A) \Rightarrow \triangleright A$; thus $\mathcal{L} \vdash w$, which contradicts to that $w \in W^{\Gamma}_{\mathcal{L}}$ (i.e., w was an $\mathcal{L}$-underivable sequent).

Similarly, $\mathcal{L} \nvdash \Pi A \Rightarrow$, for otherwise $\mathcal{L} \vdash \Pi \Rightarrow \overline{A}$ and we can apply the same argument, additionally taking into account that $\overline{A} \in \natural\Gamma$ and $\triangleright\overline{A} \in |w\rangle$.

It follows that the sequents $\Pi A \Rightarrow$ and $\Pi \Rightarrow A$ can be embedded inth some thin $\widehat{\Gamma}$-saturated sequents $x, y \in W^{\Gamma}_{\mathcal{L}}$. Then $w{\uparrow}x, y$, since $\sharp w = \Pi \subseteq \langle x|, \langle y|$. By construction, $A \in \langle x|$ and $A \in |y\rangle$.

<u>Case $\mathfrak{s} = \mathbf{4}$.</u> Consider $\Pi := \sharp w$, $\Phi := \boxtimes\Pi$. Then $\mathcal{L} \nvdash \Pi\Phi \Rightarrow A$, for otherwise by the rule $(\Rightarrow^{\triangleright}_{\mathbf{K4}})$ we could derive: $\mathcal{L} \vdash \triangleright(\Pi \vee A), \Phi \Rightarrow \triangleright A$, since $\Phi = \triangleright\Sigma$, for some Σ. As in the case for $\mathfrak{s} = \mathbf{K}$, we have that $\triangleright(\Pi \vee A) \subseteq \langle w|$. Furthermore, $\Phi = \boxtimes\sharp w \subseteq \langle w|$. Hence $\mathcal{L} \vdash w$, which is impossible.

Now, the sequent $\Pi\Phi \Rightarrow A$ is embedded in some $y \in W^{\Gamma}_{\mathcal{L}}$. It remains to check that $w{\uparrow}y$. We have:

$$\sharp w = \Pi \subseteq \langle y|,\ \sharp w = \Pi \subseteq \sharp\boxtimes\Pi = \sharp\Phi \subseteq \sharp y, \text{ since } \Phi \subseteq \langle y|.$$

Similarly, one can show that $\mathcal{L} \nvdash \Pi\Phi A \Rightarrow$ and build the required x.

<u>Case $\mathfrak{s} = \mathbf{L}$.</u> As in the case for $\mathfrak{s} = \mathbf{4}$, we build Π, Φ; then we prove, using the rule $(\Rightarrow^{\triangleright}_{\mathbf{GL}})$, that the sequent $\triangleright A, \Pi, \Phi \Rightarrow A$ is not derivable in $\mathcal{L} = [\mathbf{GL}^{\triangleright}]$, and finally, we embed this sequent to some sequent $y \in W^{\Gamma}_{\mathcal{L}}$. As in the case for $\mathfrak{s} = \mathbf{4}$, we can show that $\sharp w \subseteq \big(\sharp y \cap \langle y|\big)$; in addition, we have $\triangleright A \in |w\rangle$ and $\triangleright A \in \langle y|$. Hence $w{\uparrow}y$ (where the relation $\uparrow$ is defined according to $\langle 3^{\triangleright}_{\mathbf{L}}\rangle$). Similarly we build the required x. ◀

It remains to prove that the frame $F^{\Gamma}_{\mathcal{L}}$ we built above is an L-frame. The case for $\mathfrak{s} = \mathbf{K}$ is trivial.

Case $\mathfrak{s} = \mathbf{4}$: if $w{\uparrow}x{\uparrow}y$, then $\sharp w \subseteq \big(\sharp x \cap \langle x|\big) \subseteq \sharp x \subseteq \big(\sharp y \cap \langle y|\big)$ and $w{\uparrow}y$; hence $\uparrow$ is transitive.

Case $\mathfrak{s} = \mathbf{L}$: the irreflexivity of $\uparrow$ follows from the second line of its definition $\langle 3^{\triangleright}_{\mathbf{L}}\rangle$ and from the underivability of the sequents in $\mathcal{L}$. Now let us prove that $\uparrow$ is transitive.

Suppose that $w{\uparrow}x{\uparrow}y$. Then, similarly to the above, $\sharp w \subseteq \big(\sharp y \cap \langle y|\big)$. Furthermore, $\exists A{\in}\natural\Gamma$: $\triangleright A \in |w\rangle$ and $\triangleright A \in \langle x|$. Let us show that $\triangleright A \in \langle y|$. By the Dichotomy lemma, two cases are possible:

a) $A \in \sharp x$, but $\sharp x \subseteq \sharp y$, hence $A \in \sharp y$; in particular, $\triangleright(A \vee A) \in y$. Since $\mathcal{L} \vdash \triangleright(A \vee A) \Leftrightarrow \triangleright A$, we obtain $\triangleright A \in \langle y|$.

b) $\overline{A} \in \sharp x$. The same argument is applicable, since $\overline{A} \in \natural\Gamma$ and $\mathbf{K}^{\triangleright} \vdash \triangleright A \leftrightarrow \triangleright\overline{A}$.

This completes the proof of our Theorem. □

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